Al-Qadisiyah Journal of Pure Science

Volume 28 | Number 1 Article 14

2023

Study of transitive random dynamical systems

Salam T. Abbas
University of Al-Qadiysih, College of Science, Diyala, Beldroz, Iraq, salamalsaedi7@gmail.com

Ihsan J. Kadhim
University of Al-Qadiysih, College of Science, Najaf, Iraq

ARTICLE

Study of Transitive Random Dynamical Systems

Salam T. Abbas [a,*], Ihsan J. Kadhim [b]

[a] University of Al-Qadiysih, College of Science, Diyala, Beldroz, Iraq
[b] University of Al-Qadiysih, College of Science, Najaf, Iraq

Abstract

In this paper we state the definition of transitivity of random dynamical systems and give some equivalents to definition that explains to us the concept of transitive and study some new properties and give theorems to fully illustrate the concept of transitive of random dynamical systems.

Keywords: Random dynamical system, Random set, Transitivity of random dynamical system, Weakly dense, Nowhere weakly dense, Trajectory, Separable

1. Introduction

The transitivity is one of the effectual notions in the qualitative theory of dynamical systems. It is intensively advanced in recent years and has become an important part of the dynamical systems and there are a lot of researchers who have studied transitivity in dynamical systems look, for example [1-4,5,6,8,10,11,13].

The importance of the subject of transitivity, we did a study of transitivity of random dynamical systems .

Our study dealt with the subject of transitivity of random dynamical systems (shorty transitivity ($\mathcal{RDS}$ (θ, φ)) through a new definition of transitive and give many definition equivalents let's get more flexibility to deal with weakly dense orbit and open, closed random set .

We have also introduced new theorems for transitivity ($\mathcal{RDS}$ (θ, φ)) and we proved that transitivity of ($\mathcal{RDS}$ (θ, φ) is topological property .

We'll give a definition of random hausdorff and definition the hitting time sets and prove it is an infinite .

We'll give a definition of the isolated random point.

We're going to prove that (θ, φ)on $\mathcal{X}$ is transitive if $\mathcal{X}$ has no isolated points and (θ, φ) on $\mathcal{X}$ has a point with weakly dense orbit and some other theorems.

We'll give a definition of the separable and we're going to prove that (θ, φ) on $\mathcal{X}$ has a point with weakly dense orbit if $\mathcal{X}$ is separable and second category and(θ, φ) on $\mathcal{X}$ is transitive.

Definition (1.1) (metric dynamical system) [7,12]:

The 5-tuple $(T, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system

(Shortly $\mathcal{MDS}$) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and

(i) $\theta : T \times \Omega \to \Omega$ is $(\mathcal{B}(G) \otimes \mathcal{F}, \mathcal{F})-$ measurable,
(ii) $\theta(0, \omega) = \omega, \ \forall \omega \in \Omega$
(iii) $\theta(t+s, \omega) = \theta(t, \theta(s, \omega)) \ \forall, t, s \in T, \omega \in \Omega$
(iv) $\mathbb{P}(\theta_t(F)) = \mathbb{P}(F)$, for every $F \in \mathcal{F}$ and every $t \in T$.

Note that we write $\theta : T \times \Omega \to \Omega$ either in the form $\theta(t, \omega)$ (as a function of two variable) or in the form $\theta_t \omega$.

Definition (1.2) (random dynamical system) [14]:

A measurable random dynamical system on the measurable space $(X, \mathcal{B}(\mathcal{X}))$ over (or covering, or extending) an $\mathcal{MDS}$ $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ with time is a mapping $\varphi : \mathbb{T} \times \ \Omega \times \ \mathcal{X} \to \mathcal{X}$, with the following properties:

(i) Measurability, φ is $\mathcal{B}(\mathbb{T}) \bigotimes \mathcal{F} \bigotimes \mathcal{B}, \mathcal{B}-$measurable.
(ii) Cocycle property: The mappings $\varphi(t, \omega) := \varphi(t, \omega, \cdot) : \mathcal{X} \to \mathcal{X}$ form a cocycle over $\theta(\cdot)$, i. e. they satisfy

$\varphi(0, \omega, x) = x$ for all $\omega \in \Omega$ (if $0 \in \mathbb{T}$),

Available online 26 May 2023

* Corresponding author.
E-mail addresses: salamalsaedi7@gmail.com (S.T. Abbas), Ihsan.kadhim@qu.edu.iq (I.J. Kadhim).

https://doi.org/10.29350/2411-3514.1013

$\varphi(t+s,\omega)=\varphi(t,\theta(s)\omega)\circ\varphi(s,\omega)$ for all $s,t\in\mathbb{T}$, $\omega\in\Omega$.

If there is no ambiguity the $\mathcal{RDS}$ is denoted by (θ,φ) rather than $(T,\Omega,\mathcal{X},\theta,\varphi)$.

Definition (1.3) (tempered random variable) [12]:

A random variable $\epsilon:\Omega\to\mathbb{R}^+$ is called tempered with respect to the MDS θ if for the associated stationary stochastic process $t\mapsto\epsilon(\theta(t)\cdot)$ the invariant set for which $\lim_{t\to\infty}\frac{1}{t}\log(\epsilon(\theta(t)\omega)=0$.

