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Faez N. Ghaffoori
*Department of Mathematics, College of Basic Education, Mustansiriyah University Baghdad -Iraq*, fng_2022@uomustansiriyah.edu.iqhttp

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# On Solvability of the Integro-Differential Equations

**Author Name**
a.Faez N. Ghaffoori

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**ABSTRACT**

In this paper, we study the existence of solution to integro-differential equations in the space of Lebesgue-integrable $L_1(R^+)$ on un-bounded interval after transforming them to nonlinear integral functional equation. The used tool is the fixed point theorem due to Schauder with weak measure of non compactness, and due to De-Blasi. In addition, we give an example which satisfies the conditions of our existence theorem.

## 1. Introduction

The integral equations is one of most important branch of mathematics because it is used for solving of many problems in many fields in applied mathematics, and physics, as well as this in models dealing with integral equations advanced, since last decades many authors have studied the integral equations, see for instance [11,12,15].

The following, we give a short note for the history of some integral equations that studied before and related to the considered integral equation in this work.

First, in 1992, G. Emmanuelle investigated the functional integral equation [11]:

$$x(t) = f_1\left(t, \int_0^1 k(t,s) f_2\left(s, x(s)\right) ds\right), \quad t \in [0,1],$$

and proved the existence theorem for this equation in the class $L_1[0,1]$.

Later, in 2008, investigated on the functional integral-equation due to W. G. El-Sayed [10]:

$$x(v) = f_1\left(v, \int_0^v k(v,u) f_2\left(u, x(\emptyset(u))\right) du\right), \quad v > 0,$$

and proved the existence theorem for this equation in the class $L_1(R^+)$.

[a] Department of Mathematics, College of Basic Education, Mustansiriyah University
Baghdad - Iraq, E-Mail: fng_2022@uomustansiriyah.edu.iq

In 2013, investigated on the quadratic integral functional equations due to Mohamed M. A. Metwali [4,5]:

$$x(t) = g\left(t, x\big(\varphi_3(t)\big)\right) + f_1\left(t, f_2\left(t, x\big(\varphi_2(t)\big)\right) \int_0^t u\left(t, s, x\big(\varphi_1(t)\big)\right) ds\right), \ t \epsilon R^+.$$

and proved the existence theorem for this equation in the class $L_1(R^+)$.

In this paper, we will investigate the solvability of the integro-differential equation of the type:

$$x(t) = q(t) + \int_0^t p(t,s)\, f\big(s, x'(s)\big) ds \ , \qquad t > 0$$

where $t \in L_1(R^+)$, which corresponds to the nonlinear integral functional equation Volterra type :

$$x(t) = g(t)\, f(t, x(t)) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds, \ t \in L_1(R^+)\ , \ t > 0.$$

This research contains the following sections: In section two, we state some basic definitions and theorems regarding the integro-differential equations theory. Section three is devoted to prove the main existence results. In the fourth section, we present an example that satisfied the conditions of the existence theorem. In the last section, we state some conclusions based on the obtained results.

## 2. Preliminaries

In this section, we state some basic concepts, definitions and theorems that we shall need in the next section to prove the main results.

Let $I \subset R$ is a fixed bounded (unbounded) interval. Denote by S $= S(I)$ the set of all Lebesgue measurable-functions acting from I into R. Let us furnish this set with the metric $\rho s(x,y) = \inf\ [a + meas\{s : |x(s) - y(s)| \geq a\} : a > 0]$, mean A stands for the Lebesgue-measure of set A. Then $S(I)$ becomes a complete M. sp.

(metric space) if we identify functions which are equal almost everywhere (a.e. in short) on $I$. In addition, it is well known that convergence in measure on the interval $I$ coincides with convergence generated by the metric $\rho s$ So, let $L^1 = L^1(I)$

denote the of Lebesgue integrable space functions in the interval $I$, normed in the standard way : $\|x\| = \|x\|_{L^1} = \int_I |x(t)|\, dt$. Compactness in the space $S(I)$ the space

$L_1(I)$ is said to be **Lebesgue space**. And when $I = R^+$ we can write $L_1$ by $L_1(R^+)$, [6].

