ON CoLC Topologies

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<th>ABSTRACT</th>
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<td>a. Reyadh Delfi Ali</td>
<td>The purpose of this work is to continue the study of Co $LC$ topologies and offer new characteristics of CoLC topologies and examine their relationships with other classes of topological spaces.</td>
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<td>b. Omsad Adgheem Ali</td>
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1. Introduction:

Gauld, M., R., and Vamanamurthy [8] introduced the Co-Lindelöf topology

$$l(\psi) = \{ \emptyset \} \bigcup \{ O \in \psi : S - O \text{ is Lindelöf in } (S, \psi) \},$$

They showed that $l(\psi)$ is a topology on $S$ with $l(\psi) \subseteq \psi$.

A. Kanibir and P. Grgino [13] introduced the Co $LC$ topology

$$lc(\psi) = \{ \emptyset \} \bigcup \{ O \in \psi : S - O \text{ is } LC - \text{subspace in } (S, \psi) \}$$

is a topology on $S$ with $lc(\psi) \subseteq \psi$.

In this paper is to study the basic properties of the Co $LC$ topology.
2. Preliminaries

Definition 2.1: \((S,\psi)\) is an LC-topological space if every Lindelöf \(F \subseteq S\) is closed [8], [16].

Also known \(L-closed\) [9], [10], [12] and [14].

Definition 2.2[24]: A topological space \((S,\psi)\) is a KC-space if every compact \(K \subseteq S\) is closed.

Definition 2.3[6]: \((S,\psi)\) is cid-space if every countable subset of \(S\) is closed and discrete.

Definition 2.4 [15]:

A topological space \((S,\psi)\) is called \(P\)-space if every \(G_\sigma\)-open set in \(S\) is open.

Definition 2.5[1]:

A topological space \((S,\psi)\) is a \(Q\)-set space if if each subset of \(S\) is an \(F_\sigma\)-closed set.

Definition 2.6 [2]: A topological space \((S,\psi)\) is a weak \(P\)-space if every countable union of regular closed sets is closed.

Definition 2.7: A topological space \((S,\psi)\) is almost Lindelöf if for every open cover \(\omega\) of \(S\) there exists a countable subfamily a countable subfamily \(\beta \subseteq \omega\) such that \(S = \bigcup_{\omega \in \beta} \overline{Y}\). From the definition that every Lindelöf space is almost Lindelöf [5], [23].

Definition 2.8 [22]:

A topological space \((S,\psi)\) is \(ALC\)-space if every subset of \(S\) which is almost Lindelöf in \(S\) is closed.

Definition 2.9: A topological space \((S,\psi)\) is weakly \(ALC\)-space \(WALC\)-space if every almost Lindelöf subset of \(S\) is closed[11]. Clearly, every \(ALC\)-space is an \(WALC\)-space and every \(WALC\)-space is an \(LC\)-space.
Definition 2.10[7]: A topological space \((S,\psi)\) is called a Locally LC - space if each point of \(S\) has a neighborhood which is an LC - subspace. Clearly every LC - space is locally LC - space. In general the converse needs not be true [4], however every regular locally LC - space is LC - space.

Definition 2.11[3]: A topological space \((S,\psi)\) is a \(R_1\) - space if \(y\) and \(z\) have disjoint neighborhoods whenever \(cl\{y\} \neq cl\{z\}\). Clearly a space is Hausdorff if and only if its \(T_1\) and \(R_1\).

Definition 2.12 [1]: A topological space \((S,\psi)\) is said to be anti - Lindelöf if each Lindelöf subset of \(M\) is countable.

Definition 2.13 [2]:

(1) If \(\forall\) Lindelöf \(F_\sigma - closed\) is closed. A space \(S\) is an \(L_1\).

(2) If \(H \subseteq S\) is Lindelöf, then \(clH\) is Lindelöf. A space \(S\) is an \(L_2\).

(3) If \(\forall\) Lindelöf \(H \subseteq S\) is an \(F_\sigma - closed\). A space \(S\) is an \(L_3\).

Theorem 2.14 [2]:

(i) If \((S,\psi)\) is an LC - space, then \((S,\psi)\) is a \(L_i\) - space, \(i=1,2,3\).

(ii) If \((S,\psi)\) is an \(L_1\) - space and an \(L_3\) - space, then \((S,\psi)\) is an LC - space.

(iii) Every \(P\) - space is an \(L_1\) - space. (v) Every \(L_3\) - space is \(T_1\).

Corollary 2.15 [18]:

If \((S,\psi)\) is an LC - topological space then \((S,\ell(\psi))\) is a \(P\) - topological space.

Corollary 2.16 [18]: Every \(P\) - space \((S,\psi)\) is weak \(P\) - space.

Theorem 2.17 [2]:

For a space \((M,\Gamma)\) the following are equivalent:

(a) \((S,\psi)\) is an \(L_i\) - topological space. (b) \((S,\ell(\psi))\) is a \(P\) - topological space.
**Corollary 2.18[19]:** Every \( Q \)-set \( L_1 \)-space is an \( LC \)-space.

