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CoSemiprime Ring

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Abstract

In this paper, we introduce a new type of rings, namely cosemiprime rings as a generalization of semiprime ring, where a ring C is said to be cosemiprime ring if $I = \sqrt{I}$ for each ideal I of C . Some properties of this type of rings are obtained. Also, we determine the relationship among types of ideals in this ring.

Keywords: Prime ideal, Semiprime ideal, Semiprime ring, Semiprimary ideal, Cosemiprime ring

1. Introduction

Let R be a ring with unity. A non zero ring R is called prime ring if and only if for any a and b of R , $abr = 0$ for all r in R implies that either $a = 0$ or $b = 0$ [1]. In Ref. [2] the concept of prime ideal was introduced by Dummit, David, Foot, where an ideal I of a ring R is called prime if for every a, b in R ; $ab \in I$ implies either $a \in I$ or $b \in I$. In 1992 Park, Younk Soo and Kim, Ju. Pil introduced the concept of semiprime ideal, where an ideal I of ring R is called semiprime ideal if for each y in R ; if $y^2 \in I$ implies $y \in I$ [3], equivalently an ideal I is called semiprime ideal if $J^n \subseteq I$ for some ideal J of R and some positive integer n , then $J \subseteq I$ [4]. Lam, Tsit-Yuea introduced in Ref. [5] a ring R is called semiprime ring if the zero ideal is a semiprime ideal. Let I be an ideal of ring R , then the nilradical of I denoted by \sqrt{I} , where $\sqrt{I} = \{x \in R; x^n \in I\}$ [6]. That a nilradical of I need not be equal I for example: Let $I = [4]$ in Z_8 , then $\sqrt{[4]} = [2]$, so $I \neq \sqrt{I}$. In this paper we study rings satisfying the condition $\sqrt{I} = I$ for each ideal I of this ring, which we called cosemiprime ring. We prove many properties for this kind of rings. Every zero divisor of C/I is a nilpotent element, where I be an ideal of Cosemiprime ring C .

This paper consists of two sections. In section one, we study the basic properties of cosemiprime ring, in section two we devoted to study the generalization of the concepts: prime, primary, semiprime and semiprimary ideals in cosemiprime ring, we

prove some theorems and give some examples to show the relationships among them.

2. Basic properties of cosemiprime rings

In this section, we introduce the notion of cosemiprime ring as a generalization of semiprime ring and we study some properties of this kind of rings.

Definition (2.1): A ring C is called cosemiprime ring if $I = \sqrt{I}$ for each non zero ideal I of C .

Remarks and Examples (2.2).

- 1 -It is clear that every ideal in cosemiprime ring is semiprime ideal.
- 2 - Z_n is cosemiprime ring if and only if n is a prime number.

Proof: it is obvious.

- 3 - Z_n is cosemiprime ring if and only if n can be written as a product of two prime numbers.

Proof: let $n = p_1 p_2$; p_1, p_2 be prime number; $I_1 = (p_1)$ and $I_2 = (p_2)$, where $\sqrt{(p_1)} = (p_1)$ and $\sqrt{(p_2)} = (p_2)$, thus Z_n is cosemiprime ring.

Conversely: is obvious.

- 4 -It is clear that every simple ring is cosemiprime ring.
- 5 -Every cosemiprime ring is ring, but the converse is not true in general for examples $(Z_{12} + Z_{12}, Z_{12})$ is

ring but not cosemiprime ring, since there exist an ideal $I = [4]$ of the ring $(Z_{12}, +_{12}, \cdot_{12})$, where $\sqrt{[4]} = [2]$.

6 -It is clear that every sub cosemiprime ring is cosemiprime ring.

7 - Z is not cosemiprime ring.

8 -Every field is cosemiprime ring.

Proof: it is obvious.

But the converse is not true for example: $(Z_6, +_6, \cdot_6)$ is cosemiprime ring, by remark and example (2.2(3)), but it is not field by Ref. [7].

9 -Every prime ring is cosemiprime ring.

Proof: every prime ring is finite field [7] and by remarks and example (2.2(8)) we obtain the result.

Recall that a subring B of a ring R is called a direct summand of R if and only if there exist a subring A of R such that $R = B \oplus A$ [9].

Proposition (2.3). A direct summand of cosemiprime ring is a cosemiprime ring.

Proof: let C be a cosemiprime ring, let W be a subring of C ; $C = W \oplus X$ for subring X of C , then W is cosemiprime ring by [remarks and examples 2.2(6)].

