Common fixed point theorems by using two mappings in b-rectangular metric space

Qasim K. Kadhim
Department of Mathematics, University of Al-Qadisiyah, Diwaniyah, Iraq, ma20.post16@qu.edu.iq

Follow this and additional works at: https://qjps.researchcommons.org/home

Part of the Biology Commons, Chemistry Commons, Computer Sciences Commons, Environmental Sciences Commons, Geology Commons, Mathematics Commons, and the Nanotechnology Commons

Recommended Citation
Kadhim, Qasim K. (2023) "Common fixed point theorems by using two mappings in b-rectangular metric space," Al-Qadisiyah Journal of Pure Science: Vol. 28 : No. 1 , Article 4.
Available at: https://doi.org/10.29350/2411-3514.1003

This Article is brought to you for free and open access by Al-Qadisiyah Journal of Pure Science. It has been accepted for inclusion in Al-Qadisiyah Journal of Pure Science by an authorized editor of Al-Qadisiyah Journal of Pure Science. For more information, please contact bassam.alfarhani@qu.edu.iq.
ARTICLE

Common Fixed Point Theorems by Using Two Mappings in b-Rectangular Metric Space

Qasim K. Kadhim*, Alia S. Kurdi

Department of Mathematics, University of Al-Qadisiyah, Diwaniyah, Iraq

Abstract

In this paper, the aim to obtain the existence and uniqueness of some common fixed point theorem on b-rectangular metric space by using new conditions including rational, maximum, minimum and combination for two mappings.

Keywords: Common fixed-point, b-Rectangular metric space

1. Introduction

One of the most important topics in the development of nonlinear analysis is fixed point theory. Fixed point theory has also been successfully applied to a variety of other fields of research, including chemistry, biology, economics, computer science, engineering, and a variety of others. It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping \( Q: E \rightarrow E \) where \((E, d)\) is a metric space, is said to be a contraction if there exists \( \alpha \in [0, 1) \) such that for all \( t, u \in E \);

\[
d(Qt, Qu) \leq \alpha d(t, u) \quad (1.1)
\]

The mapping fulfilling (1.1) has a unique fixed point if the metric space \((E, d)\) is complete. Inequality (1.1) implies continuity of \( Q \).

In 1989, Backhtin [1] introduced the concept of b-metric space. In 1993, Czerwik [7] extended the results of b-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the b-metric space. Boriceanu [3], Czerwik [7], Bota [4] extended the fixed point theorem in b-metric space, many authors initiated and studied many existing fixed point theorems in such spaces. Also, the concept of b-rectangular metric space is introduced as a generalization of b-metric space and rectangular (generalized) metric space by George et al. [8].

Common fixed point theorems have been studied by many researchers like Chandok [6]. A point \( t \in E \) is a common fixed point of self mappings \( Q, R : E \rightarrow E \) of a metric space \((E, d)\) if \( Qt = Rt = t \).

2. Preliminaries

Definition 2.1. [2] Let \( X \) be a nonempty set and \( Q : E \rightarrow E \) a self map. We say that \( t \in E \) is a fixed point of \( Q \) if \( Qt = t \) and denote by \( F_Q \) or \( \text{Fix}(Q) \) the set of all fixed points of \( Q \).

Let \( E \) be any set and \( Q : E \rightarrow E \) a self map. For any given \( t \in E \), we define \( Q_n(t) \) inductively by \( Q_0(t) = t \) and \( Q_{n+1}(t) = Q(Q^n(t)) \); we recall \( Q_n(t) \) the \( n \)-th iterative of \( t \) under \( Q \).

For any \( t_0 \in E \), the sequence \( \{t_n\}_{n \geq 0} \subset E \) given by

\[
t_n = Q_{n-1}t_0, \quad n = 1, 2, \ldots
\]

is called the sequence of successive approximations with the initial value \( t_0 \). It is also known as the Picard iteration starting at \( t_0 \).

Definition 2.2. [1,7] Let \( X \) be a nonempty set and \( s \geq 1 \) be a given real number. A function \( d : X \times X \rightarrow [0, \infty) \) is a b-metric on \( X \) if, for all \( x, y, z \in X \), the following conditions hold:

(1) \( d(x, y) = 0 \) if and only if \( x = y \);
(2) \( d(x, y) = d(y, x) \);
(3) \( d(x, y) \leq sd(x, z) + s^{-1}d(z, y) \) for all \( x, y, z \in X \).
(3) \(d(x, z) \leq s[d(x, y) + d(y, z)]\) \quad \text{(b-triangular inequality)}.