**Theorem 2.19 [2]:** Every \( T_1 \), anti- Lindelöf \( L_1 \)-space is an \( LC \)-space.

**Corollary 2.20 [18]:** If \((S, \psi)\) is Lindelöf space then \( l(\psi) = \psi \).

**Theorem 2.21 [17]:** A regular almost Lindelöf space is Lindelöf.

**Theorem 2.22[19]:** Every locally compact \( LC \)-space is a Hausdorff.

**Corollary 2.23[19]:**

(i) Every \( R_1 \) \( LC \)-space is a Hausdorff. (ii) Every regular \( LC \)-space is a Hausdorff.

**Theorem 2.24[15]:** Every Hausdorff \( P \)-space is an \( LC \)-space.

**Theorem 2.25[2]:** For a Hausdorff space \( S \) the following are equivalent:

(a) \( S \) is an \( LC \)-space. \hspace{1cm} (b) \( S \) is an \( L_1 \)-space and an \( L_2 \)-space.

**Theorem 2.26 [19]:** For a Hausdorff Lindelöf space \( S \) the following are equivalent:

(a) \( S \) is an \( LC \)-space. \hspace{1cm} (b) \( S \) is an \( L_1 \)-space.

**Corollary 2.27 [11]:** For a Hausdorff Lindelöf space \( S \), the following are equivalent:

(a) \( S \) is an \( ALC \)-space. \hspace{1cm} (b) \( S \) is an \( WALC \)-space. \hspace{1cm} (c) \( S \) is \( LC \)-space. \hspace{1cm} (d) \( S \) is a \( P \)-space. \hspace{1cm} (e) \( S \) is a weak \( P \)-space.

**Theorem 2.28 [20]:** For a regular space \( S \) the following are equivalent:

(a) \( S \) is an \( LC \)-space. \hspace{1cm} (b) \( S \) is an \( WALC \)-space.

**Theorem 2.29 [19]:** If \((S, \psi)\) is a topological space and \( N \subseteq S \), \( N = \bigcup_{i=1}^{n} N_i \), where \( N_i \), \( i = 1, 2, \ldots, n \) are closed and \( LC \)-subspaces in \( S \), so \( N \) is \( LC \)-space.

**Proposition 2.30 [13]:** If \((S, \psi)\) is an \( LC \)-space, then \( \psi = lc(\psi) \).
Corollary 2.31 [13]:

Let \((S, \psi)\) be a topological space.

If \((S, lc(\psi))\) has no Lindelöf dense subset then \((S, lc(\psi))\) is an LC-space.

Theorem 2.32 [13]: For a space \((S, \psi)\) the following are equivalent:

(a) \((S, \psi)\) is an LC-space.

(b) \((S, lc(\psi))\) is an LC-space.

Proposition 2.33 [13]:

If \((S, lc(\psi))\) is disconnected, then \((S, \psi)\) is an LC-space.

Corollary 2.34 [13]:

If \((S, lc(\psi))\) is disconnected, then \((S, lc(\psi))\) is an LC-space.

Theorem 2.35 [13]: Let \((M, \Gamma)\) be a topological space.

Then \((S, \psi)\) is an LC-space and disconnected if and only if \((S, lc(\psi))\) is disconnected.

Proposition 2.36 [13]: If \((S, \psi)\) is \(T_1\), then \((S, lc(\psi))\) is \(T_1\).

Proposition 2.37 [13]:

(i) If \((S, \psi)\) is an LC-space, then \(l(\psi) \subseteq lc(\psi)\).

(ii) If \((S, \psi)\) is a Lindelöf space, then \(lc(\psi) \subseteq l(\psi)\).

Note that the reverse inclusions of (i) and (ii) are false in general [11].

Example 2.38: If \((S, \psi)\) is the discrete topological space and \(S\) is an uncountable set, so \(\psi = lc(\psi)\) (\(S\) is an LC-space and Proposition 2.30). Since \(\psi \neq l(\psi)\), hence \(lc(\psi) \subset l(\psi)\).

Example 2.39: If \((\mathbb{R}, \Omega)\) is the usual topology, \(\mathbb{N} = \{0\} \cup \{\frac{1}{j} : j = 1, 2, 3, 4, 5, \ldots\}\) and if \(\psi = \Omega_N\).

\(\Theta(\mathbb{N}, \Omega_N)\) is compact, it is Lindelöf. Therefore \(\Omega_N = l(\Omega_N)\). Take \(\frac{1}{2} \in \Omega_N\) and \(B = \left\{\frac{1}{3}, \frac{1}{4}, \ldots\right\}\). Then \(B \subseteq L = \mathbb{N} - \left\{\frac{1}{2}\right\}\) is a Lindelöf subset of \((\mathbb{L}, \Omega_L)\), and not closed in \((\mathbb{L}, \Omega_L)\). Therefore \(L\) is not an LC-subspace of \((\mathbb{N}, \Omega_N)\). So \(\left\{\frac{1}{2}\right\} \notin lc(\Omega_N)\). Hence \(l(\Omega_N) \subset lc(\Omega_N)\).