Definition (1.4) (random set) [12]:

Let $(\Omega,\mathcal{F})$ be a measurable space and $(\mathcal{X},d)$ be a metric space which is considered a measurable space with Borel σ- algebra $\mathcal{B}(\mathcal{X})$. The set-valued function : $\Omega\to\mathcal{B}(\mathcal{X}),\omega\to A(\omega)$, is said to be random set if for each $x\in\mathcal{X}$ the function $\omega\to d(x,A(\omega))$ is measurable. If $A(\omega)$ is connected closed (compact) for all $\omega\in\Omega$, it is called a random connected closed (compact) set .

Proposition (1.5) (some properties of random set) [9]:

(i) If A is a random closed set, then so is $\overline{A^c}$ (the closure of A^c).
(ii) If $\mathcal{U}$ is a random open set, then $\overline{\mathcal{U}}$ is a random closed set.
(iii) If A is a random closed set, then Int(A), the interior of A, is a random open set.

If A_1 and A_2 are random compact sets, then so is $A_1\cap A_2$.

Definition (1.6) (invariance property) [7]:

Let (θ,φ) be a measurable $\mathcal{RDS}$ and A *multifunction* $\omega\to\mathrm{D}(\omega)$ is said to be:

(i) Forward invariant *with respect to* (θ,φ) *if* $\varphi(\mathrm{t},\omega)\mathrm{D}(\omega)\subseteq\mathrm{D}(\theta_\mathrm{t}\omega)$ *for all* $\mathrm{t}>0$ *and* $\omega\in\Omega$, *i.e. if* $\mathrm{x}\in\mathrm{D}(\omega)$ *implies* $\varphi(\mathrm{t},\omega)x\mathrm{D}(\omega)\in\mathrm{D}(\theta_\mathrm{t}\omega)$ *for all* $\mathrm{t}\geq 0$ *and* $\omega\in\Omega$;
(ii) Backward invariant *with respect to* (θ,φ) *if* $\varphi(\mathrm{t},\omega)\mathrm{D}(\omega)\supseteq\mathrm{D}(\theta_\mathrm{t}\omega)$ *for all* $\mathrm{t}>0$ *and* $\omega\in\Omega$, *i.e. for every* $\mathrm{t}>0$, $\omega\in\Omega$ *and* $\mathrm{y}\in\mathrm{D}(\theta_\mathrm{t}\omega)$ *there exists* $\mathrm{x}\in\mathrm{D}(\omega)$ *such that* $\varphi(\mathrm{t},\omega)x=\mathrm{y}$;

Invariant *with respect to* (θ,φ) *if* $\varphi(\mathrm{t},\omega)\mathrm{D}(\omega)=\mathrm{D}(\theta_\mathrm{t}\omega)$ *for all* $\mathrm{t}>0$ *and* $\omega\in\Omega$, *i.e. if it is both forward and backward invariant .*

Definition (1.7) trajectories [7]:

Let $D:\omega\mapsto D(\omega)$ be a multifunction. We call the multifunction

$$\omega\mapsto\gamma_D^t(\omega):=\bigcup_{k\geq t}\varphi(k,\theta_{-k}\omega)\S_0(\theta_{-k}\omega)$$

The tail (from the moment t) of the pull back trajectories emanating from D. if $D(\omega)=\{\sqsubseteq(\omega)\}$ is single valued function

Then $\omega\mapsto\gamma_v(\omega)\equiv\gamma_D^0(\omega)$ is said to be the (pull back) trajectory (or orbit) emanating from v.

Definition (1.8): (equivalence of $\mathcal{RDS}$) [7]:

Let (θ,φ) and (θ,ψ) be two $\mathcal{RDS}$'s over the same $\mathcal{MDS}$ θ with phase spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively .

These $\mathcal{RDS}$ ((θ,φ) and (θ,ψ) are said to be (topologically) equivalent (or conjugate) if there exists a mapping:

$g:\Omega\times\mathcal{X}_1\to\mathcal{X}_2$ with the properties:

1) the mapping $x\to g(\omega,x)$ is a homeomorphism from $\mathcal{X}_1$ onto $\mathcal{X}_2$ for every $\omega\in\Omega$;
2) the mappings $\omega\to g(\omega,x_1)$ and $\omega\to\omega\to g^{-1}(\omega,x_2)$ are measurable for every $x_1\in\mathcal{X}_1$ and $x_2\in\mathcal{X}_2$;
3) the cocycles φ and ψ are cohomologous, i.e.

$\psi(t,\omega,g(\omega,x)=g(\theta_t,\omega,\varphi(t,\omega,x))$ for any $x\in\mathcal{X}_1$.

2. Transitivity

Definition (2.1):

A $\mathcal{RDS}$ (θ,φ)*on* $\mathcal{X}$ is said to be transitive, if for every nonempty random open sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ of $\mathcal{X}$, there is k $\in\mathbb{N}$ such that

$\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\varnothing$,

Definition (2.2):

A random set $\mathcal{U}(\omega)$ in $\mathcal{X}$ is said to be weakly dense of $\mathcal{X}$, if $\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\varnothing$ for every nonempty random open set $\mathcal{V}(\omega)$ of $\mathcal{X}$.