In this case, the pair \((X, d)\) is called a b-metric space.

**Definition 2.3.** [5] Let \(X\) be a nonempty set, and let \(d : X \times X \to [0, \infty)\) be a mapping such that for all \(x, y, z \in X\) and all distinct points \(u, v \in X\); each distinct from \(x\) and \(y\):

(1) \(d(x, y) = 0\) if and only if \(x = y\);
(2) \(d(x, y) = d(y, x)\);
(3) \(d(x, z) \leq s[d(x, u) + d(u, v) + d(v, z)]\) \quad \text{(rectangular inequality)}.

Then \((X, d)\) is called rectangular or generalized metric space.

**Definition 2.4.** [8] Let \(X\) be a nonempty set, \(s \geq 1\) be a given real number and let \(d : X \times X \to [0, \infty)\) be a mapping such that for all \(x, y, z \in X\) and distinct points \(u, v \in X\); each distinct from \(x\) and \(y\):

(1) \(d(x, y) = 0\) if and only if \(x = y\);
(2) \(d(x, y) = d(y, x)\);
(3) \(d(x, z) \leq s[d(x, u) + d(u, v) + d(v, z)]\) \quad \text{(b-rectangular inequality)}.

Then \((X, d)\) is called a b-rectangular metric space or a b-generalized metric space (b-g.m.s.).

**Definition 2.5.** [8] Let \((X, d)\) be a b-rectangular metric space and \(\{x_n\}\) be a sequence in \(X\) and \(x \in X\). Then

(i) The sequence \(\{x_n\}\) is said to be convergent in \((X, d)\) and converges to \(x\) if for every \(\varepsilon > 0\); there exists \(n_0 \in \mathbb{N}\) such that \(d(x_n, x) < \varepsilon\) for all \(n > n_0\) and this fact is represented by

\[
\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty.
\]

(ii) The sequence \(\{x_n\}\) is said to be b-rectangular-Cauchy in \((X, d)\) if for every \(\varepsilon > 0\); there exists \(n_0 \in \mathbb{N}\) such that \(d(x_n, x_{n+p}) < \varepsilon\) for all \(n > n_0; p > 0\) or equivalently, \(\lim_{n \to \infty} d(x_n, x_{n+p}) = 0\) for all \(p > 0\).

(iii) \((X, d)\) is said to be complete if every b-rectangular-Cauchy sequence in \((X, d)\) converges to an element of \(X\).

**Lemma 2.1.** [9] Let \((E, d)\) be a b- metric space with coefficient \(s \geq 1\) and \(Q : E \to E\) be a mapping. Suppose that \(\{t_n\}\) is a sequence in \(E\) induced by \(t_{n+1} = Qt_n\) such that

\[
d(t_n, t_{n+1}) \leq \alpha d(t_{n-1}, t_n),
\]

for all \(n \in \mathbb{N}\), where \(\alpha \in [0, 1)\) is a constant. Then \(\{t_n\}\) is a Cauchy sequence.

**Definition 2.6.** Let \(Q, R : E \to E\) be self-mappings on complete b-rectangular metric space \((E, d)\) with \(s \geq 1\). The following condition is called \((\alpha, \beta, \gamma)\)-contraction:

\[
d(Qt, Ru) \leq \alpha d(t, u) + \beta d(t, Qt)d(t, QRu) + [d(u, Qt)]^2
\]

\[
+ \gamma d(t, Qt)
\]

\(\text{(2.1)}\)

Where \(d(t, Qt) + d(u, Qt) \neq 0\), \(\alpha, \beta, \gamma > 0\), with \(0 \leq (\alpha + s\beta + \gamma) < 1\) for all \(t, u \in E\).

3. Main results

**Theorem 3.1.** Let \((E, d)\) be a complete b-rectangular metric space with \(s \geq 1\), and \(Q, R : E \to E\) be two mappings on \(E\) satisfying the condition

\[
d(Qt, Ru) \leq \alpha M(t, u) + \beta N(t, u) \tag{3.1}
\]

Where

\[
M(t, u) = \max \left\{d(t, u), \frac{d(t, Qt)}{1+d(Qt, Ru)}, \frac{d(u, Ru)}{1+d(Qt, Ru)} \right\}
\]

And

\[
N(t, u) = \min \left\{d(t, Qt), d(t, Ru), d(u, Qt) \right\}
\]

For all \(t, u \in E\) and \(\alpha, \beta \in [0, 1)\) with \(s \leq 1\). Then \(Q\) and \(R\) have a unique common fixed point.