Corollary 2.40 [2]: Every Hausdorff locally Lindelöf \(L_1\)-space is an LC-space.

Theorem 2.41 [19]: For anti-Lindelöf space \(S\) the following are equivalent:
(a) $S$ is an $LC$–space .

(b) $S$ is a $cid$–space.

**Theorem 2.42 [21]:** Every Hausdorff $P$–space is an $WALC$–space.

### 3. CoLC Topologies

**Corollary 3.1:** (i) If $(S, lc(\psi))$ is an $LC$–space then $(S, l(\psi))$ is a $P$–space.

(ii) If $(S, lc(\psi))$ is an $LC$–space then $(S, l(\psi))$ is a weak $P$–space.

**Proof.** Obvious by Theorem 2.32, Corollary 2.15 and Corollary 2.16.

**Theorem 3.2:** For a $Q$–set space $(S, \psi)$ the following are equivalent:

(a) $(S, lc(\psi))$ is an $LC$–space .

(b) $(S, l(\psi))$ is a $P$–space.

**Proof.**

$\Longleftrightarrow$: Obvious by Corollary 3.1(i).

$\Rightarrow$: If $(S, l(\psi))$ is a $P$–space, so $(S, \psi)$ is an $L_1$–space by Theorem 2.17. Since $(S, \psi)$ is a $Q$–set, therefore $(S, \psi)$ is an $LC$–space (Corollary 2.18). Thus $(S, lc(\psi))$ is an $LC$–space (Theorem 2.32).

**Theorem 3.3:** For an anti-Lindelöf $T_1$ space $(S, \psi)$ the following are equivalent:

(a) $(S, \psi)$ is an $LC$–space .

(b) $(S, lc(\psi))$ is an $LC$–space .

(c) $(S, l(\psi))$ is a $P$–space .

**Proof.**

(a) $\Rightarrow$ (b): Obvious (Theorem 2.32). (b) $\Rightarrow$ (c): This is obvious (Corollary 3.1(i)).

(c) $\Rightarrow$: If $(S, l(\psi))$ is a $P$–space, so $(S, \psi)$ is an $L_1$–space (Theorem 2.17). Since $(S, \psi)$ is an anti-Lindelöf $T_1$ space, therefore $(S, \psi)$ is an $LC$–space (Theorem 2.19). Thus $(S, lc(\psi))$ is an $LC$–space (Theorem 3.1).
**Theorem 3.4:** For a locally Lindelöf Hausdorff space \((S, \psi)\) the following are equivalent:

(a) \((S, \text{lc}(\psi))\) is an LC – space.
(b) \((S, \text{l}(\psi))\) is a P – space.

**Proof.** \(\Leftarrow: \) Obvious (Corollary 3.1(i)). \(\Rightarrow: \) If \((S, \text{l}(\psi))\) is a P – space, so \((S, \psi)\) is an \(L_1\) – space (Theorem 2.17). Since \((S, \psi)\) is a locally Lindelöf Hausdorff space, therefore \((S, \psi)\) is an LC – space (Corollary 2.40). Thus \((S, \text{lc}(\psi))\) is an LC – space (Theorem 2.32).

**Corollary 3.5:** (i) If \((S, \text{lc}(\psi))\) is Hausdorff, then \((S, \text{l}(\psi))\) is a P – space.
(ii) If \((S, \text{lc}(\psi))\) is Hausdorff, then \((S, \text{l}(\psi))\) is a weak P – space.

**Proof.** Obvious.

**Corollary 3.6:** (i) If \((S, \text{lc}(\psi))\) is disconnected, then \((S, \text{l}(\psi))\) is a P – space.
(ii) If \((S, \text{lc}(\psi))\) is disconnected, then \((S, \text{l}(\psi))\) is a weak P – space.

**Proof.** Obvious.

**Corollary 3.7:**

(i) If \((S, \psi)\) is an LC – space, then \((S, \text{lc}(\psi))\) is a \(T_1\) – space.
(ii) If \((S, \psi)\) is a KC – space, then \((S, \text{lc}(\psi))\) is a \(T_1\) – space.
(iii) If \((S, \psi)\) is locally LC – space, then \((S, \text{lc}(\psi))\) is a \(T_1\) – space.
(iv) If \((S, \psi)\) is infinite cid – space, then \((S, \text{lc}(\psi))\) is a \(T_1\) – space.

**Proof.** Obvious by Theorem 2.36.

**Corollary 3.8:** If \((S, \psi)\) is a Lindelöf LC – space, then \(\psi = \text{lc}(\psi) = \text{l}(\psi)\).

**Proof.** Obvious by Proposition 2.30 and Corollary 2.20.

**Corollary 3.9:**

(i) If \((S, \psi)\) is a LC Lindelöf space, then \((S, \text{lc}(\psi))\) is a P – space.
(ii) If \((S, \psi)\) is a LC Lindelöf space, then \((S, \text{l}(\psi))\) is a weak P – space.
(iii) If \((S, \psi)\) is a LC Lindelöf space, then \((S, \text{l}(\psi))\) is a P – space.
(iv) If \((S,\psi)\) is an LC Lindelöf space, then \((S, l(\psi))\) is a weak \(P - \text{space}\).