Theorem (2.3): for any $\mathcal{RDS}$ $(\boldsymbol{\theta},\boldsymbol{\varphi})$ the following statements are equivalent:

1) (θ,φ) is transitive,
2) for every non-empty open random set $\mathcal{U}(\omega)\subset\mathcal{X}$, $\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}$ is weakly dense in $\mathcal{X}$,
3) for every pair of non-empty open random sets $\mathcal{U}(\omega),\mathcal{V}(\omega)\subset\mathcal{X}$ there is an integer $k\in\mathbb{N}$ *for which* $\varphi(-k,\omega)\mathcal{V}(\omega)\cap\mathcal{U}(\omega)\neq\varnothing$,
4) for every non-empty open random set $\mathcal{V}(\omega)\subset\mathcal{X}$,

$\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}(\omega)\}$is weakly dense in $\mathcal{X}$,

5) every closed, invariant, proper subset $\mathcal{X}$ has empty interior.

Proof: $(1\Rightarrow 2)$

Let (θ,φ) is transitive then for every non-empty open random sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ of $\mathcal{X}$, there is k $\in\mathbb{N}$ such that $\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\varnothing$

$\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}\cap\mathcal{V}(\omega)\neq\varnothing$

$\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}$ is weakly dense in $\mathcal{X}$.

$(2 \Rightarrow 3)$

Let $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ be non-empty open random sets of, there is k $\in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}\cap\mathcal{V}(\omega)\neq\emptyset$

There is $k_0 \in \mathbb{N}$ such that $\varphi(k_0,\theta_{-k_0}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\emptyset(*)$

Note that $[\varphi(k_0,\theta_{-k_0}\omega)\mathcal{U}(\omega)]^{-1} = \varphi(-k_0,\omega)\mathcal{U}(\omega)$

Then from (*) we have

$$\mathcal{U}(\omega)\cap\varphi(-k_0,\omega)\mathcal{V}(\omega)\neq\emptyset$$

$$Or\ \varphi(-k_0,\omega)\mathcal{V}(\omega)\cap\mathcal{U}(\omega)\neq\emptyset$$

$(3 \Rightarrow 4)$

Let $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ be non-empty open random sets of $\mathcal{X}$, there is k $\in \mathbb{N}$ such that $\varphi(-k,\omega)\mathcal{V}(\omega)\cap\mathcal{U}(\omega)\neq\emptyset$

Then $\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{V}(\omega)\}\cap\mathcal{U}(\omega)\neq\emptyset$

$\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{V}(\omega)\}$ is weakly dense in $\mathcal{X}$.

$(4 \Rightarrow 5)$

Let $C(\omega)$ be closed, invariant, proper subset $\mathcal{X}$

suppose $C(\omega)$ has non-empty interior if $int(C(\omega))\neq\emptyset$

then there is open set $\mathcal{U}$ subset of $C(\omega)$

since $C(\omega)$ is closed then $C^c(\omega)$ is open in $\mathcal{X}$

$$\varphi(-k_0,\omega)C^c(\omega)\cap\mathcal{U}\neq\emptyset$$

$$\Rightarrow [\mathcal{X}/\varphi(-k_0,\omega)C(\omega)]\cap\mathcal{U}\neq\emptyset$$

$$\Rightarrow [\mathcal{X}\,/\,C(\theta_{-k_0}\omega)]\cap\mathcal{U}\neq\emptyset$$

$$\Rightarrow C^c(\theta_{-k_0}\omega)\cap\mathcal{U}\neq\emptyset$$

$\because \mathcal{U}$ subset of $C(\omega)$

$$\Rightarrow \mathcal{U}\cap C^c(\omega)=\emptyset$$

This is a contradiction

Then $\mathrm{int(C)}(\omega) = \emptyset$

$\Rightarrow C(\omega)$ has empty interior

$(5 \Rightarrow 1)$

Suppose every closed, invariant, proper subset $\mathcal{X}$ has empty interior

Let $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ be non-empty open random sets of $\mathcal{X}$, such that for every k $\in \mathbb{N}$,

suppose $\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega) = \emptyset$

$$\Rightarrow \varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\subset\mathcal{X}\,/\,\mathcal{V}(\omega)$$

$$\Rightarrow \bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}\subset\mathcal{X}\,/\,\mathcal{V}(\omega)$$

$$\Rightarrow \overline{\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}}\subset\mathcal{X}\,/\,\mathcal{V}(\omega)$$

We have $\overline{\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}}$ is closed and $\neq\mathcal{X}$

Also $\therefore \varphi(k,\theta_{-k}\omega)$ is homeomorphism and $\mathcal{U}(\omega)$ is open

$\Rightarrow \varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)$ is open

$\Rightarrow \underline{\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}}$ is open

$\Rightarrow \bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\}$ has non-empty interior

$\Rightarrow$ This is a contradiction

Then $\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\emptyset \Rightarrow (\theta,\varphi)$ is transitive .