Corollary (2.4). the direct sum of two cosemiprime ring is a cosemiprime ring.

Proof: let $C = R \oplus W$; R, W be two cosemiprime rings, let I be an ideal of R, Y be an ideal of W , so (I, Y) be an ideal of $C = R \oplus W$. Thus $\sqrt{(I, Y)} = \{(a, b) \in C; (a, b)^n \in (I, Y)\} = \{(a, b) \in C; (a^n, b^n) \in (I, Y)\} = (\sqrt{I}, \sqrt{Y}) = (I, Y)$, since R, W be two cosemiprime rings.

Theorem (2.5). suppose that C, C_1 be two rings and let $f: C \rightarrow C_1$ epimorphism such that $\ker f \subseteq I$, where I is an ideal of C , if C is cosemiprime ring, then C_1 is cosemiprime ring.

Proof: let $x \in \sqrt{f(I)}$, so there exist $n \in \mathbb{Z}^+$; $x^n \in f(I)$, thus $x^n = f(y)$; $y \in I$. Since f is onto, implies for each $x \in C_1$ there exist $a \in C$; $x = f(a)$. $f(a^n - y) = 0$, which mean $(a^n - y) \in \ker f \subseteq I$, implies $(a^n - y) \in I$, so $a^n - y = i$ for some $i \in I$. Thus $a^n = i + y \in I$, so $a \in \sqrt{I}$, but $x = f(a)$, so $x \in f(\sqrt{I}) = f(I)$, which mean $\sqrt{f(I)} \subseteq f(I)$, but $f(I) \subseteq \sqrt{f(I)}$ [8]. Which mean C_1 is cosemiprime ring.

Corollary (2.6). the homomorphic image of a cosemiprime ring is cosemiprime ring.

Let $f: C \rightarrow C_1$; be homomorphism function, I_1 be cosemiprime ideal of C_1 , let $x \in \sqrt{f^{-1}(I_1)}$, then $x^n \in f^{-1}(I_1)$, so $f(x^n) \in I_1$, but f is homomorphism, so $f^n(x) \in I_1$, implies $f(x) \in \sqrt{I_1} = I_1$, so $f(x) \in I_1$, thus $x \in f^{-1}(I_1)$, which mean $\sqrt{f^{-1}(I_1)} \subseteq f^{-1}(I_1)$, but $f^{-1}(I_1) \subseteq \sqrt{f^{-1}(I_1)}$. Thus $f^{-1}(I_1) = \sqrt{f^{-1}(I_1)}$.

Nathan in Ref. [8] we call an ideal I is principle if is generated by single element. Thus $a \in R$ is called principle ideal and denoted by $(a) = \{ra; r \in$

$R\} = Ra$, so a ring R is called principle ideal ring (P.I.R) if and only if every ideal of R is principle ideal.

Proposition (2.7). A cosemiprime ring is P.I.R.

Proof: let I be an ideal of cosemiprime ring C , if $I = \{0\}$, then $I = (0)$. If $I \neq \{0\}$, we must prove $I = (a)$; $a \in C$, a is the small positive integer in I . It is clear that $(a) \subseteq I$. Let $x \in I = \sqrt{I}$, so $x^n \in I$, by division algorithm $x^n = aq + r$; $0 < r < a$, implies $r = x^n - aq \in I$, which is a contradiction since a is a smallest element, so $r = 0$, implies $x^n = aq$, thus $x^n \in (a)$, so $x \in \sqrt{(a)} = (a)$. Thus $I = (a)$.

But the converse is not true for example: $(Z, +, \cdot)$ is P.I.R [9], but is not cosemiprime ring by remarks and examples (2.2-(7)).

3. Prime ideal, primary ideal, semiprime ideal and semiprimary ideal of cosemiprime ring

In this section we generalize the concept of (prime, primary, semiprime and semiprimary) ideal to (cosprime, cosprimary, cossemiprime and cosemiprimary) ideal in cosemiprime ring, then we will discuss several propositions and theorems.

Definition (3.1). An ideal I of cosemiprime ring C is called cosprime ideal if for each $a, b \in C$; $ab \in \sqrt{I}$ implies either $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

Theorem (3.2). suppose I be cosprime ideal of cosemiprime ring C , then every zero divisor of $C \setminus I$ is a nilpotent element.