**Corollary 3.10:** If \((S,\psi)\) is anti-Lindelöf \(cid - \text{space}\), then \((S, l(\psi))\) is an LC-\(space\).

**Proof.** Since \((S,\psi)\) is anti-Lindelöf \(cid - \text{space}\), so \((S, l(\psi))\) is an LC-\(space\) (Theorem 2.41), therefore \((S, l(\psi))\) is an LC-\(space\) (Theorem 2.32).

**Corollary 3.11:** If \((S, l(\psi))\) is a Hausdorff, then \((S,\psi)\) is a locally LC-\(space\).

**Proof.** Since \((S, l(\psi))\) is a Hausdorff, then \((S,\psi)\) is an LC-\(space\) (Proposition 2.33). Therefore \((S,\psi)\) is a locally LC.

**Corollary 3.12:**

(i) If \((S, l(\psi))\) is a Hausdorff, then \((S,\psi)\) is \(cid - \text{space}\).

(ii) If \((S, l(\psi))\) is a Hausdorff, then \((S,\psi)\) is a KC-\(space\).

**Corollary 3.13:**

(i) If \((S,\psi)\) is a regular locally LC-\(space\), then \(l(\psi) \subseteq l(\psi)\).

(ii) If \((S,\psi)\) is a regular almost Lindelöf, then \(l(\psi) \subseteq l(\psi)\).

**Proof.** (i) Obvious by Definition 2.01 and Proposition 2.30.

(ii) Obvious by Theorem 2.21 and Corollary 2.20.

**Theorem 3.14:** For a locally compact \((S,\psi)\) the following are equivalent:

(a) \((S,\psi)\) is an LC-\(space\).

(b) \((S, l(\psi))\) is a Hausdorff.

**Proof.** =>: If \((S,\psi)\) is an LC-\(space\), so \(\psi = l(\psi)\) proposition 2.30. Since \((S,\psi)\) is a locally compact, then \((S, l(\psi))\) is a Hausdorff by Theorem 2.22.

<=: Obvious by Proposition 2.33.

**Theorem 3.15:** For a regular \((S,\psi)\) the following are equivalent:

(a) \((S,\psi)\) is an LC-\(space\).

(b) \((S, l(\psi))\) is a Hausdorff.
Proof. $\Rightarrow$: Let $(S, \psi)$ be an LC-space, then $\psi = lic(\psi)$ Proposition 2.30. Since $(S, \psi)$ is a regular, then $(S, lc(\psi))$ is a Hausdorff.

$\Leftarrow$: Obvious by Proposition 2.33.

**Theorem 3.16:** For a $R_1 (S, \psi)$ the following are equivalent:

(a) $(S, \psi)$ is an LC-space. (b) $(S, lc(\psi))$ is a $T_2$.

Proof. $\Leftarrow$: If $(S, \psi)$ is an LC-space, then $\psi = lc(\psi)$ Proposition 2.30. Since $(S, \psi)$ is a $R_1$, then $(S, lc(\psi))$ is a $T$ by Corollary 2.23(i).

$\Rightarrow$: Obvious.

**Corollary 3.17:** If $(S, lc(\psi))$ is Hausdorff, then $(S, \psi)$ is a Hausdorff.

Proof. Since $(S, lc(\psi))$ is Hausdorff, then $(S, \psi)$ is an LC-space by Proposition 2.33, so $\psi = lc(\psi)$. Hence $(S, \psi)$ is a Hausdorff.

**Corollary 3.18:** If $(S, \psi)$ is Hausdorff $P$-space, then $(S, lc(\psi))$ is a Hausdorff $P$-space.

Proof. Since $(S, \psi)$ is Hausdorff $P$-space, hence $(S, \psi)$ is an LC-space by Theorem 2.24, so $\psi = lc(\psi)$. Therefore $(S, lc(\psi))$ is a Hausdorff $P$-space.

4. CoLC and $L_i$-Spaces

**Corollary 4.1:** Let $(S, lc(\psi))$ be an LC-space, then $(S, li(\psi))$ is an $L_i$.

Proof. Obvious (Corollary 3.1 and Theorem 2.14 (iii)).

**Theorem 4.2:** For a $L_3$-space $(S, \psi)$ the following are equivalent:

(a) $(S, li(\psi))$ is a $P$-space. (b) $(S, lc(\psi))$ is an LC.
Proof. $\Leftarrow$: Obvious (Corollary 3.1). $\Rightarrow$: If $\left(S,l\left(\psi\right)\right)$ is a $P$-space, so $\left(S,\psi\right)$ is an $L_1$ (Theorem 2.17). Since $\left(S,\psi\right)$ is an $L_3$, therefore $\left(S,\psi\right)$ is an $LC$ (Theorem 2.14(ii)). Hence $\left(S,lc\left(\psi\right)\right)$ is an $LC$ (Theorem 2.32).