Definition (2.4): nowhere weakly dense

A $\mathcal{X}$ is said to be nowhere weakly dense if it has an empty interior (or $\mathcal{X}$ no has a point with weakly dense orbit)

Theorem (2.5): for any $\mathcal{RDS}$ $(\boldsymbol{\theta},\boldsymbol{\varphi})$ the following statements are equivalent:

1) (θ,φ) is transitive,
2) if $\mathcal{C}(\omega)\subset\mathcal{X}$ is closed random set and $\varphi(k,\theta_{-k}\omega)\ \mathcal{C}(\omega)\subset\mathcal{C}(\omega)$, then $\mathcal{C}(\omega)=\mathcal{X}$ or $\mathcal{C}(\omega)$ is nowhere weakly dense .

Proof: $(1 \Rightarrow 2)$

Let (θ,φ) is transitive then for every non-empty open random sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ of $\mathcal{X}$, there is k $\in \mathbb{N}$ such that $\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\emptyset$ and let $\mathcal{C}(\omega)\subset\mathcal{X}$ is closed random set and $\varphi(k,\theta_{-k}\omega)\ \mathcal{C}(\omega)\subset\mathcal{C}(\omega)$

Assume $\mathcal{C}(\omega)\neq\mathcal{X}$ and $\mathcal{C}(\omega)$ has non-empty interior

Define $\mathcal{G}(\omega)=\mathcal{X}/\ \mathcal{C}(\omega) \Rightarrow \mathcal{G}(\omega)$ is open random set (since $\mathcal{C}(\omega)$ is closed)

Let $\mathcal{H}(\omega)\subset\mathcal{C}(\omega)$ be open random set (since $\mathcal{C}(\omega)$ has non-empty interior)

We have $\varphi(k,\theta_{-k}\omega)\mathcal{H}(\omega)\subset\ \mathcal{C}(\omega)$ (since $\varphi(k,\theta_{-k}\omega)\ \mathcal{C}(\omega)\subset\mathcal{C}(\omega)$)

$\varphi(k,\theta_{-k}\omega)\mathcal{H}(\omega)\cap\mathcal{G}(\omega)=\emptyset$ for every k $\in \mathbb{N}$

$\Rightarrow$ This is a contradiction since (θ,φ) is transitive

Hence $\mathcal{C}(\omega)=\mathcal{X}$ or $\mathcal{C}(\omega)$ is nowhere weakly dense.

$(2 \Rightarrow 1)$

Let $\mathcal{G}(\omega)$ be a non-empty open random set in.

Suppose (θ,φ) is not transitive for every non-empty open random sets in $\mathcal{X}$

Then from Theorem (2.1.3) part (4) we have $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}$ is not weakly dense, but $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}$ is open random set.

Define $(\omega)=\mathcal{X}/\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}$, then $\mathcal{C}(\omega)$ *is closed random set* in $\mathcal{X}$ and $\mathcal{C}(\omega)\neq\ \mathcal{X}$,

Claim $(k,\theta_{-k}\omega)\ \mathcal{C}(\omega)\subset\mathcal{C}(\omega)$.

Suppose $\varphi(k,\theta_{-k}\omega)\ \mathcal{C}(\omega)$ is not subset of $\mathcal{C}(\omega)$

This implies $\varphi(k,\ \theta_{-k}\omega)\ \mathcal{C}(\omega)\cap\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}\neq\emptyset$

This further implies

$$\varphi(-k,\omega)\circ[\varphi(k,\theta_{-k}\omega)\mathcal{C}(\omega)\cap\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}]$$

$$=\mathcal{C}(\omega)\cap\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}\neq\emptyset$$

This is a contradiction to definition of $\mathcal{C}(\omega)$

Thus $\varphi(k,\theta_{-k}\omega)\,\mathcal{C}(\omega)\subset\mathcal{C}(\omega)$

Since $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}$ is not weakly dense, there exists a non-empty open random set $\mathcal{H}(\omega)$ in $\mathcal{X}$ such that $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}\cap\mathcal{H}(\omega)=\varnothing$

This implies $\mathcal{H}(\omega)\subset\mathcal{C}(\omega)$

$\mathcal{C}(\omega)\cap\mathcal{H}(\omega)\neq\varnothing$

This is a contradiction to the fact that $\mathcal{C}(\omega)$ is nowhere weakly dense

Hence (θ,φ) is transitive .

Theorem (2.6): for any $\mathcal{RDS}$ (θ,φ) the following statements are equivalent:

1) (θ,φ) is transitive,
2) if $\mathcal{G}(\omega)\in\mathcal{X}$ is open random set and $\varphi(k,\theta_{-k}\omega)\,\mathcal{G}(\omega)\subset\mathcal{G}(\omega)$, then $\mathcal{G}(\omega)=\varnothing$ or $\mathcal{G}(\omega)$ is weakly dense in $\mathcal{X}$.

