Al-Qadisiyah Journal of Pure Science

Volume 28 | Number 1 Article 10

2023

Omitted - value transformation for a Complex Differential Equations

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ARTICLE

Omitted - Value Transformation for Complex Differential Equations

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Abstract

The purpose of this paper is to show the effects of an equivalence relation on an Omitted-value transformation defined in the unit disk $\boldsymbol{Đ}(0;1)=\{\check{z}\in\mathbb{C}:|\check{z}|<1\}$ in the form $\breve{g}(\check{z})=\frac{\tilde{\omega}ʃ}{\tilde{\omega}-ʃ}$; $\tilde{\omega}\notin ʃ(\boldsymbol{Đ})$ to generate non-linear complex differential equations for second, third, ..., nth orders, and to solve these equations. This work also provides a theorem that shows how to restrict any infinite power series for the holomorphic, univalent function to satisfy the Omitted-value transformation property when $\tilde{\omega}\notin ʃ(\boldsymbol{Đ})$.

MSC: 30C35, 34 M04.

Keywords: Complex differential equations, Conformal mapping, Univalent function

1. Introduction

A growing number of studies have revealed connect between complex analysis and one of the most well-known branches of mathematics, differential equations, which are studied separately in various scientific specializations.

The theory of complex differential equations in the unit disk has been developed since the 1980s [1], through the numerous studies on the solutions of nonlinear complex differential equations have been conducted [2]. By introducing the definition of function spaces (cf [2]. [3]), has assisted in the application of growth and oscillation theory to complex differential equations when the coefficients are analytic functions in the unit disk.

Conformal mapping is a holomorphic function that preserves local angles between two curves and their orientations, or to take other concept, any holomorphic function with a nonzero derivative is a conformal map (cf.[3],[4]). Furthermore, all of the valuable particular types of M $\ddot{o}$ bius transformations, such as (linear, rotation, inversion, and expanding) transformation generators for some groups in algebra such as PGL (n,F_l), which have a wide range of applications in mathematics and science [5].

In addition, S is a class of all holomorphic and univalent functions which defined in the open unit disk $\boldsymbol{Đ}(0;1)=\{\check{z}\in\mathbb{C}:|\check{z}|<1\}$, with normalized under specific conditions $ʃ(0)=0, ʃ'(0)=1$, in the form:

$$ʃ(\check{z})=\sum\nolimits_{n=0}^{\infty}\breve{e}_n\check{z}^n. \tag{1}$$

Notice that Taylor expansion series is expressed for every ʃ belongs to S [14].

As a result, the conformal mapping in below is belong to class S.

$$\breve{g}(\check{z})=\frac{\tilde{\omega}ʃ}{\tilde{\omega}-ʃ} \tag{2}$$

where $ʃ(\check{z})\neq\tilde{\omega}$.

In [15] the transformation $ʃ\rightarrow\breve{g}$ is known in study univalent function since $\tilde{\omega}\rightarrow\infty$ belongs to $\mathbb{C}^*=\mathbb{C}-\{0\}$., that is why $\breve{g}(\check{z})$ in (2) is called Omitted −value transformation which is composition ʃ with M $\ddot{o}$ bius transformation [6], [7].

Hence, $\breve{g}(\check{z})$ can be written as $\breve{g}(\check{z})=(\mathfrak{h}\circ ʃ)(\check{z})$, where $\mathfrak{h}(\check{z})=\frac{\tilde{\omega}\check{z}}{\tilde{\omega}-\check{z}}$ and $ʃ(\check{z})$ is univalent and holomorphic in $\boldsymbol{Đ}$, the univalence of $\mathfrak{h}(\check{z})$ satisfies when

Received 19 February 2022; accepted 24 April 2022.
Available online 26 May 2023

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https://doi.org/10.29350/2411-3514.1009

$\mathfrak{h}(\check{z}_1) = \mathfrak{h}(\check{z}_2)$. Then, $\frac{\tilde{\omega}\check{z}_1}{\tilde{\omega}-\check{z}_1} = \frac{\tilde{\omega}\check{z}_2}{\tilde{\omega}-\check{z}_2}$. This implies to $\tilde{\omega}^2\check{z}_1 - \tilde{\omega}\check{z}_1\check{z}_2 = \tilde{\omega}^2\check{z}_2 - \tilde{\omega}\check{z}_1\check{z}_2$ such that $\check{z}_1 = \check{z}_2$.

As a result, for each $\check{z} \neq \tilde{\omega}$, $\mathfrak{h}$ is a holomorphic function. Hence, $\breve{g}$ is a composition of univalent and holomorphic functions in $\boldsymbol{Đ}$. ([8],[9]).

In[10], the author introduced an equivalent class

$$\check{E}(\mathfrak{f}) = \{\breve{g} \in \check{S}, \breve{g} \sim \mathfrak{f}\}, \tag{3}$$

where $\check{E}(\mathfrak{f})$ is the set of all $\mathfrak{f} \in \check{S}$ and satisfies the equivalent relation $(\sim)$ in $\check{S}$ when $(\sim)$ be Reflexivity, Symmetry and Transitivity. As well as being used to complete the solution for a specific first order of non-linear complex differential equation.

In this paper, we only used Omitted-value transformation to generate nonlinear complex differential equations of the second, third, and nth orders as our knowledge because it omitted one or more than one point that tends to infinity to obtain a smooth domain able to produce a solution from a possible singular point, which be evident during the proofing process. Furthermore, we show how to use a non-linear differential equation to limit any infinite power series for a holomorphic, univalent function in order to satisfy the Omitted-value transformation property when $\tilde{\omega} \to \infty$.