**Theorem 2.1** [2].

The super-position operator $F$ generated by a function $f$ maps continuously the space $L^1(I)$ into itself if and only if $|f(t,x)| \leq a(t) + b|x|$ for all $t \in I$ and $x \in R$, where $a(t)$ is a function the from $L^1(I)$ and $b$ is a non-negative constant.

**Definition 2.1** (Linear integral operator) [13].
Consider the following linear integral operator:

$$(Kx)(t) = \int_a^b k(x,t) x(s) ds \ , \qquad t \in (a,b),$$

where $x$ is a function and $k(x,t)$ is the kernel. Obviously instead of an arbitrary bounded interval $(a,b)$ we will consider the interval $[0,\infty)$, simplicity.
Assume that $k : R^+ \times R^+ \rightarrow$ R is **measurable** (w.r.t.) both variables.
**Theorem 2.2** (Lusin theorem) [14] .

If $m : I \to R$ be a measure function. $\forall\, \varepsilon > 0\ \exists$ a closed sub set $D_\varepsilon$ of the interval $I$ s.t. meas$(D_\varepsilon^c) \le \varepsilon$ and $m|_{D_\varepsilon}$ is continuous.

**Theorem 2.3** (Dragoni ) [7].

Let $A$ is a compact **metric space**, $B$ a separable **metric space** and $C$ a **Banach space**. If $H : A \times B \to C$ is a function satisfies caratheodory conditions, then for every $\varepsilon > 0$, there exists a measurable closed sub set $D_\varepsilon$ of the interval $A$ such that meas$(A/ D_\varepsilon) < \varepsilon$ as $H|_{D_\varepsilon \times B}$ is **continuous**.

**Definition 2.2** ( Weak measure of noncompactness ) [4].

A function $\mu : \mathfrak{M}_E \to R^+$, is called a measure of weak non compactness if it hold the following conditions:

1. $\gamma = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ *is* nonempty and ker $\gamma \subset \aleph_E^\omega$
2. $X \subset Y \implies \mu(X) \le \mu(Y)$,
3. $\mu(Conv\, X) = \mu(X)$,
4. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y), \lambda \in [0,1]$,
5. if $X_n \in \mathfrak{M}_E, X_n = \overline{X}_n^\omega$ and $X_{n+1} \subset X_n$ $for$ $n = 1,2,\ldots\ldots$ and if

$$\lim_{n\to\infty} \gamma(X_n) = 0, \qquad \text{then } X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varphi$$

**Definition2.3** [8].

Let $X$ be a non-empty and bounded sub-set of $E$, then the **Hausdorff** measure of non-compactness $\mathcal{X}(X)$ is define as :

$\mathcal{X}(X) = \inf\, [\, r > 0 :$ there exist a finite su bset $Y$ of $E$ s.t. $X \subset Y + B_r]$

On the other hand, the 1st. important example of a measure of weak non- compactness has been defined by **De-Blasi** [9]:

$\beta(X) = \inf\{\, r > 0 :$ there exist a weakly compact subset $W$ of $E$ s.t. $X \subset W + B_r\}$ Both the Hausdorff measure $\mathcal{X}$ and the De Blasi measure $\beta$ are sures regular in the sense of the above accepted definitions. These measures play a significant role in several branches of mathematical analysis and find numerous applications [8].

**Definition 2.4** ( Banach space ).

If every Cauchy in a Normed Space $(X, \|.\|)$ converges to an element of that same

space $X$ then that Normed Space $(X, \|.\|)$ is said to be complete in the metric induced by the norm. A complete Normed Space $(X, \|.\|)$ is called a **Banach Space**.

**Definition 2.5** ( Convex set ) [13].

A set $S \subset R$ is said to be convex if : $\forall\, \lambda \in [0,1]$ and $\forall\, x, y \in S$, $\lambda x + (1-\lambda)y \in S$.

If $x, y \in R$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y$ is said to be a **convex combination** of x and y. Simply says that S is a **Convex set** if any **Convex Combination** of every two element of S is also in S.