Proof: $(1\Rightarrow2)$

Let (θ,φ) is transitive then for every non-empty open random sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ of $\mathcal{X}$, there is k $\in\mathbb{N}$ such that $\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\varnothing$

and $\mathcal{G}(\omega)\subset\mathcal{X}$ is open random set and $\varphi(k,\theta_{-k}\omega)\,\mathcal{G}(\omega)\subset\mathcal{G}(\omega)$

Assume that $\mathcal{G}(\omega)\neq\varnothing$ and $\mathcal{G}(\omega)$ is not weakly dense in $\mathcal{X}$

Then there exists non-empty open random set $\mathcal{H}(\omega)$ *in* $\mathcal{X}$ such that

$\mathcal{G}(\omega)\cap\mathcal{H}(\omega)=\varnothing$. further $\varphi(k,\theta_{-k}\omega)\,\mathcal{G}(\omega)\cap\mathcal{H}(\omega)=\varnothing$ for all k $\in\mathbb{N}$

$\Rightarrow$ This is a contradiction since (θ,φ) is transitive

Hence $\mathcal{G}(\omega)=\varnothing$ or $\mathcal{G}(\omega)$ is weakly dense in.

$(2\Rightarrow1)$

Suppose (θ,φ) is not transitive for every non-empty open random sets in $\mathcal{X}$

Let $\mathcal{G}(\omega)\subset\mathcal{X}$ and let $\mathcal{D}=\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}$. it is non-empty open and not weakly dense in $\mathcal{X}$

$$\Rightarrow(\mathcal{D})^{-1}=[\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{G}(\omega)\}]^{-1}$$
$$=\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{G}(\omega)\}$$

$$\Rightarrow\bigcup_{k\in\mathbb{N}}\{\varphi(k,\theta_{-k}\omega)\mathcal{G}(\omega)\}\subset\mathcal{G}(\omega)$$

$$\Rightarrow\varphi(k,\theta_{-k}\omega)\,\mathcal{G}(\omega)\subset\mathcal{G}(\omega)\Rightarrow\mathcal{G}(\omega)=\varnothing\Rightarrow\ \mathcal{D}=\varnothing$$

This is a contradiction since $\mathcal{D}\neq\varnothing$, hence (θ,φ) is transitive .

Theorem (2.7):

Transitivity of $\mathcal{RDS}$ is topological property.

Proof:

Let (θ,φ) and (θ,ψ) be two $\mathcal{RDS}$ s on $\mathcal{X}$ and $\mathcal{Y}$ respectively such that (θ,φ)Equivalence(θ,ψ).

Then there exists a homeomorphism mapping $g:(\theta,\ \varphi)\rightarrow(\theta,\ \psi)$. We must prove that if (θ,φ) is transitive, then (θ,ψ) is transitive. Let $\mathcal{G}(\omega)$ and $\mathcal{H}(\omega)$ be two random open sets of $\mathcal{Y}$. From continuity of g, we have $g^{-1}(\mathcal{G}(\omega))$ and $g^{-1}(\mathcal{H}(\omega))$ are two random open sets of $\mathcal{X}$.

Put $\mathcal{U}(\omega)=g^{-1}(\mathcal{G}(\omega))$ and $\mathcal{V}(\omega)=g^{-1}(\mathcal{H}(\omega))$,

and by using Definition (2.1.1), there is k $\in\mathbb{N}$ such that

$\varphi(k,\theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega)\neq\varnothing$

It follows that $g(\varphi(k,\ \theta_{-k}\omega)\mathcal{U}(\omega)\cap\mathcal{V}(\omega))\neq\varnothing$. Then,

$(\psi(k,\theta_{-k}\omega)\mathcal{G}(\omega)\bigcap\mathcal{H}(\omega))\neq\varnothing$

By using Definition (2.1.1) again, we have (θ,ψ) is transitive .

Notations (2.8):

For any two nonempty random open sets $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ of $\mathcal{X}$ for a $\mathcal{RDS}$ and for every $\omega\in\Omega$ define the hitting time sets:

$$n_{\varphi}(\mathcal{U}(\omega),\mathcal{V}(\omega))=\{n\in\mathbb{N}:\varphi(n,\theta_{-n}\omega)\mathcal{U}(\omega)\bigcap\mathcal{V}(\omega)\neq\varnothing\}$$

$$\mathcal{N}_{\varphi}(\mathcal{U}(\omega),\mathcal{V}(\omega))=\{n\in\mathbb{Z}:\varphi(n,\theta_{-n},\omega)\mathcal{U}(\omega)\bigcap\mathcal{V}(\omega)\neq\varnothing\}$$

Definition (2.9):

A $\mathcal{X}$ is said to be random hausdorff, if any two distinct points of $\mathcal{X}$ always lie in disjoint open random sets .

Theorem (2.10):

Let (θ,φ) be a transitive $\mathcal{RDS}$ on $\mathcal{X}$, where $\mathcal{X}$ is a Hausdorff space is a weakly dense in itself. Then for every two random open subsets $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ of $\mathcal{X}$ the set $\mathcal{N}_{\varphi}(\mathcal{U}(\omega),\mathcal{V}(\omega))$ is an infinite .