**Proof:** For any arbitrary \(t_0 \in E\). Define the sequence \(\{t_n\}\) in \(E\) such that

\[
t_{2n+1} = Qt_{2n}
\]
\[
t_{2n+2} = Rt_{2n+1} = QRt_{2n}
\]
\[
t_{2n+3} = Qt_{2n+2} = QRt_{2n+1}, \text{ for all } n \in \mathbb{N}
\]

Assume that there is some \(n \in \mathbb{N}\) such that \(t_n = t_{n+1}\). If \(n = 2k\), then \(t_{2k} = t_{2k+1}\) and from (3.1),

\[
d(t_{2k+1}, t_{2k+2}) = d(Qt_{2k}, Rt_{2k+1})
\]

\[
\leq \alpha M(t_{2k}, t_{2k+1}) + \beta N(t_{2k}, t_{2k+1})
\]

Where

\[
M(t_{2k}, t_{2k+1}) = \max \left\{d(t_{2k}, t_{2k+1}), \frac{d(t_{2k}, Qt_{2k})}{1+d(Qt_{2k}, Rt_{2k+1})}, \frac{d(t_{2k+1}, Rt_{2k+1})}{1+d(t_{2k-1}, Rt_{2k+1})} \right\}
\]
Thus we have
\[
\begin{aligned}
&\frac{d(t_{2k+1}, QRt_{2k+1})d(t_{2k+1}, Qt_{2k})}{2} \\
&= \max \left\{ \frac{d(t_{2k}, t_{2k+1})}{1 + d(t_{2k}, t_{2k+2})}, \frac{d(t_{2k+1}, t_{2k+2})}{1 + d(t_{2k+1}, t_{2k+2})} \right\} \\
&\frac{d(t_{2k+1}, t_{2k+2})d(t_{2k+1}, t_{2k+1})}{2} \\
&= \max \left\{ 0, \frac{d(t_{2k+1}, t_{2k+2})}{1 + d(t_{2k+1}, t_{2k+2})} \right\} \\
&= \frac{d(t_{2k+1}, t_{2k+2})}{1 + d(t_{2k+1}, t_{2k+2})}
\end{aligned}
\]
And
\[
N(t_{2k}, t_{2k+1}) = \min \left\{ d(t_{2k}, Qt_{2k}), d(t_{2k}, Rt_{2k+1}), d(t_{2k+1}, Qt_{2k}) \right\}
\]
\[
= \min \{ d(t_{2k}, t_{2k+1}), d(t_{2k}, t_{2k+2}), d(t_{2k+1}, t_{2k+1}) \} = 0
\]
Thus we have
\[
d(t_{2k+1}, t_{2k+2}) \leq \alpha \frac{d(t_{2k+1}, t_{2k+2})}{1 + d(t_{2k+1}, t_{2k+2})} \leq \alpha d(t_{2k+1}, t_{2k+2})
\]
\[
d(t_{2k+1}, t_{2k+2}) \leq \alpha d(t_{2k+1}, t_{2k+2})
\]
Which is a contradiction
\[
d(t_{2k+1}, t_{2k+2}) = 0
\]
Therefore \( t_{2k+1} = t_{2k+2} \). Hence we have \( t_{2k} = t_{2k+1} = t_{2k+2} \).
It means that \( t_{2k} = Qt_{2k} = Rt_{2k} \) \( t_{2k} \) is a common fixed point of \( R \).
If \( n = 2k+1 \), then using same influences, can be presented that \( t_{2k+1} \) is a common fixed point of \( Q \) and \( R \).
Now, assume \( t_n \neq t_{n+1} \) for all \( n \in \mathbb{N} \).
\[
d(t_{2n+1}, t_{2n+2}) = d(Qt_{2n}, Rt_{2n+1})
\]
\[
d(t_{2n+1}, t_{2n+2}) \leq \alpha M(t_{2n}, t_{2n+1}) + \beta N(t_{2n}, t_{2n+1})
\]
Where
\[
M(t_{2n}, t_{2n+1}) = \max \left\{ d(t_{2n}, t_{2n+1}), \frac{d(t_{2n}, Qt_{2n})}{1 + d(RQt_{2n}, Rt_{2n+1})}, \frac{d(t_{2n+1}, Rt_{2n+1})}{1 + d(t_{2n+1}, R_{2n+1})} \right\}
\]
\[
\frac{d(t_{2n+1}, QRt_{2n+1})d(t_{2n+1}, Qt_{2n})}{2}