**Theorem 4.3:** For a $L_2$ Hausdorff space $\left(S,\psi\right)$ the following are equivalent:

(a) $\left(S,l\left(\psi\right)\right)$ is a $P$-space.

(b) $\left(S,lc\left(\psi\right)\right)$ is an $LC$.

Proof. $\Leftarrow$: Obvious (Corollary 3.1).

$\Rightarrow$: If $\left(S,l\left(\psi\right)\right)$ is a $P$-space, so $\left(S,\psi\right)$ is an $L_1$-space (Theorem 2.17). Since $\left(S,\psi\right)$ is a $L_2$ Hausdorff space, therefore $\left(S,\psi\right)$ is an $LC$ (Theorem 2.25). Hence $\left(S,lc\left(\psi\right)\right)$ is an $LC$ (Theorem 2.32).

**Theorem 4.4:** For a Lindelöf Hausdorff space $\left(S,\psi\right)$ the following are equivalent:

(a) $\left(S,lc\left(\psi\right)\right)$ is an $LC$-space.

(b) $\left(S,l\left(\psi\right)\right)$ is a $P$-space.

(c) $\left(S,l\left(\psi\right)\right)$ is a weak $P$-space.

(d) $\left(S,\psi\right)$ is an $L_1$-space.

Proof.

(a) $\Rightarrow$(b): Obvious (Corollary 3.1).

(b) $\Rightarrow$(a): If $\left(S,l\left(\psi\right)\right)$ is a $P$-space, so $\left(S,\psi\right)$ is an $L_1$ (Theorem 2.17). Since $\left(S,\psi\right)$ is a Lindelöf Hausdorff space, therefore $\left(S,\psi\right)$ is an $LC$ (Theorem 2.26). Hence $\left(S,l\left(\psi\right)\right)$ is an $LC$ (Theorem 2.32).

(b) $\Rightarrow$(c): Obvious (Corollary 2.16). $(c) \Rightarrow(b)$: If $\left(S,l\left(\psi\right)\right)$ is a weak $P$-space. Since $\left(S,\psi\right)$ is a Hausdorff Lindelöf and $\psi = l\left(\psi\right)$, then $\left(S,\psi\right)$ is an $L_1$ (Theorem 2.27 and Theorem 2.14 (i)). Hence $\left(S,l\left(\psi\right)\right)$ is a $P$-space (Theorem 2.17).
(c) $\Rightarrow$ (d): If $(S,l(\psi))$ is a weak $P$-space, since $(S,\psi)$ is a Lindelöf, then $\psi = l(\psi)$, so $(S,l(\psi))$ is a $P$-space (Theorem 2.27). Hence $(S,l(\psi))$ is an $L_1$ (Theorem 2.14 (viii)).

(d) $\Rightarrow$ (c): Obvious (Theorem 3.2.2 and Corollary 3.1.26).

**Corollary 4.5:** If $(S,lc(\psi))$ is Hausdorff, then $(S,l(\psi))$ is an $L_1$-space.

**Proof.** Obvious (Proposition 2.33, Corollary 2.15 and Theorem 2.14 (iii)).

**Corollary 4.6:** If $(S,lc(\psi))$ is disconnected, then $(S,l(\psi))$ is an $L_1$-space.

**Proof.** Obvious (Theorem 2.35, Corollary 2.15 and Theorem 2.14 (iii)).

**Corollary 4.7:** (i) If $(S,\psi)$ is an $L_3$-space, then $(S,lc(\psi))$ is a $T_1$-space.

(ii) If $(S,\psi)$ is a $Q$-set space, then $(S,lc(\psi))$ is a $T_1$-space.

**Proof.** Obvious (Proposition 2.36).

**Theorem 4.8:** For a locally compact $L_3$-space $(S,\psi)$ the following are equivalent:

(a) $(S,l(\psi))$ is a $P$-space.

(b) $(S,lc(\psi))$ is a Hausdorff.

**Proof.** $\Leftarrow$: Obvious (Corollary 3.5(i)).

$\Rightarrow$: If $(S,l(\psi))$ is a $P$-space, then $(S,\psi)$ is an $L_1$-space (Theorem 2.17). Since $(S,\psi)$ is an $L_3$-space, therefore $(S,\psi)$ is an $LC$-space (Theorem 2.14(ii)). So $\psi = lc(\psi)$. Since $(S,\psi)$ is a locally compact space, hence $(S,\psi) = (S,lc(\psi))$ is a Hausdorff (Theorem 2.22).

**Theorem 4.9:** For a regular $L_3$-space $(S,\psi)$ the following are equivalent:

(a) $(S,l(\psi))$ is a $P$-space.