Proof:

Let $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ be two random open subsets of $\mathcal{X}$ and let $n\in\mathbb{N}$. We will show that the set $\mathcal{N}_{\varphi}(\mathcal{U}(\omega),\mathcal{V}(\omega))$ has n elements. First note that there exist n pair-wise disjoint nonempty random subsets $\mathcal{V}_1(\omega)$, $\mathcal{V}_2(\omega)$, …, $\mathcal{V}_n(\omega)$ of $\mathcal{V}(\omega)$. Indeed, as $\mathcal{X}$ is Hausdorff space and dense in itself, there exist disjoint random open subsets $\mathcal{V}_1(\omega)$ and $\mathcal{H}_1(\omega)$ of $\mathcal{V}(\omega)$. Similarly, there exists a nonempty disjoint random open subsets $\mathcal{V}_2(\omega)$ and $\mathcal{H}_2(\omega)$ of $\mathcal{H}_1(\omega)$ and nonempty disjoint random open subsets $\mathcal{V}_3(\omega)$ and $\mathcal{H}_3(\omega)$ of $\mathcal{H}_2(\omega)$ and so on.

Let $\mathcal{U}_1(\omega)=\mathcal{U}(\omega)$. From transitivity of (θ,φ), there exist k $\in\mathbb{N}$ such that $\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}_1(\omega)\cap\mathcal{V}_1(\omega)\neq\varnothing$

Let $\mathcal{U}_2(\omega)=\varphi(k_2,\theta_{-k_2}\omega)\mathcal{U}_1(\omega)\cap\mathcal{V}_1(\omega)\neq\varnothing$

As $\mathcal{V}_1(\omega)\bigcap\mathcal{V}_2(\omega)=\varnothing$, we have $k_1\neq k_2$.

Let $\mathcal{U}_3(\omega) = \varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega) \cap \mathcal{V}_2(\omega) \neq \varnothing$ and there exists $k_3 \in \mathbb{Z}$ such that:

$$\mathcal{U}_3(\omega) = \varphi(k_3, \theta_{-k_3}\omega)\mathcal{U}_3(\omega) \cap \mathcal{V}_3(\omega) \neq \varnothing$$

As $\mathcal{V}_1(\omega) \bigcap \mathcal{V}_3(\omega) = \varnothing$, and $\mathcal{V}_2(\omega) \bigcap \mathcal{V}_3(\omega) = \varnothing$, we have $k_1 \neq k_3$ and $k_2 \neq k_3$.

Continuing is this way, we obtain n distinct elements:

$$k_1, k_2, \ldots, k_n \in \mathcal{N}_\varphi(\mathcal{U}(\omega), \mathcal{V}(\omega))$$

Thus, the set contains at most k^2 elements, which gives the required contradiction.

Definition (2.11):

A $\mathcal{RDS}(\theta, \varphi)$ on $\mathcal{X}$ has a point with weakly dense orbit, if for every $\S_0 \in \mathcal{X}$ and for every a non-empty open random $\mathcal{U}(\omega)$ subsets of, there is $k \in \mathbb{N}$ such that

$$\varphi(-k, \omega)\S_0 \in \mathcal{U}(\omega)$$

Definition (2.12): isolated random point

Let $\mathcal{D}$ a non-empty random open subset of , then $x \in \mathcal{D}$ is said to be an isolated random point of $\mathcal{D}$, if there exists an open random set containing x which does not contain any point of $\mathcal{D}$ different from x .

In other words, a point $x \in \mathcal{D}$ is said to be an isolated random point of , if there exists an random open set $\mathcal{U}$ containing x such that $\mathcal{D} \cap \mathcal{U} = \{x\}$ (opposite of this,$\mathcal{X}$ has no isolated random points)

Theorem (2.13):

Let (θ, φ) be $\mathcal{RDS}$ on $\mathcal{X}$, if $\mathcal{X}$ has no isolated points and (θ, φ) has a point with weakly dense orbit then (θ, φ) is transitive .

Proof:

Let $\mathcal{X}$ has no isolated points and (θ, φ) has a point with dense orbit

$\S_0 \in \mathcal{X}$ such that $\gamma_{\S_0}^{t(\omega)} := \bigcup_{\mathcal{K} \geq t} \varphi(k, \theta_{-k}\omega)\S_0(\theta_{-k}\omega)$ is weakly dense in $\mathcal{X}$

Let $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ be two non-empty open random subsets of $\mathcal{X}$

Since $\mathcal{U}(\omega)$ be a non-empty open random set, there is $k \in \mathbb{N}$ such that

$$\varphi(-k, \omega)\S_0 \in \mathcal{U}(\omega)$$

$$\Rightarrow \S_0(\omega) \in \varphi(k, \theta_{-k}\omega)\mathcal{U}(\omega)$$

Now consider the set

$\mathcal{G}(\omega) = \mathcal{V}(\omega)/\{\S_0, \S_1, \S_2, \ldots, \S_n\}$ where $\S_j(\omega) = \varphi(j, \theta_{-j}\omega)\ \S_0$