\]
\[
= \max \left\{ d(t_{2n}, t_{2n+1}), \frac{d(t_{2n}, t_{2n+1})}{1 + d(t_{2n}, t_{2n+2})}, \frac{d(t_{2n+1}, t_{2n+2})}{1 + d(t_{2n+1}, t_{2n+2})} \right\}
\]
\[
\frac{d(t_{2n+1}, t_{2n+2})d(t_{2n+1}, t_{2n+1})}{2}
\]
\[
= \max \left\{ d(t_{2n}, t_{2n+1}), d(t_{2n}, t_{2n+1}), \frac{d(t_{2n+1}, t_{2n+2})}{1 + d(t_{2n+1}, t_{2n+2})}, 0 \right\}
\]
\[
= \max \left\{ d(t_{2n}, t_{2n+1}), \frac{d(t_{2n+1}, t_{2n+2})}{1 + d(t_{2n+1}, t_{2n+2})} \right\}
\]
And
\[
N(t_{2n}, t_{2n+1}) = \min \{ d(t_{2n}, Qt_{2n}), d(t_{2n}, Rt_{2n+1}), \}
\]
\[
= \min \{ d(t_{2n}, t_{2n+1}), d(t_{2n}, t_{2n+2}), d(t_{2n+1}, t_{2n+1}) \} \]
\[
= 0
\]
If \( (t_{2n}, t_{2n+1}) = (t_{2n}, t_{2n+1}) \), then by (3.3)
\[
d(t_{2n+1}, t_{2n+2}) \leq \alpha d(t_{2n}, t_{2n+1})
\]
If \( (t_{2n}, t_{2n+1}) = \frac{d(t_{2n}, t_{2n+2})}{1 + d(t_{2n}, t_{2n+2})} \), then by (3.3)
\[
d(t_{2n+1}, t_{2n+2}) \leq \alpha \frac{d(t_{2n+1}, t_{2n+2})}{1 + d(t_{2n+1}, t_{2n+2})} \leq \alpha d(t_{2n+1}, t_{2n+2})
\]
Which is a contradiction
By induction we get
\[
d(t_{r+1}, t_n) \leq \alpha^2 d(t_1, t_0)
\]
(3.4)
Thus from (3.4), we obtain that
\[
\lim_{n \to \infty} d(t_n, t_{n+1}) = 0,
\]
Hence \( \{t_n\} \) is a b-rectangular-Cauchy sequence in \((E, d)\). By completeness of \((E, d)\), there exists \(r \in E\) such that \( t_n = Q_{n-1} \to r \) as \( n \to \infty \).
\[
d(r, Qr) \leq s[d(r, t_n) + d(t_n, t_{n+1}) + d(t_{n+1}, Qr)]
\]
\[
\frac{1}{s}d(r, Qr) \leq d(r, t_n) + d(t_n, t_{n+1}) + d(R_{t_n}, Qr)
\]
\[
\frac{1}{s}d(r, Qr) \leq d(r, t_n) + d(t_n, t_{n+1}) + \alpha d(t_n, t_n) + \beta N(t_n, t_n)
\]
(3.5)
Where
\[
M(r, t_n) = \max \left\{ d(r, t_n), \frac{d(r, Qr)}{1 + d(R_{t_n}, R_{t_n})}, \frac{d(t_n, R_{t_n})}{1 + d(t_n, R_{t_n})} \right\}
\]
\[
\frac{d(t_n, R_{t_n})d(t_n, Qr)}{2}
\]
Therefore, we have a unique common fixed point $r$.

**Corollary 3.1.** Let $(E, d)$ be a complete b-rectangular metric space with $s > 1$, and $Q, R : E \to E$ be two mappings on $E$ satisfying the condition

$$d(Qt, Ru) \leq \alpha M(t, u) + \beta N(t, u) \quad (3.1)$$

where

$$M(t, u) = \max \left\{ \frac{d(t, Qt)}{1 + d(RQu, Ru)}, \frac{d(u, Ru)d(u, Qt)}{2} \right\}$$

and

$$N(t, u) = \min \{d(t, Qt), d(t, Ru), d(u, Qt)\}$$

For all $t, u$ in $E$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta < 1$, then $Q$ and $R$ have a unique common fixed point.