**Theorem 2.4** (Dieudonne' theorem ) [5].

A bounded set $X \subset L^1$ is relatively weakly compact if and only if

(i) $\forall \varepsilon > 0, \exists\, \delta > 0$ s.t. if meas $D \leq \delta$ then: $\int_D |x(t)|dt \leq \varepsilon,\ x \in X,$

(ii) $\forall \varepsilon > 0,\ \exists\, T > 0$, such that $\int_T^{\infty} |x(t)|dt \leq \varepsilon, \forall\, x \in X,$

Now, for a non- empty and bounded sub set $X$ of $L^1$ define:

$$c(X) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left[\int_D |x(t)|\, dt, D \subset R^+, meas\ D \leq \varepsilon\right]\right\}\right\} \quad (2.1)$$

$$d(X) = \lim_{T\to\infty}\left\{\sup\left[\int_T^{\infty} |x(t)|dt \leq \varepsilon,\ \ x \in X,\right]\right\} \quad (2.2)$$

Further, let us put : $\gamma(X) = c(X) + d(X)$. (2.3)

**Theorem 2.5** [3].

The function $\gamma(X)$is a **regular measure** of weak non compactness in the Space $L^1$ such that $\beta(X) \leq \gamma(X) \leq 2\beta(X)$, where $\beta$ denotes the De-Blasi measure in the space $L^1$. Moreover, $\gamma(B) = 2$.

**Theorem 2.6** [1].

Let $X$ be a non-empty, bounded and compact in measure sub set of $L^1$. Then $\chi(X) \leq \gamma(X) \leq 2\chi(X)$, where $\chi$ denotes Hausdorff measure of noncompactness.

**Definition 2.6**

Let $X$ be a subset of a Banach space $E$. A mapping $T: X \subset E \to X$ has a fixed point if there is an $x \in X$ such that $T(x) = x$.

**Theorem 2.7** [1].
Let $X$ be a convex sub set of a Banach space $E$ and $T : X \to X$ is compact, continuous map. Then $T$ has at least one fixed point in $X$.

## 3. Main Results

We take the following integro-differential Equation:

$$x(t) = q(t) + \int_0^t p(t,s)\, f\left(s, x'(s)\right)ds \quad (3.1)$$

By differentiating both sides of equation (3.1) w. r. t. $t$ yields that

$$x'(t) = q'(t) + p(t,t)\, f\left(t, x'(t)\right) + \int_0^t \frac{\partial p}{\partial t}(t,s)\, f\left(s, x'(s)\right)ds$$

Put $y(t) = x'(t)$, and $q'(t) = h(t)$, and $\frac{\partial p}{\partial t}(t,s) = k(t,s)$, and $p(t,t) = g(t)$
Then we get: $y(t) = g(t)\, f(t, y(t)) + h(t) + \int_0^t k(t,s)\, f\left(s, y(s)\right)ds$ (3.2)

Now, we will investigate the solvability of the nonlinear integral functional equation with Volterra (3.2), where $t \in L_1(R^+)$.

We will define the Operator $\boldsymbol{H}$ associated with integral equation (3.2) take the following form.

$$\boldsymbol{Hx} = \boldsymbol{Ax} + \boldsymbol{Bx} \quad (3.3)$$

where

$(Ax)(t) = g(t)\, f\big(t, x(t)\big)$ ,

$(Bx)(t) = h(t) + \int_0^t k(t,s)\, f\,\big(s, x(s)\big) ds$

$= h(t) + KFx,$ where $(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds,$

$Fx = f(t,x),$ are L.O. at super position respect tively.

We shall treat the equation (3.3) under the following conditions

1) g : $R^+ \to R$ is bounded function such that : $M = \sup_{t\in R^+} |g(t)|$, and

$h : R^+ \to R$, such that $h \in L_1(R^+)$.

2) $f$:$R^+ \to R$ satisfies the caratheodory conditions and there are positive function $a \in L_1$ and constant $b \geq 0$ such that : $|f(t,x)| \leq a(t) + b|x|$.

