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ARTICLE

Finite Time Blow Up of Solutions for the m-Laplacian Equation With Variable Coefficients

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Abstract

In this work, we deal with the m- Laplacian equation with time dependent variable coefficients. Under suitable conditions on variable coefficients, we prove the blow up of solutions for finite time with negative initial energy. These results partially generalize and extend some recent ones in previous literature.

MSC: 35A01, 35B44, 35L05

Keywords: Blowup, m- Laplacian equation, Variable coefficients

1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + \mu_1(t)|u_t|^{p-2}u_t = \mu_2(t)|u|^{q-2}u, & x\in\Omega, t>0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = 0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n $(n \in N)$ with a smooth boundary $\partial\Omega$, and $m \geq 2, p \geq 2, q > 2,$ $\mu_1(t)$ is a non-negative function of t and $\mu_2(t)$ is a positive functions of t. The quantity $|u_t|^{p-2}u_t$ is a damping term which assures global existence, and $|u|^{q-2}u$ is the source term which contributes to nonexistence of global solutions. $\mu_1(t)$ and $\mu_2(t)$ can be regarded as two control buttons which can dominate the polarity between damping term and source term.

When $\mu_1(t) = \mu_2(t)\equiv 1$ and $m = 2$, then the problem (1) can be reduced to the following wave equation

$$u_{tt} - \Delta u + |u_t|^{p-2}u_t = |u|^{q-2}u.$$

Many authors have been established the existence, nonexistence and decay of solutions, see [2,4–6,12,13]. The interaction between nonlinear damping $(|u_t|^{p-2}u_t)$ and the source term $(|u|^{q-2}u)$ makes the problem more interasting. Levine [4,5] first studied the interaction between the linear damping $(p = 2)$ and source term by using Concavity method. But this method can't be applied in the case of a nonlinear damping term. Georgiev and Todorova [2] extended Levine's result to the nonlinear case $(p > 2)$. They showed that solutions with negative initial energy blow up in finite time. Later, Vitillaro in [13] extended these results to situations where the nonlinear damping and the solution has positive initial energy.

Pişkin and Fidan [9] considered

$$u_{tt} - \Delta u - \Delta u_t + \mu_1(t)|u_t|^{p-2}u_t = \mu_2(t)|u|^{q-2}u,$$

with initial-boundary conditions, and proved a blow up of solutions.

Messaoudi [7], studied the following problem

$$u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u. \tag{2}$$

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He studied decay of solutions of the problem (2). Then the problem (2) was studied by Wu and Xue [14] and Pişkin [8].

Zheng et al. [16] considered the Petrovsky equation in a bounded domain. They proved the blow up of solutions.

$$u_{tt} - \Delta^2 u + k_1(t)|u_t|^{m-2}u_t = k_2(t)|u|^{p-2}u$$

In this paper, we established the blow up of solutions. To our best knowledge, the blow up of solutions of m-Laplacian equation with variable coefficients have not yet studied. By using the same techniques in [16].

This paper is organized as follows: In the next section, we present some lemmas, not notations and local existence theorem. In section 3, the blow up of solutions are given.

2. Preliminaries

In order to state the main results to problem (1) more clearly, we start to our work by introducing some notations lemmas which will be used in this paper. Throughout this paper $\|u\|_p = \|u\|_{L^p(\Omega)}$ and $\|u\|_2 = \|u\|$ denote the usual $L^p(\Omega)$ norm and $L^2(\Omega)$ norm, respectively. Also, $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ is a Hilbert spaces (see [1, 11], for details).

Lemma 2.1. [3] Assume that

$$\begin{cases} m \le q < \infty, n \le m, \\ m < q < \dfrac{nm}{n-m}, n > m. \end{cases}$$

Then, there exist a positive constant C>1, depending on Ω only, such that

$$\|u\|_q^s \le C\left(\|\nabla u\|_m^m + \|u\|_q^q\right) \tag{3}$$

For any $u \in W_0^{1,m}(\Omega)$ and $m \le s \le q$.

Lemma 2.2. Suppose that $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t and $\mu_2'(t) \ge 0$. Let u(t) be a solution of problem (1) then the energy functional E(t) is non- increasing, namely $E'(t) \le 0$.

Proof. Multiplying the equation (1) with u_t and integrating with respect to x over the domain Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{\mu_2(t)}{q}\|u\|_q^q\right) = -\mu_1(t)\|u_t\|_p^p - \frac{\mu_2'(t)}{q}\|u\|_q^q. \tag{4}$$

By the equality (4), we get

$$E'(t) = -\mu_1(t)\|u_t\|_p^p - \frac{\mu_2'(t)}{q}\|u\|_q^q \le 0,$$

and $E(t) \le E(0)$. Where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{m}\|\nabla u\|_m^m - \frac{\mu_2(t)}{q}\|u\|_q^q \tag{5}$$

and

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{m}\|\nabla u_0\|_m^m - \frac{\mu_2(0)}{q}\|u_0\|_q^q.$$

In order to obtain our main results, we set

$$H(t) = -E(t) \tag{6}$$

In the following remark, C denotes a generic constant that varies from line to line. Combining (3), (5) and (6), we obtain

Remark 2.1. Assume that

$$\begin{cases} m \le q < \infty, n \le m, \\ m < q < \dfrac{nm}{n-m}, n > m, \end{cases}$$

and energy functional $E(t) < 0$. Then, there exist a positive constant C, depending only on Ω, such that

$$\|u\|_q^s \le C\left(H(t) + \|u_t\|^2 + \left(\frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right) \tag{7}$$

for any $u \in W_0^{1,m}(\Omega)$ and $m \le s \le q$.