(b) $(S,lc(\psi))$ is a Hausdorff.
Proof. $\Rightarrow$: Obvious (Corollary 3.5(i)). $\Leftarrow$: If $(S,l(\psi))$ is a $P$-space, so $(S,\psi)$ is an $L_1$ (Theorem 2.17). Since $(S,\psi)$ is an $L_3$, therefore $(S,\psi)$ is an LC (Theorem 2.14(ii)), so $\psi = lc(\psi)$. Since $(S,\psi)$ is a regular, Thus $(S,\psi) = (S,lc(\psi))$ is a Hausdorff by Corollary 2.23(ii).

**Theorem 4.10:** For a $R_1 L_3 - space (S,\psi)$ the following are equivalent:

(a) $(S,l(\psi))$ is a $P$-space.
(b) $(S,lc(\psi))$ is a Hausdorff

Proof. $\Leftarrow$: Obvious by Corollary 3.5(i).

$\Rightarrow$: If $(S,l(\psi))$ is a $P$-space, so $(S,\psi)$ is an $L_1$ (Theorem 2.17). Since $(S,\psi)$ is an $L_3$, therefore $(S,\psi)$ is an LC (Theorem 2.14(ii)), so $\psi = lc(\psi)$. Since $(S,\psi)$ is a $R_1$-space, hence $(S,\psi) = (S,lc(\psi))$ is a Hausdorff by Corollary 2.23(i).

**Corollary 4.11:** Let $(S,\psi)$ be a $Q$-set space.

If $(S,l(\psi))$ is a $P$-space, then $(S,lc(\psi))$ is $T_1$.

Proof. Since $(S,l(\psi))$ is a $P$-space, so $(S,\psi)$ is an $L_1$-space by Theorem 2.17. Since $(S,\psi)$ is a $Q$-set, therefore $(S,\psi)$ is an LC (Corollary 2.18), then $(S,lc(\psi))$ is $T_1$ (Proposition 2.36).

**Corollary 4.12:** Let $(S,\psi)$ be a $L_3$-space topological space.

If $(S,l(\psi))$ is a $P$-space, then $(S,lc(\psi))$ is $T_1$.

Proof. Since $(S,l(\psi))$ is a $P$-space, so $(S,\psi)$ is an $L_1$-space by Theorem 2.17. Since $(S,\psi)$ is an $L_3$, therefore $(S,\psi)$ is an LC (Theorem 2.14(ii)), thus $(S,lc(\psi))$ is $T_1$ (Proposition 2.36).

**Corollary 4.13:** (i) If $(S,\psi)$ is a Linelof LC-space, then $(S,lc(\psi))$ is an $L_1$-space.
(ii) If $(S,\psi)$ is a Lindelof LC-space, then $(S,l(\psi))$ is an $L_1$-space.

**Corollary 4.14:** If $(S,lc(\psi))$ has no Lindelof-dense subset, then $(S,lc(\psi))$ is $T_1$. 
**Proof.** Since \((S, lc(\psi))\) has dense - no Lindelöf subset, therefore \((S, lc(\psi))\)is an \(L_3\) (Corollary2.31 and Theorem 2.14(i)), hence \((S, lc(\psi))\)is \(T_1\) (Theorem 2.14(v)).

**Theorem 4.15:** For a locally compact \(Q – set\) space \((S, \psi)\) the following are equivalent:

(a) \((S, lc(\psi))\) is a \(P – space\).

(b) \((S, lc(\psi))\) is a Hausdorff.

**Proof:** \(\Rightarrow\): If \((S, lc(\psi))\) is a \(P – space\), so \((S, \psi)\) is an \(L_i\) (Theorem 2.17). Since \((S, \psi)\) is a \(Q – set\), therefore \((S, \psi)\) is an \(LC\) (Corollary 2.18) and \(\psi = lc(\psi)\). Since \((S, \psi)\) is a locally compact, hence \((S, lc(\psi))\) is a Hausdorff by Theorem 2.22.

\(\Leftarrow\): Obvious (Corollary 3.5(i)).

**Theorem 4.16:** For a \(R_i\ \(Q – set\) space \((S, \psi)\) the following are equivalent:

(a) \((S, l(\psi))\) is a \(P – space\).

(b) \((S, lc(\psi))\) is a Hausdorff.

**Proof:** \(\Rightarrow\): If \((S, l(\psi))\) is a \(P – space\), so \((S, \psi)\) is an \(L_i\) (Theorem 2.17). Since \((S, \psi)\) is a \(Q – set\), therefore \((S, \psi)\) is an \(LC\) (Corollary 2.18) and \(\psi = lc(\psi)\). Since \((S, \psi)\) is a \(R\). Hence \((S, lc(\psi))\) is a Hausdorff (Corollary 2.23(i)).

\(\Leftarrow\): Obvious (Corollary 3.5(i)).

**Theorem 4.17:** For a regular \(Q – set\) space \((S, \psi)\) the following are equivalent:

(a) \((S, l(\psi))\) is a \(P – space\).

(b) \((S, lc(\psi))\) is a Hausdorff.