Then $\mathcal{G}(\omega)$ is non-empty open random set, there is $k \in \mathbb{N}$ such that

$$\mathcal{G}(\omega) \cap \gamma_{\S}^{t(\omega)} \neq \varnothing$$

There exists $\dagger \in \mathcal{G}(\omega) \wedge \dagger \in \gamma_{\S}^{t(\omega)}$

$$\Rightarrow \dagger \in \mathcal{G}(\omega) \wedge \dagger \in \left\{ \bigcup_{\mathcal{K} \geq t} \varphi(k, \theta_{-k}\omega)\S_0(\theta_{-k}\omega) \right\}$$

$$\Rightarrow \dagger \in \mathcal{G}(\omega) \wedge \exists k \in \mathbb{N} \text{ such that } \dagger = \varphi(k, \theta_{-k}\omega)\S_0(\theta_{-k}\omega)$$

$$\Rightarrow \exists k \in \mathbb{N} \text{ such that } \varphi(-k, \omega)\dagger = \S_0(\omega)$$

$$\Rightarrow \S_0(\omega) \in \mathcal{G}(\omega)$$

$$\Rightarrow \S_0(\omega) \in \varphi(k, \theta_{-k}\omega)\mathcal{U}(\omega) \cap \mathcal{G}(\omega)$$

$$\Rightarrow (k, \theta_{-k}\omega)\mathcal{U}(\omega) \cap \mathcal{G}(\omega) \neq \varnothing$$

$\Rightarrow (\theta, \varphi)$ is transitive .

Theorem (2.14):

Let (θ, φ) be $\mathcal{RDS}$ on $\mathcal{X}$, if for each a non-empty open random $\mathcal{U}(\omega)$ subset of $\mathcal{X}$, there is $k \in \mathbb{N}$ such that $\bigcup_{k \geq t} \varphi(k, \theta_{-k}\omega)\ \mathcal{U}(\omega) = \mathcal{X}$ Then (θ, φ) is transitive .

Proof:

Let $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ be two non-empty open random subsets of $\mathcal{X}$

Then there is $k \in \mathbb{N}$ such that $\bigcup_{k \geq t} \varphi(k, \theta_{-k}\omega)\ \mathcal{U}(\omega) = \mathcal{X}$

Since $\mathcal{V}(\omega)$ subset of $\mathcal{X}$ then there is $\mathcal{Z}(\omega) \in \mathcal{U}(\omega)$ such that

$\varphi(k, \theta_{-k}\omega)\ \mathcal{Z}(\omega) \in \mathcal{V}(\omega)$ for some k

Then $\varphi(k, \theta_{-k}\omega)\mathcal{U}(\omega) \cap \mathcal{V}(\omega) \neq \varnothing$

$\Rightarrow (\theta, \varphi)$ is transitive .

Theorem (2.15):

Let (θ, φ) be a continuous $\mathcal{RDS}$ on $\mathcal{X}$ then (θ, φ) is transitive if and only if for each random variables $\S$, $\dagger \in \mathcal{X}$ and for each $\epsilon(\omega) > 0$ there exists Random variable $\ddagger \in \mathcal{X}$ and $k \in \mathbb{N}$ such that $d(\S, \ddagger) < \epsilon(\omega)$ and

$$d(\varphi(k, \theta_{-k}\omega)\ddagger, \dagger) < \epsilon(\omega)$$

Proof: ($\Leftarrow$)

Let $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ be two non-empty open random subsets of $\mathcal{X}$

And $\in \mathcal{U}(\omega)$, $\dagger \in \mathcal{V}(\omega)$

then there exist $\epsilon_1(\omega) > 0$ such that $\mathcal{N}(\S,\ \epsilon_1(\theta_k\omega)) \subseteq \mathcal{U}(\omega)$, there exist $\epsilon_2(\omega) > 0$ such that $\mathcal{N}(\dagger,\ \epsilon_2(\theta_k\omega)) \subseteq \mathcal{V}(\omega)$

let $\epsilon(\omega) = \{\epsilon_1(\omega), \epsilon_2(\omega)\}$

then $\mathcal{N}(\S,\ \epsilon(\theta_k\omega)) \subseteq \mathcal{U}(\omega)$ and $\mathcal{N}(\dagger,\ \epsilon(\theta_k\omega)) \subseteq \mathcal{V}(\omega)$

then there exists random variable $\ddagger \in \mathcal{X}$

and $k \in \mathbb{N}$ such that $d(\S, \ddagger) < \epsilon(\omega)$ and $d(\varphi(k, \theta_{-k}\omega)\ddagger, \dagger) < \epsilon(\omega)$

then $\in \mathcal{U}(\omega)$, $\varphi(k, \theta_{-k}\omega)\ddagger \in \mathcal{V}(\omega)$

Then $\varphi(k, \theta_{-k}\omega)\mathcal{V}(\omega) \cap \mathcal{U}(\omega) \neq \varnothing$

then (θ, φ) is transitive .