Proof: let $a \in C \setminus I$ be the zero divisor in $C \setminus I$, so $a \neq 0_{C \setminus I}$ then there exist $b \neq 0_{C \setminus I}$; $a \cdot b = 0_{C \setminus I}$.

$a \cdot b \in I$, which mean $ab \in I = \sqrt{I}$, so $(ab)^n \in I$, so $a^n b^n \in I$, but I is a cosprime ideal so either $a^n \in I$ or $b^n \in I$. Thus $(a+I)^n = I$ or $(b+I)^n = I$, which mean $a \in I$ is a nilpotent of $C \setminus I$.

Definition (3.3). an ideal J is called cosprimary ideal of cosemiprime ideal C if and only if $\forall a, b \in C$; $ab \in \sqrt{J}$ implies either $a \in \sqrt{J}$ or $b^m \in \sqrt{J}$ for some positive integer m .

Definition (3.4). an ideal J is called cossemiprime of a cosemiprime ring C if and only if for each $a \in C$, if $a^{2n} \in J$ implies $a^n \in J$.

Definition (3.5). let I be an ideal of a cosemiprime ring, then I is called a cossemiprimary ideal of a cosemiprime ring C if and only if $\forall x, y \in C$; $xy \in \sqrt{I}$ implies either $x^m \in \sqrt{I}$ or $y^m \in \sqrt{I}$ for some $m \in \mathbb{Z}^+$.

Proposition (3.6). let C be cosemiprime ring and I be cosprime ideal, then I is a cosprimary ideal of C .

Proof: let $x, y \in C$, since I is cosprime ideal, then if $xy \in \sqrt{I}$ implies either $x \in \sqrt{I}$ or $y \in \sqrt{I}$ which mean I is a cosprimary ideal.

Theorem (3.7). let C be a cosemiprime ring, I is an ideal of C . Then I is a cosprime ideal of C if and only if it is cosprimary ideal.

Proof: \rightarrow is directly from proposition 3.6

Conversely, let $x, y \in C$ such that $xy \in \sqrt{I}$, since I is coprimary ideal implies either $x \in \sqrt{I}$ or $y^m \in \sqrt{I}$ for some $m \in \mathbb{Z}^+$. Thus either $x \in \sqrt{I}$ or $y^{mn} \in I$, but I is ideal in cosemiprime ring, so either $x \in \sqrt{I}$ or $y \in \sqrt{I}$ for some $mn \in \mathbb{Z}^+$, which mean I is a coprime ideal.

Theorem (3.8). suppose C be cosemiprime ring, I be an ideal of C , then I is coprimary ideal if and only if I is cosemiprime ideal.

Proof: let I be coprimary ideal, for each $x, y \in C$; $xy \in \sqrt{I}$ implies either $x \in \sqrt{I}$ or $y^m \in \sqrt{I}$. If $x \notin \sqrt{I}$, $y^{2n} \in \sqrt{I}$ (by definition 3.4), so $y^{2n} \in I$, implies $y^2 \in \sqrt{I}$, which mean $y \in \sqrt{I}$ so by theorem (2.7) either $y \in \sqrt{I}$ or $y \in \sqrt{I}$ which mean $y^n \in I$, so I is cosemiprime ideal. Conversely, if I is cosemiprime ideal to prove I is coprimary ideal, if $xy \in \sqrt{I}$, suppose $x \notin \sqrt{I}$, if $y^{2n} \in \sqrt{I}$, then $y \in \sqrt{I}$ which mean $y^n \in I = \sqrt{I}$, for some $n \in \mathbb{Z}^+$. Thus I is coprimary ideal.

Theorem (3.9). suppose C be cosemiprime ring, I be an ideal of C . Then I is cosemiprimary ideal if and only if I is coprimary ideal.

Proof: it is clear that every coprimary ideal is cosemiprimary ideal. Conversely, if $ab \in \sqrt{I}$ such that $a^n b^n \in I$, but I is cosemiprimary ideal so either $a^n \in \sqrt{I}$ or $b^n \in \sqrt{I}$ which is mean is coprimary.

Corollary (3.10). let C be cosemiprime ring, I be an ideal of C . Then I is cosemiprimary ideal if and only if I is cosemiprime ideal.

Proof: is directly from theorem (3.8) and theorem (3.9).

Corollary (3.11). suppose C be cosemiprime ring, I be an ideal of C . Then I is coprime ideal if and only if I is cosemiprime ideal.

Proof: it follows directly from theorem (3.7) and theorem (3.8).

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