**Proof.** \(\Rightarrow\): If \((S, l(\psi))\) is a \(P – space\), so \((S, \psi)\) is an \(L_i\) (Theorem 2.17). Since \((S, \psi)\) is a \(Q – set\), therefore \((S, \psi)\) is an \(LC\) (Corollary 2.18) and \(\psi = lc(\psi)\). Since \((S, \psi)\) is a regular. Hence \((S, lc(\psi))\) is a Hausdorff (Corollary 2.23(i)). \(\Leftarrow\): Obvious (Corollary 8.1.20(i)).

5. CoLC and WALC

**Corollary 5.1:** (i) If \((S, \psi)\) is an \(WALC – space\), then \((S, lc(\psi))\) is a \(T_1 – space\).
(ii) If \((S, \psi)\) is an ALC - space, then \((S, lc(\psi))\) is a \(T_1\) - space.

**Proof.** Obvious ) Definition 2.9 and Proposition 2.36.

**Theorem 5.2:** Let \((S, \psi)\) be a regular topological space. If \((S, lc(\psi))\) has no Lindelöf -dense subset, then \((S, \psi)\) and \((S, lc(\psi))\) are WALC - spaces.

**Proof.** Since \((S, lc(\psi))\) has dense -no Lindelöf subset, so \((S, \psi)\) is an LC ) Corollary 2.31 and Theorem 2.32. Since \((S, \psi)\) is a regular, therefore \((S, \psi)\) is an WALC ) Theorem 2.28. Since \(\psi = lc(\psi)\) then \((S, lc(\psi))\) is an WALC.

**Corollary 5.3:** Let \((S, \psi)\) be a regular topological space. If \((S, lc(\psi))\) is a Hausdorff, then \((S, \psi)\) is an WALC - space.

**Proof.** Since \((S, lc(\psi))\) is a Hausdorff, so \((S, \psi)\) is an LC ) Proposition 2.33. Since \((S, \psi)\) is a regular, then \((S, \psi)\) is an WALC - space Theorem 2.28.

**Theorem 5.4:**

If \((S, \psi)\) is an WALC - space, then \((S, lc(\psi))\) is an WALC - space and an LC - space.

**Proof.** Since \((S, \psi)\) is an WALC, so \((S, \psi)\) is an LC. \(\psi = lc(\psi)\). Hence \((S, lc(\psi))\) is an WALC and an LC.

**Theorem 5.5:** If \((S, lc(\psi))\) is \(T_1\) - space, then \((S, \psi)\) is an WALC - space.

**Proof.** Let \(w, z \in S\) and \(w \neq z\). Since \((S, lc(\psi))\) is Hausdorff, \(\exists O, V \in lc(\psi) \ni w \in O, z \in V\) and \(O \cap V = \emptyset\). Thus \(S = (S - O)Y(S - V)\) and since \(S - O\), \(S - V\) are two closed, LC - subspaces of \((S, \psi)\). Therefore \((S, lc(\psi))\) is an LC (Theorem 2.29). Hence \((S, lc(\psi))\) is an WALC (Theorem 2.28).

**Theorem 5.6:** For a regular space \((S, \psi)\) the following are equivalent:

(a) \((S, \psi)\) is an WALC - space.

(b) \((S, lc(\psi))\) is an LC - space.
Proof. $\Rightarrow$: Obvious (Theorem 5.4). $\Leftarrow$: If $\left( S, l_{c}(\psi) \right)$ is an $LC$, so $\left( S, \psi \right)$ is an $LC$ (Theorem 2.32). Since $\left( S, \psi \right)$ is a regular space, hence $\left( S, \psi \right)$ is an $WALC$ (Theorem 2.28).

**Theorem 5.7:** For a regular space $\left( S, \psi \right)$ the following are equivalent:

(a) $\left( S, \psi \right)$ is an $WALC$ - space.

(b) $\left( S, l_{c}(\psi) \right)$ is an $WALC$ - space.

Proof. $\Rightarrow$: Obvious (Theorem 5.4). $\Leftarrow$: If $\left( S, l_{c}(\psi) \right)$ is an $LC$, so $\left( S, \psi \right)$ is an $LC$ (Theorem 2.32). Since $\left( S, \psi \right)$ is a regular space, hence $\left( S, \psi \right)$ is an $WALC$ (Theorem 2.28).

**Corollary 5.8:** (i) If $\left( S, \psi \right)$ is an $WALC$ - space, then $l(\psi) \subseteq l_{c}(\psi)$.

(ii) If $\left( S, \psi \right)$ is an $ALC$ - space, then $l(\psi) \subseteq l_{c}(\psi)$.

Proof. (i) and (ii) Obvious (Definition 2.9 and Proposition 2.37(i)).

**Theorem 5.9:** Let $\left( S, \psi \right)$ be a regular space. Then $\left( S, l_{c}(\psi) \right)$ is an $WALC$ - space if it can be written as the union of two closed sets which are not equal to $S$.

Proof.