($\Rightarrow$) Let $\S, \dagger \in \mathcal{X}$, $\epsilon(\omega) > 0$ and

let $\mathcal{U}(\omega) = \mathcal{N}(\S,\ \epsilon(\omega))$, $\mathcal{V}(\omega) = \mathcal{N}(\dagger,\ \epsilon(\omega))$

since (θ, φ) is transitive, then there exist $k \in \mathbb{N}$ such that

$\varphi(k,\theta_{-k}\omega)\mathcal{V}(\omega)\cap\mathcal{U}(\omega)\neq\varnothing$

Then there exist $\ddagger\in\mathcal{U}(\omega)$ such that $\varphi(k,\theta_{-k}\omega)\ddagger\in\mathcal{V}(\omega)$

Therefor $d(\S,\ddagger)<\epsilon(\omega)$ and $d(\varphi(k,\theta_{-k}\omega)\ddagger,\dagger)<\epsilon(\omega)$.

Definition (3.1):

A $\mathcal{X}$ is said to be Separable if it has a countable weakly dense subset .

Definition (3.2):

A $\mathcal{X}$ is said to be nowhere weakly dense if it has an empty interior (or $\mathcal{X}$ no has a point with weakly dense orbit)

Definition (3.3):

A $\mathcal{X}$ which can be written as a countable union of nowhere weakly dense sets is called a first category space .

Definition (3.4):

If $\mathcal{X}$ is not first category space then it is a second category space .

Theorem (3.5):

Let (θ,φ) be $\mathcal{RDS}$ on $\mathcal{X}$ if $\mathcal{X}$ is separable and second category and (θ,φ) is transitive, then (θ,φ) has a point with weakly dense orbit .

Proof:

Let (θ,φ) be $\mathcal{RDS}$ on $\mathcal{X}$ if $\mathcal{X}$ is separable and second category and (θ,φ) is transitive

Assume that there is no point with weakly dense orbit

Let $\{\mathcal{V}_n(\omega)\}_{n=1}^{\infty}$ be a countable base .

Then for every $x\in\mathcal{X}$ there exist $\mathcal{U}_x(\omega)$ a non-empty random open set such that

$$\bigcup_{\mathcal{K}\geq t}\varphi(k,\theta_{-k}\omega)x(\theta_{-k}\omega)\cap\mathcal{U}_x(\omega)=\varnothing,$$

Then there exist $\mathcal{V}_x(\omega)$ subset $\mathcal{U}_x(\omega)$ such that

$$\bigcup_{\mathcal{K}\geq t}\varphi(k,\theta_{-k}\omega)x(\theta_{-k}\omega)\cap\mathcal{V}_x(\omega)=\varnothing,$$

Since (θ,φ) is transitive then by theorem (2.1.3) part (4)

We have $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}_x(\omega)\}$is weakly dense in $\mathcal{X}$,

Define $\mathcal{A}_{n(x)}(\omega) = \mathcal{X}/\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}_x(\omega)\}$

And because $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}_x(\omega)\}$ is open random set and it should meet every open random set in $\mathcal{X}$ (since (θ,φ) is transitive)

Then $\mathcal{A}_{n(x)}(\omega)$ is closed random set, nowhere weakly dense and $x\in\mathcal{A}_{n(x)}(\omega)$

Then $\mathcal{X}=\bigcup_{n(x)=1}^{\infty}\mathcal{A}_{n(x)}(\omega)$,a countable union of nowhere weakly dense sets

Then $\mathcal{X}$ is first category space. This is a contradiction to the fact that $\mathcal{X}$ is second category . Hence (θ,φ) has a point with weakly dense orbit .

References

[1] Barzanouni À, Shah E. Chain transitivity for maps on G-spaces. Matematicki Vesnik Matematiqki Vesnik 2019;71(4): 326−37. December 2019.

[2] Gorodnik A. Dynamical systems and ergodic theory. University of Bristol; 2017/18.

[3] Kameyama A. Topological transitivity and strong transitivity. Acta Math Univ Comen 2002;LXXI(2). pp. 139−14.

[4] Akin E, Auslander J, Nagar A. Variations on the concept of topological transitivity. The City College, New York City, NY 10031, USA: Mathematics Department; 2016.

[5] Acosta G, Illanes A, Mendez-Lango H. The transitivity of induced maps. Mexico: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, México D.F.; 2008, 04510.

[6] Cairns G, kolganova A, Nielsen A. Topological transitivity and mixing notions for group actions. Rocky Mt J Math 2007; 37. Number 2.

[7] Chueshov I. Monotone random systems theory and applications. Springer-Verlag Berlin Heidelberg Germany; 2002.

[8] Jabbar Kadhim I, Kadhim Jebur S. Study of some dynamical concepts in general topological spaces. College of Computer Science Information Technology, University of Al-Qadiysia; 2016.

[9] Molchanov I. Theory of random sets. London: Springer; 2005.

[10] de Vries J. Topological dynamical systems, retired from CWI. Centrum voor Wiskunde & Informatica; 2014.

[11] Alongi JM. Gail S Nelson recurrence and Topology. American Mathematical Society Providence, Rhode Island; 2007.

[12] Arnold L. Random dynamical systems. Springer-Verlag Berlin Heidelberg Germany; 1998.

[13] Das R, Garg M. Average chain transitivity and the AL most average shadowing property. Commun Korean Math 2016;(1):201−14

[14] Ash RB. Probability and measure theory. New Yourk: Academic Press; 2000.