**Theorem 3.2.** Let $(E, d)$ be a complete b-rectangular metric space. Then $(\alpha, \beta, \gamma)$-contraction satisfy a unique common fixed point.

Proof: For any arbitrary point, $t_0 \in E$. Define sequence $\{t_n\}$ in $E$ such that

$$t_{2n+1} = Qt_{2n}$$

$$t_{2n+2} = Rt_{2n+1}$$

$$t_{2n+3} = Qt_{2n+2} = QRt_{2n+1}, \text{ for all } n \in \mathbb{N} \quad (3.1)$$

Suppose that there is some $n \in \mathbb{N}$ such that $t_n = t_{n+1}$. If $n = 2k$, then $t_{2k} = t_{2k+1}$ and from the condition (2.1) with $t = t_{2k}$ and $u = t_{2k+1}$ we have

$$d(t_{2k+1}, t_{2k+2}) = d(Qt_{2k}, Rt_{2k+1})$$

$$\leq \alpha d(t_{2k}, t_{2k+1})$$

$$+ \beta \frac{d(t_{2k}, Qt_{2k})d(t_{2k}, QRt_{2k+1}) + [d(t_{2k+1}, Qt_{2k})]^2}{d(t_{2k}, Qt_{2k+1}) + d(t_{2k+1}, Qt_{2k})}$$

$$+ \gamma d(t_{2k}, Qt_{2k}) \quad (3.2)$$

$$= \beta \frac{d(t_{2k}, t_{2k+1})d(t_{2k}, t_{2k+2}) + [d(t_{2k+1}, t_{2k+1})]^2}{d(t_{2k}, t_{2k+3}) + d(t_{2k+1}, t_{2k+1})}$$

$$+ \gamma d(t_{2k}, t_{2k+1})$$

$$= 0$$

therefore $d(t_{2k+1}, t_{2k+2}) = 0$. Hence $t_{2k+1} = t_{2k+2}$. Thus we have $t_{2k} = t_{2k+1} = t_{2k+2}$. By (3.1), it means $t_{2k} = Qt_{2k} = Rt_{2k}$, that is, $t_{2k}$ is a common fixed point of $Q$ and $R$.

If $n = 2k + 1$, then using the same arguments as in the case $t_{2k} = t_{2k+1}$, it can be shown that $t_{2k+1}$ is a common fixed point of $Q$ and $R$.

From now on, we suppose that $t_n \neq t_{n+1}$ for all $n \in \mathbb{N}$.
We will show that
\( d(t_n,t_{n+1}) \leq (\alpha + \beta + \gamma)d(t_{n-1},t_n) \), for all \( n \in \mathbb{N} \) (3.3)
\[ = sd(r,t_{2n+1}) + sd(t_{2n+1},t_{2n+2}) + sd(t_{2n+2},QR) \]
\[ = sd(r,t_{2n+1}) + sd(t_{2n+1},t_{2n+2}) + sd(Rt_{2n+1},QR) \]

\[ \leq a d(t_{2k},t_{2k+1}) + \beta \frac{d(t_{2k},QR_{2k})d(t_{2k},QR_{2k+1}) + [d(t_{2k+1},QR_{2k})]^2}{d(t_{2k},QR_{2k+1}) + d(t_{2k+1},QR_{2k})} + \gamma d(t_{2k},QR_{2k}) \]
\[ = a d(t_{2k},t_{2k+1}) + \beta \frac{d(t_{2k},t_{2k+1})d(t_{2k},t_{2k+3}) + [d(t_{2k+1},t_{2k+1})]^2}{d(t_{2k},t_{2k+3}) + d(t_{2k+1},t_{2k+1})} + \gamma d(t_{2k},t_{2k+1}) \]
\[ = a d(t_{2k},t_{2k+1}) \]
\[ + \beta \frac{d(t_{2k},t_{2k+1})[sd(t_{2k},t_{2k+1}) + sd(t_{2k+1},t_{2k+2}) + sd(t_{2k+2},t_{2k+3})]}{sd(t_{2k},t_{2k+1}) + sd(t_{2k+1},t_{2k+2}) + sd(t_{2k+2},t_{2k+3})} \]

There are two cases which we have to consider.