2. Preliminaries

This section begins by stating some of the basic definitions and Remarks that have proved effective in our work.

Definition (2.1) [11]A function $\mathfrak{f}$ on $\mathbb{C}$ is said to be univalent (1-1) in a domain $Đ \subset \mathbb{C}$ if for $\check{z}_1, \check{z}_2 \in Đ$, such that $\mathfrak{f}(\check{z}_1) = \mathfrak{f}(\check{z}_2)$ implies to $\check{z}_1 = \check{z}_2$.

Also, $\mathfrak{f}$ is said to be locally univalent at a point $\check{z}_0 \in Đ$ if it is univalent in some neighbourhood of $\check{z}_0$. For holomorphic functions $\mathfrak{f}$, the condition $\mathfrak{f}'(\check{z}_0) \neq 0$ is equivalent to local univalent at $\check{z}_0$.

Example (2.1) [11]

Consider the function $\mathfrak{f}(\check{z}) = \check{z}^2$ defined in the region $\mathbb{C} \setminus \{0\}$. For $\check{z} \neq 0$ could be $\mathfrak{f}'(\check{z}) = 2\check{z} \neq 0$. Hence $\mathfrak{f}(\check{z}) = \check{z}^2$ is locally univalent in $\mathbb{C}\{0\}$. But, we noticed that $\mathfrak{f}(-\check{z}) = (-\check{z})^2 = \check{z}^2 = \mathfrak{f}(\check{z})$; so this function is not univalent in the whole region $\mathbb{C} \setminus \{0\}$. However, $\mathfrak{f}(\check{z}) = \check{z}^2$ is just univalent on $\{\check{z} \in \mathbb{C} : \Re(\check{z}) > 0\}$

Theorem (2.1) [Koebe one- Quarter Theorem] [6]

The range of every $\mathfrak{f} \in S$ contains the disk $\{\tilde{\omega} \in \mathbb{C} : |\tilde{\omega}| < \frac{1}{4}\}$, i.e; $dist(0, \partial\mathfrak{f}(\boldsymbol{Đ})) \geq \frac{1}{4}$

Definition (2.3) [12]A conformal mapping is a transformation $\omega = \mathfrak{f}(\check{z})$ that preserves the magnitude of angles, and their orientations. However, if $\mathfrak{f}$ is a holomorphic function defined in the domain $Đ$ containing $\check{z}_0$, and if $\mathfrak{f}'(\check{z}) \neq 0$, then $\omega = \mathfrak{f}(\check{z})$ is conformal at $\check{z}_0$.

Remark (2.2) [13]

1. A conformal mapping is a transformation $\omega = \mathfrak{f}(\check{z})$ that preserves local angles. It is also known as a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map.
2. A mapping that preserves the magnitude of angles, but not their orientation is called an isogonal mapping

3. Main results

Here are some interesting results that prove the effect of Omitted - value transformation on the generation of non-linear complex differential equations. We will begin with second and third orders to ensure that the property of this transformation is able to generate differential equations in those orders, and then proceed to generate nth-order nonlinear complex differential equations using the Second Cauchy Integral Formula, as stated below, with theorem showing the limiting range of holomorphic functions represented by the infinity power series.

Theorem (3.1). If $\mathfrak{f} \in \check{S}$. Consider the second order nonlinear complex differential equation

$$\frac{\breve{g}''}{\breve{g}'} = \frac{\mathfrak{f}''}{\mathfrak{f}'} + \frac{2\mathfrak{f}'}{\tilde{\omega} - \mathfrak{f}} \tag{4}$$

on the open unit disk $Đ$, then $\check{E}(f) = \{\breve{g} \in \check{S}, \breve{g} \sim \mathfrak{f}\}$ generated the solution in $\check{S}$.

Proof . Consider $\breve{g} = \mathfrak{f}_{\tilde{\omega}} = \frac{\tilde{\omega}\mathfrak{f}}{\tilde{\omega}-\mathfrak{f}}$, $\tilde{\omega} \notin \mathfrak{f}(Đ)$ since $\breve{g} \in \check{E}(f)$.

Differentiate the transformation (2) with respect to $\check{z}$ twice as follows

$$\breve{g}' = \frac{(\tilde{\omega}-\mathfrak{f})\tilde{\omega}f' - \tilde{\omega}\mathfrak{f}(-\mathfrak{f}')}{(\tilde{\omega}-\mathfrak{f})^2} = \frac{\tilde{\omega}(\tilde{\omega}-\mathfrak{f})\mathfrak{f}' + \tilde{\omega}\mathfrak{f}\mathfrak{f}'}{(\tilde{\omega}-\mathfrak{f})^2} = \frac{\tilde{\omega}^2 f' - \tilde{\omega}\mathfrak{f}\mathfrak{f}' + \tilde{\omega}\mathfrak{f}\mathfrak{f}'}{(\tilde{\omega}-\mathfrak{f})^2}$$

So, $\breve{g}' = \frac{\tilde{\omega}^2\mathfrak{f}'}{(\tilde{\omega}-\mathfrak{f})^2}$ is the first derivative of $\breve{g}$

Continue to get the next derivative order

$$\breve{g}'' = \frac{\{(\tilde{\omega}-\mathfrak{f})^2[\tilde{\omega}^2\mathfrak{f}'']\} - \{\tilde{\omega}^2\mathfrak{f}'(2(\tilde{\omega}-\mathfrak{f}) - \mathfrak{f}')\}}{(w-\mathfrak{f})^4}$$

