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ON CoLC Topologies

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ABSTRACT

The purpose of this work is to continue the study of CoLC topologies and offer new characteristics of CoLC topologies and examine their relationships with other classes of topological spaces.

1. Introduction:

Gauld, M., R, and Vamanamurthy [8] introduced the Co- Lindelöf topology $l(\psi)=\{\phi\}\cup\{O\in\psi : S-O is \quad Lindelof \quad in \quad (S,\psi)\}$.They showed that $l(\psi)$ is a topology on S with $l(\psi)\subseteq\psi$,

A. Kanibir and P. Girgino [13] introduced the CoLC topology $lc(\psi)=\{\phi\}\cup\{O\in\psi : S-O is \quad LC-subspace \quad in \quad (S,\psi)\}$ is a topology on S with $lc(\psi)\subseteq\psi$.

In this paper is to study the basic properties of the CoLC topology .

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2. Preliminaries

Definition2.1: (S,ψ) is an $LC-topological$ $space$ if every Lindelöf $F \subset S$ is closed $[8],[16]$. Also known $L-closed$ $[9]$, $[10]$, $[12]$ and $[14]$.

Definition 2.2[24]: A topological space (S,ψ) is a $KC-space$ if every compact $K \subset S$ is closed .

Definition 2.3[6] :

(S,ψ) is $cid-space$ if every countable subset of S is closed and discrete.

Definition2.4 [15]:

A topological space (S,ψ) is called $P-space$ if every $G_\sigma-open$ set in S is open.

Definition 2.5[1]:

A topological space (S,ψ) is a $Q-set$ space if if each subset of S is an $F_\sigma-closed$ sets.

Definition2.6 [2]: A topological space (S,ψ) is a weak $P-space$ if every countable union of regular closed sets is closed.

Definition2.7: A topological space (S,ψ) is almost Lindelöf if for every open cover ω of S there exists a countable subfamily a countable subfamily $\beta \subset \omega$ such that $S = \bigcup_{O\in\beta} \overline{O}$. From the definition that every Lindelöf space is almost Lindelöf $[5]$, $[23]$.

Definition2.8 [22]:

A topological space (S,ψ) is $ALC-space$ if every subset of S which is almost Lindelöf in S is closed.

Definition2.9: A topological space (S,ψ) is weakly $ALC-space$ $WALC-space$ if every almost Lindelöf subset of S is closed $[11]$ .Clearly, every $ALC-space$ is an $WALC-space$ and every $WALC-space$ is an $LC-space$.

Definition2.10[7]: A topological space (S,ψ) is called a Locally $LC-space$ if each point of S has a neighborhood which is an $LC-subspace$.Clearly every $LC-space$ is locally $LC-space$ .In general the converse needs not be true $[4]$, however every regular locally $LC-space$ is $LC-space$..

Definition2.11[3]: A topological space (S,ψ) is a $R_1-space$ if y and z have disjoint neighborhoods whenever $cl\{y\}\neq cl\{z\}$ .Clearly a space is Hausdorff if and only if its T_1 and R_1.

Definition2.12 [1]: A topological space (S,ψ) is said to be anti – Lindelöf if each Lindelöf subset of M is countable.

Definition2.13 [2] :

(1) If $\forall$ Lindelöf F_σ $-closed$ is closed.A space S is an L_1.

(2) If $H\subseteq S$ is Lindelöf, then clH is Lindelöf. A space S is an L_2.

(3) If $\forall$ Lindelöf $H\subseteq S$ is an F_σ $-closed$. A space S is an L_3

Theorem2.14 [2]:

(i) If (S,ψ) is an $LC-space$, then (S,ψ) is a $L_i-space$, i=1,2,3.

(ii) If (S,ψ) is an $L_1-space$ and an $L_3-space$, then (S,ψ) is an $LC-space$.

(iii) Every $P-space$ is an $L_1-space$. (v) Every $L_3-space$ is T_1.

Corollary2.15 [18]:

If (S,ψ) is an $LC-topological\quad space$ then $(S,l(\psi))$ is a $P-topological\quad space$.

Corollary2.16 [18]: Every $P-space$ (S,ψ) is weak $P-space$.

Theorem2.17 [2]:

For a $space$ (M,Γ) the following are equivalent:

(a) (S,ψ) is an $L_1-topological\quad space$. (b) $(S,l(\psi))$ is a $P-topological\quad space$.

Corollary 2.18[19]: Every $Q-set$ $L_1-space$ is an $LC-space$.

Theorem2.19 [2]: Every T_1, anti- Lindelöf $L_1-space$ is an $LC-space$.