3) $k$: $R^+ \times R^+ \to R$ satisfies caratheodory conditions such that the linear ope rator K defined as

$$(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds, \qquad t > 0 \tag{3.4}$$

maps the space $L_1$ into itself ( note that due to this assumptions the linear operator K will be continuous whose norm $\|K\|$).

4) q = $bM + b\|K\| < 1$.

**Theorem 3.1:** Let the conditions (1)-(4) are holds, then (3.3) has at least one Integrable solution on $L_1(R^+)$.

**Proof :** $1^{st}$. T.P. $H$ maps continuously $L_1(R^+) \to L_1(R^+)$.

$$\int_0^\infty |(Hx)(t)| dt = \int_0^\infty |\, g(t)\, f\big(t, x(t)\big) + h(t) \int_0^t k(t,s)\, f\,\big(s, x(s)\big) ds$$

$$\leq \int_0^\infty \left|g(t)\, f\big(t, x(t)\big)\right| dt + \int_0^\infty \left| h(t) + \int_0^t k(t,s)\, f\,\big(s, x(s)\big) ds \right| dt$$

$$\leq \int_0^\infty |g(t)| [a(t) + b|x(t)|] dt + \int_0^\infty |h(t)| dt + \|K\| [a(s) + b|x(s)|] ds$$

$$\leq M\|a\| + bM \int_0^\infty |x(t)| dt + \|h\| + \|K\|\|a\| + b\|K\| \int_0^\infty |x(s)| ds$$

$$\leq [M + \|K\|]\, \|a\| + \|h\| + [bM + b\|K\|\,] \int_0^\infty |x(t)| dt < \infty$$

Then : $H : L_1(R^+) \to L_1(R^+)$ continuously.

Now, let $x$ be an arbitrary function in the ball $B_r \subset L_1(R^+)$. In view of our cond- itions, we get

$$\|Hx\| = \int_0^\infty \left| g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t k(t,s)\, f\,\big(s, x(s)\big) ds \right| dt$$

$$= \|gF\| + \|h\| + \|KFx\| \leq |g|\, \|F\| + \|h\| + \|K\|\, \|Fx\|$$

$$\leq M \int_0^\infty \left| f\big(t, x(t)\big) \right| dt + \|h\| + \|K\| \int_0^\infty \left| f\big(t, x(t)\big) \right| dt$$

$$\leq \|h\| + M \int_0^\infty [a(t) + b|x(t)|] dt\ \|K\| \int_0^\infty [a(t) + b|x(t)|] dt$$

$$\leq \|h\| + M\|a\| + bM \int_0^\infty |x(t)|] dt + \|K\|\|a\| + b\|K\| \int_0^\infty |x(t)|] dt$$

$\leq \|h\| + [M + \|K\|]\|a\| + [bM + b\|K\|]\|x\|$

Then : $\|Hx\| \leq \|h\| + [M + \|K\|]\|a\| + [bM + b\|K\|]r \leq r$

So, $H$ transform $B_r$ into $B_r$, where $r \leq \frac{\|h\| + [M + \|K\|]\|a\|}{1 - [bM + b\|K\|]}$, using condition (4), we get $r > 0$.
Next, T. P. that $\beta(HX) \leq q\beta(X)$, ∀ bounded sub set $X$ of $B_r$. Let $\varepsilon > 0$,

and a set $D \subset R^+$ s. t. meas $(D) \leq \varepsilon$. For any $x \in X$, we get

$$\int_D |(Hx)(t)|dt = \int_D | g(t) f(t, x(t)) + h(t) + \int_0^t k(t,s) f(s, x(s))ds | dt$$

$$\leq \int_D |g(t) f(t, x(t))|dt + \int_0^\infty \left|h(t) + \int_0^t k(t,s) f(s, x(s))ds\right| dt$$

$$\leq \int_D |g(t)||f(t, x(t))|dt + \int_D |h(t)|dt + \|KFx\|_{L_{1(D)}}$$

$$\leq M \int_D [a(t) + b|x(t)|]dt + \int_D |h(t)|dt + \|K\|_{L_{1(D)}} \int_D |f(t, x(s))|ds \leq$$

$$M \int_D a(t)dt + bM \int_D |x(t)|dt + \int_D |h(t)|dt + \|K\|_{L_{1(D)}} \int_D a(s)ds + \mathrm{b}\, \|K\|_{L_{1(D)}} \int_D |x(t)|dt$$

$$\leq M \int_D a(t)dt + \int_D |h(t)|dt + [bM + b\, \|K\|_{L_{1(D)}}] \int_D |x(t)|dt,$$ where