Next, we state the local existence theorem that can be established by combining arguments of [2,10,15].

Theorem 2.1. (Local existence). Suppose that

$$\begin{cases} m \le q < \infty, n \le m, \\ m < q < \dfrac{nm}{n-m}, n > m. \end{cases}$$

Then, for any given $(u_0, u_1) \in (W_0^{1,m}(\Omega) \times L^2(\Omega))$, the problem (1) has a local solution satisfying

$$u \in C\left([0,T] : W_0^{1,m}(\Omega), u_t \in C\left(\times [0,T]; L^2(\Omega)\right) \cap L^p(\Omega, [0,T])\right)$$

for some T>0.

3. Blow up

In this section, we will consider the blow up of solutions for the problem (1).

Theorem 3.1. Let the assumptions of Lemma 2 hold. And assume that $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t, $\mu_2'(t) \geq 0$ and

$$\lim_{t\to\infty}\mu_1(t)\mu_2(t)^{\alpha(p-1)}$$

exists, where

$$0 < \alpha \leq min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}.$$

Then the solution of Eq. (1) blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}$$

if $q > p$ and the initial energy function

$$E(0) < 0,$$

where

$$L(0) = [H(0)]^{1-\alpha} + \varepsilon\int_\Omega u_0 u_1 dx > 0.$$

Proof. From $(4)-(6)$, we get

$$\frac{d}{dt}H(t) = \mu_1(t)\|u_t\|_p^p + \frac{\mu_2'(t)}{q}\|u\|_q^q \geq 0 \tag{8}$$

for almost, every $t \in [0, T)$. Also,

$$0 < H(0) \leq H(t) \leq \frac{\mu_2(t)}{q}\|u\|_q^q, t \in \left[0, T\right). \tag{9}$$

Define

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx \tag{10}$$

where $\varepsilon > 0$ is small to be chosen later, and

$$0 < \alpha \leq min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}. \tag{11}$$

Differentiating (10) with respect to t and combining the first equation of (1), we have

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \left(uu_t + u_t^2\right)dx$$

$$= (1-\alpha)H^{-\alpha}(t)H'(t)$$

$$+\varepsilon\int_\Omega \left(u div\left(|\nabla u|^{m-2}\nabla u\right) - \mu_1(t)|u_t|^{p-1}u_t = \mu_2(t)|u|^q + u_t^2\right)dx$$

$$= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|_m^m dx$$

$$-\varepsilon\mu_2(t)\|u\|_q^q - \varepsilon\mu_1(t)\int_\Omega |u_t|^{p-1}u dx. \tag{12}$$

Due to the Hölder's and Young's inequalities, we get

$$\left|\mu_1(t)\int_\Omega |u_t|^{p-1}u dx\right| \leq \mu_1(t)\int_\Omega |u_t|^{p-1}u dx$$

$$\leq \left(\int_\Omega \mu_1(t)|u_t|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \mu_1(t)|u_t|^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{p-1}{p}\mu_1(t)\delta^{-\frac{p}{p-1}}\|u_t\|_p^p + \frac{\delta^p}{p}\mu_1(t)\|u\|_p^p \tag{13}$$

where δ is positive constant to be determined later. According to the conditions $\mu_1(t) \geq 0, \mu_2'(t) \geq 0$ and (8), we obtain

$$H'(t) \geq \mu_1(t)\|u_t\|_p^p. \tag{14}$$

Combining $(5), (6), (12), (13)$ and (14), we get

$$L'(t) \geq \left[(1-\alpha)H^{-\alpha}(t) + \frac{p-1}{p}\varepsilon\delta^{-\frac{p}{p-1}}\right]H'(t)$$

$$+\varepsilon\left(qH(t) - \frac{\delta^p}{p}\mu_1(t)\|u_t\|_p^p\right)$$

$$+\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m. \tag{15}$$

Since the integral is taken over the variable x, it is reasonable to take δ depending on variable t. From (9), we get

$$0 < H^{-\alpha}(t) \leq H^{-\alpha}(0),$$

for every $t > 0$. Hence $H^{-\alpha}(t)$ is a positive function and bounded. Thus, by taking $\delta^{-\frac{p}{p-1}} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (15), we get

$$L'(t) \geq \left[(1-\alpha) - \frac{p-1}{p}\varepsilon k\right]H^{-\alpha}(t)H'(t) - \varepsilon\|\nabla u\|_m^m$$

$$+\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m$$

$$+\varepsilon\left[qH(t) - \frac{k^{1-p}}{p}\mu_1(t)H^{\alpha(p-1)}(t)\|u\|_p^p\right]. \tag{16}$$