Corollary2.20 [18]: If (S,ψ) is Lindelöf space then $l(\psi)=\psi$.

Theorem2.21 [17]: A regular almost Lindelöf space is Lindelöf.

Theorem 2.22[19]: Every locally compact $LC-space$ is a Hausdorff.

Corollary2.23[19]:

(i) Every $R_1LC-space$ is a Hausdorff. (ii)Every regular $LC-space$ is a Hausdorff.

Theorem 2.24[15]: Every Huasdorff $P-space$ is an $LC-space$.

Theorem 2.25[2]: For a Hausdorff space S the following are equivalent:

(a) S is an $LC-space$. (b) S is an $L_1-space$ and an $L_2-space$.

Theorem2.26 [19]: For a Hausdorff Lindelöf space S the following are equivalent:

(a) S is an $LC-space$. (b) S is an $L_1-space$.

Corollary2.27 [11]: For a Hausdorff Lindelöf space S, the following are equivalent:

(a) S is an $ALC-space$. (b) S is an $WALC-space$. (c) S is $LC-space$.

(d) S is a $P-space$. (e) S is a weak $P-space$.

Theorem2.28 [20]: For a regular space S the following are equivalent:

(a) S is an $LC-space$. (b) S is an $WALC-space$.

Theorem2.29 [19]: If (S,ψ) is a topological space and $N\subseteq S$, $N=\bigcup_{i=1}^{n}N_i$, where N_i , $i=1,2,...,n$ are closed and $LC-subspaces$ in S ,so N is $LC-subspace$.

Proposition2.30 [13]: If (S,ψ) is an $LC-space$, then $\psi=lc(\psi)$.

Corollary2.31 [13]: .

Let (S,ψ) be a topological space.

If $(S,lc(\psi))$ has no Lindelöf -dense subset then $(S,lc(\psi))$ is an $LC-space$.

Theorem 2.32[13]: For a space (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$. (b) $(S,lc(\psi))$ is an $LC-space$.

Proposition2.33 [13]:

If $(S,lc(\psi))$ is T_2, then (S,ψ) is an $LC-space$.

Corollary2.34 [13]:

If $(S,lc(\psi))$ is disconnected, then $(S,lc(\psi))$ is an $LC-space$.

Theorem 2.35[13]: Let (M,Γ) be a topological space.

Then (S,ψ) is an $LC-space$ and disconnected if and only if $(S,lc(\psi))$ is disconnected.

Proposition2.36 [13]: If (S,ψ) is T_1, then $(S,lc(\psi))$ is T_1.

Proposition2.37 [13]:

(i) If (S,ψ) is an $LC-space$, then $l(\psi)\subseteq lc(\psi)$.

(ii) If (S,ψ) is a Lindelöf space, then $lc(\psi)\subseteq l(\psi)$.

Note that the reverse inclusions of (i) and (ii) are false in general [11].

Example2.38: If (S,ψ) is the discrete topological space and S is an uncountable set .So $\psi=lc(\psi)$ (S is an $LC-space$ and Proposition2.30).Since $\psi\neq l(\psi)$. Hence $lc(\psi)\not\subset l(\psi)$.

Example2.39: If (R,Ω) is the usual topology , $\mathrm{N}=\{0\}\cup\left\{\frac{1}{j}: j=1,2,3,4,5...\right\}$ and if $\Psi=\Omega_N$. $\Theta(\mathrm{N},\Omega_N)$ is compact, it is Lindelöf. Therefore $\Omega_N=l(\Omega_N)$. Take $\left\{\frac{1}{2}\right\}\in\Omega_N$ and $\mathrm{B}=\left\{1,\frac{1}{3},\frac{1}{4},...\right\}$.Then $\mathrm{B}\subseteq L=\mathrm{N}-\left\{\frac{1}{2}\right\}$ is a Lindelöf subset of (L,Ω_L),and not closed in (L,Ω_L).Therefore L is not an $LC-subspace$ of (N,Ω_N). So $\left\{\frac{1}{2}\right\}\notin lc(\Omega_N)$.Hence $l(\Omega_N)\not\subset lc(\Omega_N)$.

Corollary2.40 [2]: Every Huasdorff , locally Lindelöf $L_1-space$ is an $LC-space$..

Theorem2.41 [19]: For anti – Lindelöf space S the following are equivalent:

(a) S is an $LC-space$. (b) S is a $cid-space$.

Theorem2.42[21]: Every Huasdorff $P-space$ is an $WALC-space$.

3. CoLC Topologies

Corollary3.1: (i) If $(S,lc(\psi))$ is an $LC-space$ then $(S,l(\psi))$ is a $P-space$.