$\|K\|_{L_{1(D)}}$ denotes the norm of the operator $K : L_{1(D)} \to L_{1(D)}$. Hence

$\lim_{\varepsilon \to 0}\{\sup\{\int_D |h(t)|dt : D \subset R^+,\ \text{meas(D)} \leq \varepsilon\}\} =$

$= \lim_{\varepsilon \to 0} \sup [\int_D a(t)dt : D \subset R^+,\ \text{meas(D)} \leq \varepsilon\}] = 0$. We get

$$c(HX) \leq [q = bM + b\, \|K\|_{L_{1(D)}}]\, c(X) \quad (3.5)$$

From $T > 0$, any $x \in X$, we have

$$\int_T^\infty |(Hx)(t)|dt = \int_T^\infty | g(t) f(t, x(t)) + h(t) + \int_0^t k(t,s) f(s, x(s))ds | dt$$

$$\leq \int_0^\infty |g(t) f(t, x(t))|dt + \int_T^\infty \left|h(t) + \int_0^t k(t,s) f(s, x(s))ds\right| dt$$

$$\leq \int_T^\infty |g(t)||f(t, x(t))|dt + \int_T^\infty |h(t)|dt + \int_T^\infty \left| \int_0^t k(t,s) f(s, x(s))ds\right| dt$$

$$\leq M \int_T^\infty [a(t) + b|x(t)|]dt + \|h\| + \|\mathrm{K}\| \int_{\mathrm{T}}^\infty [\mathrm{a(s)} + \mathrm{b}|\mathrm{x(s)}|]\mathrm{ds}$$

$$\leq M\, \|a\| + bM\|x\| + \|h\| + \|K\|\|a\| + b\|K\|\|x\| \leq [bM + b\|K\|]\, \|x\|$$

We get : $d(HX) \leq [q = bM + b\|K\|]\, d(X)$ (3.6)
Then we deduce that $\beta(HX) \leq q\beta(X)$. [ because Eq. (3.5), (3.6) and (2.3), and using Th. (2.5)].
Next, let Let $B_r^1 = Conv(HB_r)$, where $Conv(HB_r)$ denotes the closure of the convex $(HB_r^2) \subset B_r^2$, and so on we get a **decreasing sequence** $(B_r^n)$ of bounded, convex, close of $HB_r$ . Since $HB_r \subset B_r$ , then $B_r^1 \subset B_r$
Similarly, let $B_r^2 = Conv(HB_r^1)$ , then $B_r^2 \subset B_r^1$, also
$B_r^3 = Conv$ d subsets of $B_r$ such that $H(B_r^n) \subset B_r^n$, $n \in N$. And by using the properties due to **De-Blasi** measure $\beta$ of weak non compactness, then
$\beta(B_r^{n+1}) = \beta(ConvHB_r^n) = \beta(HB_r^n) \leq q\beta(B_r^n)$, $n \geq N$, and so on, we have $\beta(B_r^{n+1}) \leq q^n\beta(B_r^n)$, $q < 1$, $n \geq N$. Hence, as $n \to \infty$, we get $\lim_{n\to\infty} \beta(B_r^{n+1}) = 0$.