By using the $(5), (6), (9)$ and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q > p)$, we arrive at $\|u\|_p^p \leq C\|u\|_q^p$ and

$$L'(t) \geq \left[(1-\alpha) - \frac{p-1}{p}\varepsilon k\right] H^{-\alpha}(t)H'(t) - \varepsilon\|\nabla u\|_m^m$$

$$+\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{m}-1\right)\|\nabla u\|_m^m$$

$$+\varepsilon\left[qH(t) - \frac{Ck^{1-p}}{p}\left(\frac{\mu_2(t)}{q}\right)^{\alpha(p-1)}\|u\|_p^{p+q\alpha(p-1)}\right]. \quad (17)$$

From (11), we get $2 \leq s = p + q\alpha(p-1) \leq q$. Combining (5), (6), Remark 3 and (17), we obtain

$$L'(t) \geq \left[(1-\alpha) - \frac{p-1}{p}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{p}-1\right)\|\nabla u\|_m^m$$

$$+\varepsilon\left[qH(t) - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(H(t) + \|u_t\|_2^2 + \frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right]$$

$$\geq \left[(1-\alpha) - \frac{p-1}{p}\varepsilon k\right] H^{-\alpha}(t)H'(t)$$

$$+\varepsilon\left(\frac{q+2}{2} - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right)H(t)$$

$$+\varepsilon\left[\frac{q+6}{4} - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right]\|u_t\|^2$$

$$+\varepsilon\left[\frac{q-2}{2q}\mu_2(t) - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(\frac{\mu_2(t)}{q}+1\right)\right]\|u\|_q^q \quad (18)$$

where $C_1 = \frac{C}{pq^{\alpha(p-1)}}$. Since $\lim_{t\to\infty}\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ exists, $\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ is bounded for every $t>0$. Then, we choose k large enough so that the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$ in (18) are strictly positive. Therefore, we arrive at

$$L'(t) \geq \left[(1-\alpha) - \frac{p-1}{p}\varepsilon k\right] H^{-\alpha}(t)H'(t)$$

$$+\beta\left[H(t) + \|u_t\|^2 + \left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right] \quad (19)$$

where

$$\beta = min\left\{\frac{q+2}{2} - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t),\right.$$

$$\frac{q+6}{4} - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t),$$

$$\left.\frac{q-2}{2q}\mu_2(t) - C_1k^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right\}$$

is the minimum of the coefficients of $H(t)$, $\|u_t\|^2$ *and* $\|u\|_q^q$. Once k is fixed, we can take ε small enough so that $(1-\alpha) - \frac{p-1}{p}\varepsilon k \geq 0$ and

$$L(0) = [H(0)]^{1-\alpha} + \varepsilon\int_\Omega u_0u_1dx > 0. \quad (20)$$

Then (19) becomes

$$L'(t) \geq \varepsilon\beta\left[H(t) + \|u_t\|^2 + \left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right] \geq 0. \quad (21)$$

Then, we have

$$L(t) \geq L(0) > 0. \quad (22)$$

For the definition of $L(t)$ (*see* (10)) we have

$$\left|\int_\Omega uu_tdx\right| \leq \|u\|\|u_t\|$$

$$\leq C\|u\|_q\|u_t\| \quad (23)$$

using Hölder's inequality and the embedding $L^q(\Omega)\hookrightarrow L^p(\Omega)$ $(q>p)$. Thanks to Young's inequality, we have

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}} \leq C\|u\|_q^{\frac{1}{1-\alpha}}\|u_t\|^{\frac{1}{1-\alpha}}$$

$$\leq C\left(\|u\|_q^{\frac{2}{1-2\alpha}} + \|u_t\|^2\right) \quad (24)$$

from (11), we arrive at $\frac{2}{1-2\alpha} < q$.
Combining (24) and Remark 3, we get

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}} \leq C\left(H(t) + \|u_t\|^2 + \left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right). \quad (25)$$

Therefore, we obtain

$$L^{\frac{1}{1-\alpha}}(t) = \left[H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_tdx\right]^{\frac{1}{1-\alpha}}$$

$$\leq 2^{\frac{1}{1-\alpha}}\left(H(t) + \left|\varepsilon\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}}\right)$$

$$\leq C\left(H(t) + \|u_t\|^2 + \left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right). \quad (26)$$

Combining (21), (22) and (26), we have

$$L'(t) \geq \gamma L^{\frac{1}{1-\alpha}}(t) \tag{27}$$

where γ is a constant depending only on C, β *and* ε. Integrating (27), we arrive at

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{\alpha-1}\gamma t}. \tag{28}$$

If

$$t \to \left[\frac{1-\alpha}{\alpha\gamma L^{\frac{1}{1-\alpha}}(0)}\right]^{-}, L^{-\frac{1}{1-\alpha}}(0) - \frac{\alpha}{\alpha-1}\gamma t \to 0.$$

Hence, $L(t)$ blows up in finite time T^{*} and

$$T^{*} \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{1}{1-\alpha}}(0)},$$

which complete the proof of the Theorem.

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