Using a simple basic calculations can be helpful achieve the following results.

$$\breve{g}'' = \frac{\tilde{\omega}^2(\tilde{\omega}-\mathfrak{f})^2\mathfrak{f}'' - \{\tilde{\omega}^2\mathfrak{f}'(-2(\tilde{\omega}-\mathfrak{f})\mathfrak{f}')\}}{(\tilde{\omega}-\mathfrak{f})^4}$$

$$\breve{g}'' = \frac{\tilde{\omega}^2(\tilde{\omega}-\mathfrak{f}^{)}\mathfrak{f}'' + 2\tilde{\omega}^2\mathfrak{f}'^2(\tilde{\omega}-\mathfrak{f})}{(\tilde{\omega}-\mathfrak{f})^4}$$

$$\breve{g}'' = \frac{\tilde{\omega}^2 ʃ''}{(\tilde{\omega}-ʃ)^2} + \frac{2(\tilde{\omega}ʃ')^2}{(\tilde{\omega}-ʃ)^3}$$

$$\breve{g}'' = \left(\frac{\tilde{\omega}^2}{(\tilde{\omega}-ʃ)^2}\right)ʃ'' + 2\frac{\tilde{\omega}^2 ʃ'}{(\tilde{\omega}-ʃ)^2}\left(\frac{ʃ'}{(\tilde{\omega}-ʃ)}\right)$$

$$\breve{g}'' = \left(\frac{\tilde{\omega}^2}{(\tilde{\omega}-ʃ)^2}\right)ʃ'' + 2\breve{g}'ʃ'\left(\frac{1}{\tilde{\omega}-ʃ}\right)$$

$$\breve{g}'' = \left(\frac{\breve{g}'}{ʃ'}\right)ʃ'' + 2\breve{g}'\frac{\breve{g}}{\tilde{\omega}ʃ}ʃ' \tag{5}$$

Divide equation (5) by $\breve{g}$ to obtain $\frac{\breve{g}''}{\breve{g}'} = \frac{ʃ''}{ʃ'} + \frac{2ʃ'}{\tilde{\omega}-ʃ}$. Thus, $\breve{g}\in\breve{E}(f)$ is a solution of given non-linear differential equation.

The reverse direction: If $\breve{g}$ be a solution to the ordinary differential equation in S, then should be prove that $\breve{g}\in\breve{E}(f)$.

Consider.

$\breve{g} = \frac{\tilde{\omega}ʃ}{\tilde{\omega}-ʃ}, \tilde{\omega}\notin ʃ(Đ)$

Then $(\tilde{\omega}-ʃ)\breve{g} = \tilde{\omega}ʃ$ which implies to $\tilde{\omega}(\breve{g}-ʃ) = \breve{g}ʃ$, so that $\tilde{\omega} = \frac{\breve{g}ʃ}{\breve{g}-ʃ}$

Set, $ɦ = \frac{\breve{g}ʃ}{\breve{g}-ʃ}$, $ɦ' = \frac{ʃ^2\breve{g}'-ʃ'\breve{g}^2}{(ʃ-\breve{g})^2}$ and $ɦ''$ as follows

$$ɦ'' = \frac{(ʃ-\breve{g})^2[ʃ^2\breve{g}'' + 2\breve{g}'ʃʃ' - f\breve{g}''^2 - 2\breve{g}\breve{g}'ʃ'] - (ʃ^2\breve{g}' - ʃ'\breve{g}^2)[2(ʃ-\breve{g})(ʃ'-\breve{g}')]}{(ʃ-\breve{g})^4}$$

$$ɦ'' = \frac{(ʃ-\breve{g})^2[ʃ^2\breve{g}'' - ʃ''\breve{g}^2 + 2\breve{g}'ʃ'(ʃ-\breve{g})] - 2(ʃ^2\breve{g}' - \breve{g}^2ʃ')(ʃ-\breve{g})(ʃ'-\breve{g}')}{(ʃ-\breve{g})^4} \tag{6}$$

Clearly, some terms in (6) such as $ʃ^2\breve{g}' = ʃ'\breve{g}^2$, $ʃ^2\breve{g}'' = ʃ''\breve{g}^2$, and $2\breve{g}'ʃ'ʃ = 2g'ʃ'\breve{g}$ equal to zero as follows: Assume that

$ʃ^2\breve{g}' - ʃ'\breve{g}^2 = 0$, since $ʃ^2\breve{g}' = ʃ^2.\frac{\tilde{\omega}^2ʃ'}{(\tilde{\omega}-ʃ)^2} = \frac{ʃ^2\tilde{\omega}^2ʃ'}{(\tilde{\omega}-ʃ)^2} = ʃ'\left(\frac{ʃ^2\tilde{\omega}^2}{(\tilde{\omega}-ʃ)^2}\right) = ʃ'\breve{g}^2$

$ʃ^2\breve{g}'' - ʃ''\breve{g}^2 = 0$, since

$$ʃ^2\breve{g}'' = ʃ^2\left[\frac{\tilde{\omega}^2ʃ''}{(\tilde{\omega}-ʃ)^2} + \frac{2\tilde{\omega}^2ʃ'^2}{(\tilde{\omega}-ʃ)^3}\right] = \frac{\tilde{\omega}^2ʃ^2ʃ''}{(\tilde{\omega}-ʃ)^2} + \frac{2\tilde{\omega}^2ʃ'^2ʃ^2}{(\tilde{\omega}-ʃ)^3} \tag{7}$$

and

$ʃ''\breve{g}^2 = \frac{\tilde{\omega}^2ʃ^2ʃ''}{(\tilde{\omega}-ʃ)^2}$.