So, $Y = \bigcap_{n \in N} B_r^n$ is a non empty, **closed, bounded, convex** and relatively-weakly compact sub set of $B_r$, $H(Y) \subset Y$.
In the sequel, T. P. that : $H(Y)$ is **relatively-compact** in the space $L_1(R^+)$. Now let $\{y_n\}$ be a se-quence in $Y$ and $\varepsilon > 0$ , Then by Dragoni Th. (2.3), there exists a closed measure-able sub set $D_\varepsilon$ of $[0,t]$, s.t : $m\left(\frac{[0,t]}{D_\varepsilon}\right) < \varepsilon$. And $f$ and $k|_{D_{\varepsilon \times R}}$ are continuous. Let us take a$|_{D_{\varepsilon \times R}}$ rbitrary $t_1, t_2 \in D_\varepsilon$ and assume $t_1 < t_2$.
For an arbitrary fixed $n \in N$. Let $U_n(t) = \int_0^t k(t,s)\, f\left(s, y_n(s)\right) ds$. Then, we have

$$|U_n(t_2) - U_n(t_1)| = \left| \int_0^{t_2} k(t_2,s)\, f\left(s, y_n(s)\right) ds - \int_0^{t_1} k(t_1,s) f\left(s, y_n(s)\right) ds \right|$$

$$= \left| \int_0^{t_2} k(t_2,s)\, f\left(s, y_n(s)\right) ds - \int_0^{t_2} k(t_1,s) f\left(s, y_n(s)\right) ds + \int_0^{t_2} k(t_1,s) f\left(s, y_n(s)\right) ds - \int_0^{t_1} k(t_1,s) f\left(s, y_n(s)\right) ds \right|$$

$$\leq \left| \int_0^{t_2} k(t_2,s)\, f\left(s, y_n(s)\right) ds - \int_0^{t_2} k(t_1,s) f\left(s, y_n(s)\right) ds \right| + \left| \int_0^{t_2} k(t_1,s) f\left(s, y_n(s)\right) ds - \int_0^{t_1} k(t_1,s) f\left(s, y_n(s)\right) ds \right| \leq$$

$$\int_0^{t_2} |k(t_2,s) - k(t_1,s)|\, \left| f\left(s, y_n(s)\right)\right| ds + \int_{t_1}^{t_2} |k(t_1,s)|\, \left| f\left(s, y_n(s)\right)\right| ds$$

$$\leq \int_0^{t_2} |k(t_2,s) - k(t_1,s)|\, [a(t) + b|y_n(s)|] ds + \int_{t_1}^{t_2} |k(t_1,s)|\, [a(t) + b|y_n(s)|]\, ds$$

But $t_2 - t_1$ is small enough, then $(t_2 - t_1) \to 0$ .
Now, since $\{y_n\} \subset Y$ and $Y$ is **bounded**, then $\{y_n\}$ is **bounded.**
Hence $(U_n)$ is a sequence of equi-continuous and uniformly **bounded function** in $C(D_\varepsilon)$. And so $\{B(y_n)\}$ is a sequence of equi-continuous and uniformly bounded-function in $(D_\varepsilon)$ . So $\{A(y_n)\}$ is a sequence of equi-continuous and uniformly **bounded-function** in $C(D_\varepsilon)$. Hence $\{H(y_n)\}$ uniformly bounded in $C(D_\varepsilon)$, then
$\{H(y_n)\}$ is relativelycompact in $C(D_\varepsilon)$. From which, we deduce that
$\{H(y_n)\}$ is Cauchy sequence in $C(D_\varepsilon)$.
Now , since $\{H(y_n)\}$ is Cauchy sequence in $L_1$. Then by Dieudonne' theorem (2.4) and $H(Y)$ is relatively compact in $C(D_\varepsilon)$, we deduce that for every $\sigma > 0$, there is $\delta > 0$ such that $\sup_y \int_{D_{i\delta}} |(Hy)(\mathrm{t})| dt < \frac{\sigma}{4.2^i}$. For meas( $D_{i\delta}) < \delta$, $D_{i\delta} \subset [i-1, i]$, $i = 1, \ldots, n$. Choose for each $i$, $i = 1, \ldots, n$, $r_i^* \epsilon N$ with $m([i-1,i]/D_{r_i^*}) < \delta$,

then for $n_1, n_2 \in N$, we have

$$\int_0^\infty \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt = \lim_{n\to\infty} \sum_{i=1}^n \int_{i-1}^i \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt =$$

$$\lim_{n\to\infty} \sum_{i=1}^n \int_{[i-1,i]/D_{r_i^*}} \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt + \int_{D_{r_i^*}} \left|(Hy_{n_1})(t) - (Hy_{n_2})(t)\right| dt$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{2.\sigma}{4.2^i} + \left\|Hy_{n_1} - Hy_{n_2}\right\|_{c(D_{r_i^*})}\right) \leq \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{\sigma}{2.2^i} + \frac{\sigma}{2.2^i}\right) = \sum_{i=1}^\infty \frac{\sigma}{2^i} = \sigma .$$