**Case 1.** \( n = 2k + 1, k \in \mathbb{N} \).
From the condition (2.1) with \( t = t_{2k} \) and \( u = t_{2k+1} \) we have
\[ d(t_{2k+1},t_{2k+2}) = d(QR_{2k},Rt_{2k+1}) \]
\[ + \gamma d(t_{2k},t_{2k+1}) \]
\[ = a d(t_{2k},t_{2k+1}) + \beta d(t_{2k},t_{2k+1}) + \gamma d(t_{2k},t_{2k+1}) \]
\[ d(t_{2k+1},t_{2k+2}) \leq (\alpha + \beta + \gamma) d(t_{2k},t_{2k+1}) \]
Thus we obtain that
\[ d(t_n,t_{n+1}) \leq (\alpha + \beta + \gamma)d(t_{n-1},t_n), n = 2k + 1, k \in \mathbb{N} \] (3.4)

**Case 2.** \( n = 2k, k \in \mathbb{N} \). By using the same argument as in **Case 1**, it can be proved that (3.3) holds for \( n = 2k \), that is
\[ d(t_n,t_{n+1}) \leq (\alpha + \beta + \gamma)d(t_{n-1},t_n), n = 2k, k \in \mathbb{N} \] (3.5)

From (3.4) and (3.5) we can conclude that
\[ d(t_n,t_{n+1}) \leq (\alpha + \beta + \gamma)d(t_{n-1},t_n), \text{ for all } n \in \mathbb{N} \]
Thus we obtain that (3.3) holds.
Since \( 0 \leq (\alpha + \beta + \gamma) < 1 \), by Lemma (2.1) we can say that \( \{t_n\} \) is a Cauchy sequence in \((E,d)\). Since \((E,d)\) is a complete b-rectangular metric space, \( \{t_n\} \) converges to same \( r \in E \) as \( n \to +\infty \).

**Step 2.** We will prove that \( QR = Rr = r \).
Using b-rectangular inequality and (2.1), we have
\[ d(r,QR) \leq s[d(r,t_{2n+1}) + d(t_{2n+1},t_{2n+2}) + d(t_{2n+2},QR)] \]
\[ \leq a d(r,v) + \beta \frac{d(r,QR)d(r,QRv) + [d(v,QR)]^2}{d(r,QRv) + d(v,QR)} + \gamma d(r,QR) \]

**Step 3.** We will prove that \( Q \) and \( R \) have a unique common fixed point.
Suppose now that \( r \) and \( r \) are different common fixed points of \( Q \) and \( R \), then from (2.1), we have
\[ d(r,v) = d(Qr,Rv) \]
\[ \leq a d(r,v) + \beta \frac{d(r,QR)d(r,QRv) + [d(v,QR)]^2}{d(r,QRv) + d(v,QR)} + \gamma d(r,QR) \]
\[ d(r,v) = \alpha d(r,v) + \beta \frac{d(r,v)d(r,QRv) + |d(v,r)|^2}{d(r,v) + d(v,r)} + \gamma d(r,r) \]

\[ = \alpha d(r,v) + \beta \frac{d(v,r)^2}{2d(v,r)} \]

\[ d(r,v) \leq \left( \alpha + \frac{\beta}{2} \right) d(r,v) \]

Which is a contradiction

Since \( 0 \leq (\alpha + \frac{\beta}{2}) < 1 \), we have \( d(r,v) = 0 \).

Thus proved that \( Q \) and \( R \) have a unique common fixed point \( r \) in \( E \).

Now, if \( \alpha, \gamma = 0 \) in theorem 3.2, we get the following corollary:

Corollary 3.2. Let \( (E,d) \) be a complete \( b \)-rectangular metric space with \( \geq 1 \), and \( Q,R : E \to E \) be two mappings on \( E \) satisfying the condition

\[ d(Qt,Rt) \leq \beta \frac{d(t,Qt)d(t,QRu) + |d(u,Qt)|^2}{d(t,QRu) + d(u,Qt)} \]

for all, \( t,u \) in \( E \) and \( d(t,QRu) + d(u,Qt) \neq 0 \), with \( \beta > 0 \), \( 0 \leq s\beta < 1 \). Then \( Q \) and \( R \) have a unique common fixed point.

References