Obviously, the term $\frac{2\tilde{\omega}^2ʃ'^2ʃ^2}{(\tilde{\omega}-ʃ)^3} = \frac{2\tilde{\omega}^2ʃ'^2ʃ^2}{\left[\tilde{\omega}\left(1-\frac{ʃ}{\tilde{\omega}}\right)\right]^3} = \frac{2ʃ'^2ʃ^2}{\tilde{\omega}\left(1-\frac{ʃ}{\tilde{\omega}}\right)^3} \to 0$ as $\tilde{\omega}\to\infty$.

Hence, $ʃ^2\breve{g}'' = ʃ^2\left[\frac{\tilde{\omega}^2ʃ''}{(\tilde{\omega}-ʃ)^2} + \frac{2\tilde{\omega}^2ʃ'^2}{(\tilde{\omega}-ʃ)^3}\right] = \frac{\tilde{\omega}^2ʃ^2ʃ''}{(\tilde{\omega}-ʃ)^2} = ʃ''\breve{g}^2$

$2\breve{g}'ʃ'ʃ - 2\breve{g}'\breve{g}ʃ' =$ in short calculations

Let $2\breve{g}'ʃ'(ʃ-\breve{g}) = 0$, since $ʃ\neq\breve{g}$ then $ʃ-\breve{g}\neq 0$.

Therefore must be $2\breve{g}'ʃ' = 0$, depending on both of $ʃ', \breve{g}'\in\breve{S}$ and satisfied normilization condition.

As a result of the preceding, $ɦ'' = 0$, then $ɦ' = cont$,that is ; $ɦ' = \tilde{\omega}$.

Since $ɦ' = \frac{ʃ^2\breve{g}'-ʃ'\breve{g}^2}{(ʃ-\breve{g})^2} = \tilde{\omega}$, which implies to $\tilde{\omega}(ʃ-\breve{g})^2 = ʃ^2\breve{g}' - ʃ'\breve{g}^2$,

with $ʃ^2\breve{g}' - ʃ'\breve{g}^2 = 0$, therefore $\tilde{\omega} = 0$ since $ʃ-\breve{g}\neq 0$.

Hence, the integral of $ɦ' = \tilde{\omega}$ would be $ɦ = \tilde{\omega}$ such that $\tilde{\omega} = \frac{\breve{g}ʃ}{ʃ-\breve{g}}$

Finally, $\breve{g} = \frac{\tilde{\omega}ʃ}{\tilde{\omega}-ʃ}$. The proof is complete ∎

Theorem (3.2). If $ʃ\in\breve{S}$. Consider the third order nonlinear complex differential equation

$$\frac{\breve{g}'''}{\breve{g}'} = \frac{ʃ'''}{ʃ'} + \frac{6}{\tilde{\omega}-ʃ}ʃ'' + \frac{6}{(\tilde{\omega}-ʃ)^2}ʃ'^2 \tag{8}$$

on the open unit disk $Đ$, then $\breve{E}(f) = \{\breve{g}\in\breve{S}, \breve{g}\sim ʃ\}$ generated the solution in $\breve{S}$.

Proof. Consider $\breve{g} = ʃ_{\tilde{\omega}} = \frac{\tilde{\omega}ʃ}{\tilde{\omega}-ʃ}$, $\tilde{\omega}\notin ʃ(Đ)$ since $\breve{g}\in\breve{E}(f)$.

As previously stated, Differentiate the transformation (2) with respect to $\check{z}$ three times as follows

$\breve{g}' = \frac{\tilde{\omega}^2ʃ'}{(\tilde{\omega}-ʃ)^2}$

$\breve{g}'' = \frac{\tilde{\omega}^2ʃ''}{(\tilde{\omega}-ʃ)^2} + \frac{2(\tilde{\omega}ʃ')^2}{(\tilde{\omega}-ʃ)^3}$

$$\breve{g}''' = \frac{\breve{g}'}{ʃ'}ʃ''' + \frac{6}{\tilde{\omega}-ʃ}\breve{g}'ʃ'' + 6\breve{g}'\frac{ʃ'^2}{(\tilde{\omega}-ʃ)^2} \tag{9}$$

Divide both of sides equation (9) by $\breve{g}'$ as follows

$\frac{\breve{g}'''}{\breve{g}'} = \frac{ʃ'''}{ʃ'} + \frac{6}{\tilde{\omega}-ʃ}ʃ'' + \frac{6}{(\tilde{\omega}-ʃ)^2}ʃ'^2$

Thus, $\breve{g}\in\breve{E}(f)$ is a solution of the given differential equation.

The reverse direction: If $\breve{g}$ be a solution to the ordinary differential equation in S, then should be prove that $\breve{g}\in\breve{E}(f)$.

Consider $\breve{g}=\frac{\tilde{\omega}\int}{\tilde{\omega}-\int}$, $\tilde{\omega}\notin\int(Đ)$. Then $(\tilde{\omega}-\int)\breve{g}=\tilde{\omega}\int$ which implies to $\tilde{\omega}(\breve{g}-\int)=\breve{g}\int$, so that $\tilde{\omega}=\frac{\breve{g}\int}{\breve{g}-\int}$.