(ii) If $(S,lc(\psi))$ is an $LC-space$ then $(S,l(\psi))$ is a weak $P-space$.

Proof. Obvious by Theorem 2.32, Corollary. 2.15 and Corollary 2.16.

Theorem3.2: For a $Q-set$ space (S,ψ) the following are equivalent:

(a) $(S,lc(\psi))$ is an $LC-space$. (b) $(S,l(\psi))$ is a $P-space$.

Proof.

$\Leftarrow$: Obvious by Corollary 3.1(i). $\Rightarrow$:If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an $L_1-space$ by Theorem 2,17 Since (S,ψ) is a $Q-set$, therefore (S,ψ) is an $LC-space$ (Corollary 2.18). Thus $(S,lc(\psi))$ is an $LC-space$ (Theorem 2.32).

Theorem3.3: For a anti- Lindelöf T_1 space (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$.(b) $(S,lc(\psi))$ is an $LC-space$.(c) $(S,l(\psi))$ is a $P-space$.

Proof.

(a) $\Rightarrow$(b): Obvious (Theorem 2.32).(b) $\Rightarrow$(c):This is obvious (Corollary3.1(i)).

(c) $\Rightarrow$(b): If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an $L_1-space$ (Theorem 2.17). Since (S,ψ) is anti- Lindelöf T_1 space, therefore (S,ψ) is an $LC-space$ (Theorem 2.19).Thus $(S,lc(\psi))$ is an $LC-space$ (Theorem 3.1).

Theorem3.4: For a locally Lindelöf Hausdorff space (S,ψ) the following are equivalent:

(a) $(S,lc(\psi))$ is an $LC-space$. (b) $(S,l(\psi))$ is a $P-space$.

Proof. $\Leftarrow$: Obvious (Corollary 3.1(i)). $\Rightarrow$: If $(S,l(\psi))$ is a $P-space$, so (S,ψ) is an $L_1-space$ (Theorem 2.17). Since (S,ψ) is a locally Lindelöf Hausdorff space, therefore (S,ψ) is an $LC-space$ (Corollary 2.40).Thus $(S,lc(\psi))$ is an $LC-space$ (Theorem 2.32).

Corollary3.5: (i) If $(S,lc(\psi))$ is Hausdorff, then $(S,l(\psi))$ is a $P-space$.

(ii) If $(S,lc(\psi))$ is Hausdorff, then $(S,l(\psi))$ is a weak $P-space$.

Proof. Obvious .

Corollary3.6: (i) If $(S,lc(\psi))$ is disconnected, then $(S,l(\psi))$ is a $P-space$.

(ii) If $(S,lc(\psi))$ is disconnected, then $(S,l(\psi))$ is a weak $P-space$.

Proof. Obvious .

Corollary3.7:

(i) If (S,ψ) is an $LC-space$, then $(S,lc(\psi))$ is a $T_1-space$.

(ii) If (S,ψ) is a $KC-space$, then $(S,lc(\psi))$ is a $T_1-space$.

(iii) If (S,ψ) is locally $LC-space$, then $(S,lc(\psi))$ is a $T_1-space$.

(iv) If (S,ψ) is infinite $cid-space$, then $(S,lc(\psi))$ is a $T_1-space$.

Proof. Obvious by Theorem 2.36.

Corollary3.8: If (S,ψ) is a Lindelöf $LC-space$, then $\psi=lc(\psi)=l(\psi)$.

Proof. Obvious by Proposition 2.30 and Corollary2.20.

Corollary3.9:

(i) If (S,ψ) is a LC Lindelöf space, then $(S,lc(\psi))$ is a $P-space$.

(ii) If (S,ψ) is a LC Lindelöf space, then $(S,lc(\psi))$ is a weak $P-space$.

(iii) If (S,ψ) is a LC Lindelöf space, then $(S,l(\psi))$ is a $P-space$.

(iv) If (S,ψ) is a LC Lindelöf space, then $(S,l(\psi))$ is a weak $P-space$.

Corollary3.10: If (S,ψ) is anti- Lindelöf $cid-space$, then $(S,lc(\psi))$ is an $LC-space$.

Proof. Since (S,ψ) is anti- Lindelöf $cid-space$, so (S,ψ) is an $LC-space$ (Theorem 2.41), therefore $(S,lc(\psi))$ is an $LC-space$ (Theorem 2.32).

Corollary3.11: If $(S,lc(\psi))$ is a Hausdorff, then (S,ψ) is a locally $LC-space$.

Proof. Since $(S,lc(\psi))$ is a Hausdorff , then (S,ψ) is an $LC-space$ (Proposition 2.33). Therefore (S,ψ) is a locally LC.