Let $S = K_1 \cup K_2$ and let $K_1 \neq S, K_2 \neq S$ be closed in $\left( S, l_{c}(\psi) \right)$. Then $K_1$ and $K_2$ are LC - subspace of $\left( S, \psi \right)$ by Definition CoLC. Therefore $\left( S, \psi \right)$ is an $LC$ - space (Theorem 2.29), since $K_1$ and $K_2$ are closed in $\left( S, \psi \right)$. So $\left( S, l_{c}(\psi) \right)$ is an $LC$ - space (Theorem 2.32) and $\psi = l_{c}(\psi)$. Since $\left( S, \psi \right)$ is a regular space, then $\left( S, l_{c}(\psi) \right)$ is an $WALC$ - space (Theorem 2.28).

**Corollary 5.10:** Let $\left( S, \psi \right)$ be a regular topological space. If $\left( S, l_{c}(\psi) \right)$ is disconnected, then $\left( S, l(\psi) \right)$ is an $WALC$ - space.

Proof. Obvious (Theorem 5.9).

**Theorem 5.11:** For a regular $Q$ - set space $\left( S, \psi \right)$ the following are equivalent:

(a) $\left( S, \psi \right)$ is an $LC$ - space.

(b) $\left( S, \psi \right)$ is an $WALC$ - space.
(c) \((S,l(\psi))\) is a \(P\)–space. (d) \((S,lc(\psi))\) is a Hausdorff and an \(LC\)–space.

**Proof.**

(a) \(\implies\)(b): Obvious (Theorem 2.28).

(b) \(\implies\)(a): Obvious (Definition 2.9).

(b) \(\implies\)(c): Let \((S,\psi)\) be an \(WALC\)–space, so \((S,\psi)\) is an \(LC\)–space (Theorem 2.28). Therefore \((S,l(\psi))\) is a \(P\)–space (Theorem 2.15).

(c) \(\implies\)(b): If \((S,l(\psi))\) is a \(P\)–space, so \((S,\psi)\) is an \(L_1\) (Theorem 2.17). Since \((S,\psi)\) is a \(Q\)–set, therefore \((S,\psi)\) is an \(LC\) (Corollary 2.18) and \(\psi = lc(\psi)\). Since \((S,\psi)\) is a regular. Hence \((S,lc(\psi))\) is an \(WALC\) (Theorem 2.28).

(c) \(\implies\)(d): If \((S,l(\psi))\) is a \(P\)–space, so \((S,\psi)\) is an \(L_1\) (Theorem 2.17). Since \((S,\psi)\) is a \(Q\)–set, therefore \((S,\psi)\) is an \(LC\)–space (Corollary 2.18) and \(\psi = lc(\psi)\). Since \((S,\psi)\) is a regular. Hence \((S,lc(\psi))\) is a Hausdorff (Corollary 2.23(iii)).

(d) \(\implies\)(c): Obvious (Corollary 3.5).

**Theorem 5.12:** For a regular \(P\)–space \(S\) the following are equivalent:

(a) \((S,\psi)\) is a \(T\)–space (b) \((S,\psi)\) is a Hausdorff space (c) \((S,\psi)\) is an \(WALC\)–space (d) \((S,\psi)\) is an \(LC\)–space (e) \((S,\psi)\) is a Locally \(LC\)–space (f) \((S,\psi)\) is a \(KC\)–space.

**Proof.**

(a) \(\implies\)(b): If \(S\) is a \(T_1\), since \(S\) is a regular, hence \(S\) is a Hausdorff.

(b) \(\implies\)(a): Obvious. (b) \(\implies\)(c): Obvious (Theorem 2.42).
(c) \( \implies \)(b): If \( S \) is an WALC, so \( S \) is an \( T_1 \), since \( S \) is a regular, therefore \( S \) is a Hausdorff.

(c) \( \implies \)(d): Obvious (Definition 2.9).

(d) \( \implies \)(c): Obvious (Theorem 2.28).  
(d) \( \iff \)(e): Obvious (Definition 2.10).

(e) \( \implies \)(f): Let \( S \) be a Locally LC, since \( S \) is a regular, so \( S \) is an LC (Definition 2.10), therefore \( S \) is a KC.

(f) \( \implies \)(e): Let \( S \) be a KC, then \( S \) is an \( T_1 \), since \( S \) is a regular, therefore \( S \) is a Hausdorff, so \( S \) is an LC (Theorem 2.24), therefore \( S \) is a Locally LC.

**Corollary 5.13:** For a regular \( P - space \) \((S, \psi)\) the following are equivalent:

(a) \((S, \psi)\) is a \( T - space \).
(b) \((S, \psi)\) is a Hausdorff space.
(c) \((S, \psi)\) is an WALC - space.

(d) \((S, lc(\psi))\) is an \( LC - space \) and \( T - space \).
(e) \((S, \psi)\) is an LC - space.

(f) \((S, \psi)\) is a Locally LC - space.
(g) \((S, \psi)\) is a KC - space.

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**References**