Set, $\hbar=\frac{\breve{g}\int}{\breve{g}-\int}$, $\hbar'=\frac{\int^2\breve{g}'-\int'\breve{g}^2}{(\int-\breve{g})^2}$,

$$\hbar''=\frac{(\int-\breve{g})^2[\int^2\breve{g}''-\int''\breve{g}^2+2\breve{g}'\int'(\int-\breve{g})]-2(\int^2\breve{g}'-\breve{g}^2\int')(\int-\breve{g})(\int'-\breve{g}')}{(\int-\breve{g})^4}$$

$$\hbar'''=\frac{(\int-\breve{g})^2[\int^2\breve{g}'''+2\breve{g}''\int\int'-2\int''\breve{g}\breve{g}'-\breve{g}^2\int'''+2\breve{g}'\int'(\int'-\breve{g}')]+}{(\int-\breve{g})^4}+\frac{(\int-\breve{g})[2\breve{g}'\int''+2\int'\breve{g}'']}{(\int-\breve{g})^4}-\frac{(\int^2\breve{g}''-\int''\breve{g}^2)+2\breve{g}'\int'\left(\int-\breve{g}\right))[2(\int-\breve{g})(\int'-\breve{g}')]}{(\int-\breve{g})^4} \quad (10)$$

As a result of the preceding $\int^2\breve{g}''=\int''\breve{g}^2$ such that $\int^2\breve{g}''-\int''\breve{g}^2=0$,

and $2\breve{g}'\int'(\int'-\breve{g}')=0$ since $2\breve{g}'\int'=0$ is a result of the previous theorem (2.1) as a result we obtain

$$\hbar'''=\frac{\int^2\breve{g}'''-\breve{g}^2\int'''+2\breve{g}''\int\int'-2\int''\breve{g}'\breve{g}+(\int-\breve{g})[2\breve{g}'\int''+2\int'\breve{g}'']}{(\int-\breve{g})^2} \quad (11)$$

Clearly, some of terms in (11) such as $\breve{g}'\int''+\int'\breve{g}''$, $\int^2\breve{g}'''-\int'''\breve{g}^2$, and $\breve{g}''\int\int'-\int''\breve{g}\breve{g}'$ are equal to zero as follows: Assume that

$\breve{g}'\int''+\int'\breve{g}''=0$, so that $\breve{g}'\int''=-\int'\breve{g}''$.Therefore

$$\frac{\tilde{\omega}^2\int'\int''}{(\tilde{\omega}-\int)^2}=-\frac{\tilde{\omega}^2\int'\int''}{(\tilde{\omega}-\int)^2}-\frac{2(\tilde{\omega}\int')^2\int'}{(\tilde{\omega}-\int)^3} \quad (12)$$

there is a symmetry between the first term in the left side and the first term in the right side in equation (12).

So, we have to estimate the term $\frac{-2(\tilde{\omega}\int')^2\int'}{(\tilde{\omega}-\int)^3}$ as follows

Obviously, the term $\frac{2\tilde{\omega}^2\int'^2\int^2}{(\tilde{\omega}-\int)^3}=\frac{2\tilde{\omega}^2\int'^2\int^2}{\left[\tilde{\omega}\left(1-\frac{\int}{\tilde{\omega}}\right)\right]^3}=\frac{2\int'^2\int^2}{\tilde{\omega}\left(1-\frac{\int}{\tilde{\omega}}\right)^3}\to 0$ as $\tilde{\omega}\to\infty$.

Therefore $\frac{\int'\int''}{\frac{1}{\tilde{\omega}^2}(\tilde{\omega}-\int)^2}=-\frac{\int'\int''}{\frac{1}{\tilde{\omega}^2}(\tilde{\omega}-\int)^2}$. As a result, the term $\breve{g}'\int''+\int'\breve{g}''=0$.

L et $\int^2\breve{g}'''-\int'''\breve{g}^2=0$

Rewrite the previous statement as follows

$$\frac{\tilde{\omega}^2\int^2\int'''}{(\tilde{\omega}-\int)^2}+\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^3}\int^2\int'\int''+\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^4}\int^2\int'^3=\frac{\tilde{\omega}^2\int^2\int'''}{(\tilde{\omega}-\int)^2} \quad (13)$$

there is a symmetry between the first term in the left side and that term in the right side in equation (13).

So, we have to estimate the terms $\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^3}\int^2\int'\int''$ and $\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^4}\int^2\int'^3$ as follows:

Obviously, the term $\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^3}\int^2\int'\int''=\frac{6\tilde{\omega}^2}{\left[\tilde{\omega}\left(1-\frac{\int}{\tilde{\omega}}\right)\right]^3}\int^2\int'\int''=\frac{6}{\tilde{\omega}\left(1-\frac{\int}{\tilde{\omega}}\right)^3}\int^2\int'\int''\to 0$, as $\tilde{\omega}\to\infty$.

And the same process for $\frac{6\tilde{\omega}^2}{(\tilde{\omega}-\int)^4}\int^2\int'^3\to 0$, as $\tilde{\omega}\to\infty$.

$\breve{g}''\int\int'-\int''\breve{g}\breve{g}'=0$ in short calculations

Consider $\breve{g}''\int\int'-\int''\breve{g}\breve{g}'=0$ and multiply both sides by the function $\breve{g}$ as shown follows:

$\breve{g}''\int\breve{g}\int'-\int''\breve{g}^2\breve{g}'=0$, it is clear that $\int''\breve{g}^2=\int^2\breve{g}''$. Hence, $\breve{g}''\int\breve{g}\int'-\int^2\breve{g}''\breve{g}'=0$,

such that $\breve{g}''\int\ [\breve{g}\int'-\int\breve{g}']=0$, notice that $\int\breve{g}'=\int'\breve{g}$. Finally $\breve{g}''\int\int'-\int''\breve{g}\breve{g}'=0$.