For large value of $n_1, n_2$ we deduce that $\{H(y_n)\}$ is Cauchy sequence in $L_1$.
Since $L_1$ is complete space. Then $H(y_n)$ is relatively compact in $L_1$. Finally, we can use Schauder fixed point theorem to get a fixed point for our operator $H$.
So the functional integral equation (3.1) is solvable in $L_1$.

## 4. Example on The Obtained Results

Consider the integral equation:

$$x(t) = 3e^{-t}\left[\ln\left(\frac{1}{1+t^2}\right) + e^{-2|x|}\right] + te^{-2t} + \int_0^t e^{s-t}\left[\ln\left(\frac{1}{1+t^2}\right)e^{-2|x|}\right]ds$$

$g(t) = 3e^{-t}$, $f(t,x(t)) = \left[\ln\left(\frac{1}{1+t^2}\right) + e^{-2|x|}\right]$, $h(t) = te^{-2t}$, $k(t,s) = e^{s-t}$

We will hold the assumptions of our existence Theorem 3.1.

1) Since $\sup_{t\in R^+}|g(t)| = \sup_{t\in R^+} 3e^{-t} = 3 = M$. Next, to prove that $h \in L_1(R^+)$,

we have $\int_0^\infty h(t)\,dt = \int_0^\infty te^{-2t}\,dt = \frac{-te^{-2t}}{2} - \frac{e^{-2t}}{4}\Big|_0^\infty = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < \infty$

2) $|f(t,x(t))| = \left|\ln\left(\frac{1}{1+t^2}\right) + e^{-2|x|}\right| \leq \left|\ln\left(\frac{1}{1+t^2}\right)\right| + |e^{-2|x|}| \leq \frac{1}{1+t^2} + 2|x|$

$a(t) = \frac{1}{1+t^2}, \int_0^\infty \frac{1}{1+t^2}\,dt = tan^{-1}t\,\Big|_0^\infty = \frac{\pi}{2} < \infty$, then $a(t) \in L_1(R^+)$, $b = 2 > 0$

3) $k(t,s) = e^{s-t}$ is continuous function both variables

$(Kx)(t) = \int_0^t e^{s-t}x(s)\,ds$, $||Kx|| \leq \int_0^\infty \int_0^t e^{s-t}|x(s)|\,ds\,dt$

$= \int_0^\infty \int_{t=s}^t e^{s-t}|x(s)|\,dt\,ds = \int_0^\infty -e^{s-t}|x(s)|\,|_s^\infty ds = \int_0^\infty |x(s)|\,ds = ||x||$

so, $||Kx|| < ||x|| \to ||K|| = 1$

4) $[M + ||K||] = 3 + ||K||, so, \frac{1}{5}[M + ||K||] = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} < 1,\ b = \frac{1}{5}$

Since, all conditions of Theorem (3.1) are hold and so our integral equation (3.2) is solvable in $L_1(R^+)$.

## 5. Conclusions

In this paper, we proved the existence to the solution of integro-differential equations of the type : $x(t) = q(t) + \int_0^t p(t,s)\,f(s,x'(s))ds$ , which can be transformed to nonlinear integral functional equations with Volterra type :

$$x(t) = g(t)\,f(t,x(t)) + h(t) + \int_0^t k(t,s)\,f(s,x(s))ds, \quad t \in L_1(R^+).$$

In the space of Lebesgue integrable functions on the unbounded interval $R^+ = [0,\infty)$. The main using tools are Schauder fixed point theorem and weak measure of non-compactness, due to De-Blasi. Also, we give an example which satisfies the conditions of our existence theorem. Based on the obtained results, we observe that the existence can be guaranteed for a large class of Integro-differential equations. However, the uniqueness results for such types of problems is still an open problem.

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