Corollary3.12: (i) If $(S,lc(\psi))$ is a Hausdorff, then (S,ψ) is $cid-space$.

(ii) If $(S,lc(\psi))$ is a Hausdorff, then (S,ψ) is a $KC-space$.

Corollary3.13:

(i) If (S,ψ) is a regular locally $LC-space$, then $l(\psi)\subseteq lc(\psi)$.

(ii) If (S,ψ) is a regular almost Lindelöf, then $lc(\psi)\subseteq l(\psi)$.

Proof. (i) Obvious by Definition 2.01 and Proposition2.30.

(ii) Obvious by Theorem 2.21and Corollary 2.20.

Theorem3.14: For a locally compact (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof. $\Rightarrow$: If (S,ψ) is an $LC-space$, so $\psi=lc(\psi)$ proposition 2.30. Since (S,ψ) is a locally compact, then $(S,lc(\psi))$ is a Hausdorff by Theorem 2.22.

$\Leftarrow$: Obvious by Proposition 2.33.

Theorem3.15: For a regular (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof. $\Rightarrow$:Let (S,ψ) be an $LC-space$,then $\psi = lc(\psi)$ Proposition2.30. Since (S,ψ) is a regular, then $(S,lc(\psi))$ is a Hausdorff.

$\Leftarrow$: Obvious by Proposition 2.33.

Theorem3.16: For a R_1 (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$. (b) $(S,lc(\psi))$ is a T_2.

Proof. $\Leftarrow$: If (S,ψ) is an $LC-space$,then $\psi = lc(\psi)$ Proposition2.30. Since (S,ψ) is a R_1, then $(S,lc(\psi))$ is a T_2 by Corollary2.23(i).

$\Rightarrow$: Obvious.

Corollary3.17: If $(S,lc(\psi))$ is Hausdorff, then (S,ψ) is a Hausdorff.

Proof. Since $(S,lc(\psi))$ is Hausdorff , then (S,ψ) is an $LC-space$ by Proposition 2.33, so $\psi = lc(\psi)$. Hence (S,ψ) is a Hausdorff.

Corollary3.18: If (S,ψ) is Hausdorff $P-space$, then $(S,lc(\psi))$ is a Hausdorff $P-space$.

Proof. Since (S,ψ) is Hausdorff $P-space$, hence (S,ψ) is an $LC-space$ by Theorem 2.24,so $\psi = lc(\psi)$. Therefore $(S,lc(\psi))$ is a Hausdorff $P-space$.

4. CoLC and $L_i - Spaces$

Corollary4.1: Let $(S,lc(\psi))$ be an $LC-space$,*then* $(S,l(\psi))$ is an L_1.

Proof. Obvious (Corollary 3.1 and Theorem 2.14 (iii)).

Theorem4.2: For a $L_3-space$ (S,ψ) the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is an LC.

Proof. $\Leftarrow$: Obvious (Corollary 3.1). $\Rightarrow$: If $(S,l(\psi))$ is a $P-space$,so $(S,\psi)$is an L_1 (Theorem 2.17). Since (S,ψ) is an L_3, therefore (S,ψ) is an LC (Theorem 2.14(ii)) . Hence $(S,lc(\psi))$ is an LC (Theorem 2.32).

Theorem4.3: For a L_2 Hausdorff space $(S,\psi)$the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is an LC..

Proof. $\Leftarrow$: Obvious (Corollary 3.1).

$\Rightarrow$: If $(S,l(\psi))$is a $P-space$,so $(S,\psi)$is an $L_1-space$ (Theorem 2.17). Since(S,ψ) is a L_2 Hausdorff space, therefore (S,ψ) is an LC (Theorem 2.25). Hence$(S,lc(\psi))$ is an LC (Theorem 2.32).

Theorem4.4: For a Lindelöf Hausdorff space $(S,\psi)$the following are equivalent:

(a) $(S,lc(\psi))$ is an $LC-space$. (b) $(S,l(\psi))$ is a $P-space$.

(c) $(S,l(\psi))$ is a weak $P-space$. (d) (S,ψ) is an $L_1-space$.

Proof.

(a) $\Rightarrow$(b): Obvious (Corollary 3.1).

(b)$\Rightarrow$(a): If $(S,l(\psi))$ is a $P-space$,so $(S,\psi)$is an L_1(Theorem 2.17). Since (S,ψ) is a Lindelöf Hausdorff space, therefore (S,ψ) is an LC (Theorem 2.26). Hence $(S,l(\psi))$ is an LC (Theorem 2.32).