As a result, $\hbar'''=0$ and it is so simple to get $\hbar''=\tilde{\omega}$ by taking a similar technique in theorem (2.1) in order to get $\hbar=\tilde{\omega}$ such that $\tilde{\omega}=\frac{\breve{g}\int}{\int-\breve{g}}$

Finally, $\breve{g}=\frac{\tilde{\omega}\int}{\tilde{\omega}-\int}$. The proof is complete ∎

Theorem (3.3). Let $\int\in\breve{S}$, and defined by $\breve{E}(f)=\{\breve{g}\in\breve{S},\breve{g}\sim\int\}$ into the transformation $\breve{g}(z)=\frac{\tilde{\omega}\int}{\tilde{\omega}-\int}$. Then $\breve{g}(z)$ generates a nth- order nonlinear complex differential equation

$$\breve{g}^{(m)}(\breve{z})=\left[\frac{d^m\int(\breve{z})}{d\breve{z}^m}+\sum\nolimits_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\int(\breve{z}))^n}{d\breve{z}^m}\right] \quad (14)$$

in the open unit disk $Đ$.

Proof. What we shall do now is to get the nth-order derivative of the function. In order to construct an nth-order nonlinear complex differential equation, the Second Cauchy Integral Formula must be considered in the form.

$\breve{g}^{(m)}(z_0)=\frac{m!}{2\pi!}\int\frac{\breve{g}(z)}{(\breve{z}-\breve{z}_0)^{m+1}}d\breve{z}$

with the support of the Omitted-value transformation definition, which is represented by.

$\breve{g}(z)=\frac{\tilde{\omega}\int(\breve{z})}{\tilde{\omega}-\int(\breve{z})}$; $\tilde{\omega}\notin\int(Đ)$, where $\breve{g} \in \check{E}(f) = \{\breve{g} \in \check{S}, \breve{g} \sim \mathfrak{f}\}$.

To obtain nth-order derivative of the function $\mathfrak{f}(\check{z})$ as below, we have to consider

$\mathfrak{f}(\check{z}) = \check{z} + a_2\check{z}^2 + a_3\check{z}^3 + \ldots + a_n\check{z}^n + \ldots$ in $\check{S}$.

And then take the following steps:

$\mathfrak{f}'(\check{z}) = 1 + 2a_2\check{z} + 3a_3\check{z}^2 + \ldots + na_n\check{z}^{n-1} + \ldots,$

$\mathfrak{f}''(\check{z}) = 2a_2 + 6a_3\check{z} + \ldots + n(n-1)a_n\check{z}^{n-2} + \ldots,$

$\mathfrak{f}'''(\check{z}) = 6a_3 + \ldots + n(n-1)(n-2)a_n\check{z}^{n-3} + \ldots,$

$\vdots$

$\mathfrak{f}^{(n)}(\check{z}) = (m!)a_m + \ldots + \frac{n!}{(m-n)!}a_n\check{z}^{n-m} + \ldots$

Returning to the definition of $\breve{g}(\check{z}) = \frac{\tilde{\omega}\mathfrak{f}}{\tilde{\omega}-\mathfrak{f}}$, performing a specific technique with the support of well-known series is known as a geometric series, and it will be effective in finding the key to this proof.

Let $\breve{g}(\check{z}) = \tilde{\omega}\mathfrak{f} * \frac{1}{\tilde{\omega}-\mathfrak{f}}$, then

$\breve{g}(\check{z}) = \tilde{\omega}f * \left[\frac{\frac{1}{\tilde{\omega}}}{1-\frac{\mathfrak{f}}{\tilde{\omega}}}\right]$; this function now be similar to the form of geometric series

Hence,

$$\text{Hence}, \breve{g}(\check{z}) = \tilde{\omega}\mathfrak{f} * \frac{1}{\tilde{\omega}}\left[1 + \frac{\mathfrak{f}}{\tilde{\omega}} + \left(\frac{\mathfrak{f}}{\tilde{\omega}}\right)^2 + \left(\frac{\mathfrak{f}}{\tilde{\omega}}\right)^3 + \ldots\right],$$

$$\begin{aligned}
\breve{g}(\check{z}) &= \tilde{\omega}\left(\check{z} + a_2\check{z}^2 + \ldots\right) \\
&* \frac{1}{\tilde{\omega}}\left[1 + \frac{1}{\tilde{\omega}}\left(\check{z} + a_2\check{z}^2 + \ldots\right) + \frac{1}{\tilde{\omega}^2}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^2 + \frac{1}{\tilde{\omega}^3}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^3 + \ldots\right. \\
&\left. + \frac{1}{\tilde{\omega}^n}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^n + \ldots\right] \\
\breve{g}(\check{z}) &= \tilde{\omega}\left(\check{z} + a_2\check{z}^2 + \ldots\right) \\
&* \left[\frac{1}{\tilde{\omega}} + \frac{1}{\tilde{\omega}^2}\left(\check{z} + a_2\check{z}^2 + \ldots\right) + \frac{1}{\tilde{\omega}^3}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^2 + \frac{1}{\tilde{\omega}^4}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^3\right. \\
&\left. + \ldots\frac{1}{\tilde{\omega}^{n+1}}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^n + \ldots\right] \\
\breve{g}(\check{z}) &= \left(\check{z} + a_2\check{z}^2 + \ldots\right) + \frac{1}{\tilde{\omega}}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^2 + \frac{1}{\tilde{\omega}^2}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^3 + \frac{1}{\tilde{\omega}^3}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^4 + \\
&\ldots\frac{1}{\tilde{\omega}^n}\left(\check{z} + a_2\check{z}^2 + \ldots\right)^{n+1} + \ldots
\end{aligned} \tag{13}$$

Derive (13) to (m)- times to get the goal in this step.

$$\breve{g}'(\check{z}) = (1 + 2a_2\check{z} + 3a_3\check{z}^2 + \cdots) + \frac{2}{\tilde{\omega}}(\check{z} + a_2\check{z}^2 + \cdots)(1 + 2a_2\check{z} + \cdots) + \frac{3}{\tilde{\omega}^2}(\check{z} + a_2\check{z}^2 + \cdots)^2(1 + 2a_2\check{z} + \cdots) + \frac{4}{\tilde{\omega}^3}(\check{z} + a_2\check{z}^2 + \cdots)^3(1 + 2a_2\check{z} + \cdots) + \cdots + \frac{n+1}{\tilde{\omega}^n}(\check{z} + a_2\check{z}^2 + \cdots)^n(1 + 2a_2\check{z} + \cdots) + \cdots$$

$$\breve{g}''(z) = (2a_2 + 6a_3\check{z} + \cdots) + \frac{2}{\tilde{\omega}}[(\check{z} + a_2\check{z}^2 + \cdots)(2a_2 + 6a_3\check{z} + \cdots) + (1 + 2a_2\check{z} + \cdots)(1 + 2a_2\check{z} + \cdots)] + \frac{3}{\tilde{\omega}^2}[(\check{z} + a_2\check{z}^2 + \cdots)^2(2a_2 + 6a_3\check{z} + \cdots) + 2(1 + 2a_2\check{z} + 3a_3\check{z}^2 + \cdots)(\check{z} + a_2\check{z}^2 + \cdots)(1 + 2a_2\check{z} + \cdots)] + \frac{4}{\tilde{\omega}^3}[(\check{z} + a_2\check{z}^2 + \cdots)^3(2a_2 + 6a_3\check{z} + \cdots) + 3(1 + 2a_2\check{z} + \cdots)(\check{z} + a_2\check{z}^2 + \cdots)^2(1 + 2a_2\check{z} + 3a_3\check{z}^2 + \cdots)] + \cdots + \frac{n+1}{\tilde{\omega}^n}[(\check{z} + a_2\check{z}^2 + \cdots)^n(2a_2\check{z} + 6a_3\check{z}^2 + \cdots) + n(1 + 2a_2\check{z} + \cdots)(\check{z} + a_2\check{z}^2 + \cdots)^{n-1}(1 + 2a_2\check{z} + \cdots)] + \cdots$$

$$\breve{g}^{(m)}(\check{z}) = \frac{d^{(m)}}{d\check{z}^m}(\check{z} + a_2\check{z}^2 + a_3\check{z}^3 + \cdots) + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m}{d\check{z}^m}(\check{z} + a_2\check{z}^2 + \cdots)^n \ldots\ldots(15)$$

Rewrite above relation (15) which describes m th − order nonlinear complex differential equation as follows

$$\breve{g}^{(m)}(\check{z}) = \left[\frac{d^m\mathfrak{f}(\check{z})}{d\check{z}^m} + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\mathfrak{f}(\check{z}))^n}{d\check{z}^m}\right]$$

Next, we show that if $\breve{g}$ is a solution to the ordinary differential equation in

S, then $\breve{g}\in\breve{E}(f)$, we begin by considering Second Cauchy Integral Formula

$$\breve{g}^{(m)}(\check{z}_0) = \frac{m!}{2\pi!}\int\frac{\breve{g}(\check{z})}{(\check{z}-\check{z}_0)^{m+1}}d\check{z} \qquad (16)$$

where $\check{z}_0$ denotes a singular point in the function $\hbar(\check{z}) = \frac{\breve{g}(\check{z})}{(\check{z}-\check{z}_0)^{m+1}}$.

Hence,

$$\breve{g}^{(m)}(\check{z}_0) = \left[\frac{d^m\mathfrak{f}(\check{z})}{d\check{z}^m} + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\mathfrak{f}(\check{z}))^n}{d\check{z}^m}\right]_{\check{z}=\check{z}_0} \qquad (17)$$

From (16) and (17) we get

$$\frac{m!}{2\pi!}\int\frac{\breve{g}(\check{z})}{(\check{z}-\check{z}_0)^{m+1}}d\check{z} = \left[\frac{d^m\mathfrak{f}(\check{z})}{d\check{z}^m} + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\mathfrak{f}(\check{z}))^n}{d\check{z}^m}\right]_{\check{z}=\check{z}_0} \qquad (18)$$

By differentiating both side of equation (18) we obtain

$$\frac{m!}{2\pi!}\left[\frac{\breve{g}(\check{z})}{(\check{z}-\check{z}_0)^{m+1}}\right] = \left[\frac{d^m\mathfrak{f}(\check{z})}{d\check{z}^m} + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\mathfrak{f}(\check{z}))^n}{d\check{z}^m}\right]_{\check{z}=\check{z}_0}$$

Such that,

$$\breve{g}(\check{z}) = \frac{2\pi!(\check{z}-\check{z}_0)^{m+1}}{m!}\left[\frac{d^m\mathfrak{f}(\check{z})}{d\check{z}^m} + \sum_{n=2}^{\infty}\frac{1}{\tilde{\omega}^{n-1}}\cdot\frac{d^m(\mathfrak{f}(\check{z}))^n}{d\check{z}^m}\right]_{\check{z}=\check{z}_0}$$

is a formula for the complete solution of the given equation.

The Proof is complete. ∎

The following theorem illustrates the case of limiting each infinite power series for the holomorphic, univalent function in S to satisfy the Omitted-value transformation.

Theorem (3.4). Let $\mathfrak{f}(\check{z}) = C_0 + C_n(\check{z}-\tilde{\omega})^n + C_{n+1}(\check{z}-\tilde{\omega})^{n+1} + C_{n+2}(\check{z}-\tilde{\omega})^{n+2} + \ldots$ be a holomrophic function in $Đ$, and defined by $\breve{E}(f) = \{\breve{g}\in\check{S}, \breve{g}\sim\mathfrak{f}\}$.

Consider 2nd order non-linear complex differential equation $\frac{\breve{g}''}{\breve{g}'} = \frac{\mathfrak{f}''}{\mathfrak{f}'} + \frac{2\mathfrak{f}'}{\tilde{\omega}-\mathfrak{f}}$. Then $\breve{g}(z) = C_1\check{z} + C_2$ be a solution for the given equation.

Proof.

$\mathfrak{f}'(\check{z}) = nC_0(\check{z}-\tilde{\omega})^{n-1} + (n+1)C_{n+1}(\check{z}-\tilde{\omega})^n + (n+2)C_{n+2}(\check{z}-\tilde{\omega})^{n+1} + \ldots$

$= \sum_{n=1}^{\infty}nC_n(\check{z}-\tilde{\omega})^{n-1}$

$\mathfrak{f}''(\check{z}) = n(n-1)C_n(\check{z}-\tilde{\omega})^{n-2} + n(n+1)C_{n+1}(\check{z}-\tilde{\omega})^{n-1} + (n+1)(n+2)C_{n+2}(\check{z}-\tilde{\omega})^n + \ldots$

Substituted each of $\mathfrak{f}(\check{z})$, $\mathfrak{f}'(\check{z})$, and $\mathfrak{f}''(\check{z})$ in the equation $\frac{\breve{g}''}{\breve{g}'} = \frac{\mathfrak{f}''}{\mathfrak{f}'} + \frac{2\mathfrak{f}'}{\tilde{\omega}-\mathfrak{f}}$ to find the form of the function $\breve{g}(z)$, which would play a main role in determining whether the function $\mathfrak{f}(\check{z})$ satisfies the Omitted − value transformation as follows: Consider

$$\frac{\mathfrak{f}''(\check{z})}{\mathfrak{f}'(\check{z})} = \frac{n(n-1)C_n(\check{z}-\tilde{\omega})^{n-2} + n(n+1)C_{n+1}(\check{z}-\tilde{\omega})^{n-1} + (n+1)(n+2)C_{n+2}(\check{z}-\tilde{\omega})^n + \ldots}{nC_n(\check{z}-\tilde{\omega})^{n-1} + (n+1)C_{n+1}(\check{z}-\tilde{\omega})^n + (n+2)C_{n+2}(\check{z}-\tilde{\omega})^{n+1} + \ldots}$$

After a simple calculation, the qoutiant of $\frac{\mathfrak{f}''(\check{z})}{\mathfrak{f}'(\check{z})}$ is equal to zero, which implies to $f'' = 0$. Hence, by

Integrating $f'' = 0$ twice these will get $\mathfrak{f}(\check{z}) = c_1\check{z} + c_2$, where c_1, c_2 are constant and non equal to zero.

And then, substitute $\mathfrak{f}(\check{z}) = c_1\check{z} + c_2$ in $\frac{2\mathfrak{f}'}{\tilde{\omega}-\mathfrak{f}}$ such that the qoutiant of

$\frac{2\mathfrak{f}'}{\tilde{\omega}-\mathfrak{f}} = \frac{2nC_n(\check{z}-\tilde{\omega})^{n-1}+2(n+1)C_{n+1}(\check{z}-\tilde{\omega})^{n}+2(n+2)C_{n+2}(\check{z}-\tilde{\omega})^{n+1}+\ldots}{\tilde{\omega}-C_0-C_n(\check{z}-\tilde{\omega})^{n}-C_{n+1}(\check{z}-\tilde{\omega})^{n+1}-C_{n+2}(\check{z}-\tilde{\omega})^{n+2}-\ldots}$
is equal to zero.

Hence, $\frac{\mathfrak{f}''}{\mathfrak{f}'} + \frac{2\mathfrak{f}'}{\tilde{\omega}-\mathfrak{f}} = 0$, that is; $\frac{\breve{g}''}{\breve{g}'} = 0$, such that $\breve{g}'' = 0$

Finally, Integrating $\breve{g}'' = 0$ twice with respect to $\check{z}$ to obtain

$\breve{g}(z) = C_1\check{z} + C_2$, be a solution for given equation, where $\mathfrak{f}(\check{z}) = \frac{\tilde{\omega}(C_1\check{z}+C_2)}{\tilde{\omega}+C_1\check{z}+C_2}$. ∎

4. Conclusion

In this study work.

1. We found that Omitted − value transformation with equivalent class $\check{E}(\mathfrak{f}) = \{\breve{g} \in \check{S}, \breve{g} \sim \mathfrak{f}\}$ play an important role in generation of nonlinear complex differential equations and their solutions.
2. It has been indicated how to limit any infinite power series for the holomorphic, univalent function in order to satisfy the Omitted-value transformation property when $\tilde{\omega} \notin \mathfrak{f}(Đ)$.

Author contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to express a deep thanks and gratitude to Department of Mathematics, College of Science, Mustansiriyah University for deep supporting to appear this research paper as it is now.

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