(b) $\Rightarrow$(c): Obvious (Corollary 2.16).(c)$\Rightarrow$(b):If $(S,l(\psi))$is a weak $P-space$. Since$(S,\psi)$is a Hausdorff Lindelöf and $\psi=l(\psi)$, then (S,ψ) is an L_1 (Theorem 2.27 and Theorem 2.14 (i)). Hence $(S,l(\psi))$is a $P-space$ (Theorem 2.17).

(c) $\Rightarrow$(d): If $(S,l(\psi))$ is a weak $P-space$,since (S,ψ) is a Lindelöf,then $\psi = l(\psi)$, so $(S,l(\psi))$ is a $P-space$ (Theorem 2.27).Hence $(S,l(\psi))$ is an L_1 (Theorem 2.14 (viii)).

(d) $\Rightarrow$(c): Obvious (Theorem 3.2.2 and Corollary 3.1.26).

Corollary4.5: If $(S,lc(\psi))$ is Hausdorff, then $(S,l(\psi))$ is an $L_1-space$.

Proof. Obvious (Proposition2.33,Corollary2.15andTheorem2.14(iii)).

Corollary4.6: If $(S,lc(\psi))$ is disconnected, then $(S,l(\psi))$ is an $L_1-space$.

Proof. Obvious (Theorem2.35,Corollary2.15and Theorem2.14(iii)).

Corollary4.7: (i) If (S,ψ) is an $L_3-space$, then $(S,lc(\psi))$ is a $T_1-space$.

(ii) If (S,ψ) is a $Q-set$ space, then $(S,lc(\psi))$ is a $T_1-space$

Proof. Obvious (Proposition 2.36).

Theorem4.8: For a locally compact $L_3-space$ (S,ψ) the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof. $\Leftarrow$: Oobvious (Corollary 3.5(i)).

$\Rightarrow$: If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an $L_1-space$ (Theorem 2.17).Since (S,ψ) is an $L_3-space$,therefore (S,ψ) is an $LC-space$ (Theorem 2.14(ii)), so $\psi = lc(\psi)$.Since (S,ψ) is a locally compact space, hence $(S,\psi) = (S,lc(\psi))$ is a Hausdorff (Theorem 2.22).

Theorem4.9: For a regular $L_3-space$ (S,ψ) the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof. $\Rightarrow$: Obvious (Corollary 3.5(i)). $\Leftarrow$: If $(S,l(\psi))$ is a $P-space$, so (S,ψ) is an L_1 (Theorem 2.17). Since (S,ψ) is an L_3, therefore (S,ψ) is an LC (Theorem2.14(ii)), so $\psi = lc(\psi)$ .Since (S,ψ) is a regular , Thus $(S,\psi)=(S,lc(\psi))$ is a Hausdorff by Corollary 2.23(ii).

Theorem4.10: For a R_1 $L_3-space$ (S,ψ) the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$.

(b) $(S,lc(\psi))$ is a Hausdorff

Proof. $\Leftarrow$: Obvious by Corollary 3.5(i).

$\Rightarrow$: If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an L_1 (Theorem 2.17). Since (S,ψ) is an L_3, therefore (S,ψ) is an LC (theorem 2.14(ii)),so $\psi = lc(\psi)$.Since (S,ψ) is a $R_1-space$, hence $(S,\psi)=(S,lc(\psi))$ is a Hausdorff by Corollary 2.23(i).

Corollary4.11: Let (S,ψ) be a $Q-set$ space.

If $(S,l(\psi))$ is a $P-space$, then $(S,lc(\psi))$ is T_1.

Proof. Since $(S,l(\psi))$ is a $P-space$, so (S,ψ) is an $L_1-space$ by Theorem 2.17. Since (S,ψ) is a $Q-set$, therefore (S,ψ) is an LC (Corollary 2.18), then $(S,lc(\psi))$ is T_1 (Proposition 2.36).

Corollary4.12: :_Let (S,ψ) be a $L_3-space$ topological space.

If $(S,l(\psi))$ is a $P-space$, then $(S,lc(\psi))$ is T_1.

Proof. Since $(S,l(\psi))$ is a $P-space$, so (S,ψ) is an L_1 (Theorem 2.17). Since (S,ψ) is an L_3, therefore (S,ψ) is an LC (Theorem 2.14(ii)), thus $(S,lc(\psi))$ is T_1 (Proposition 2.36).

Corollary4.13: (i)If (S,ψ) is aLindelöf $LC-space$,then $(S,lc(\psi))$ is an $L_1-space$.

(ii)If (S,ψ) is a Lindelöf $LC-space$, then $(S,l(\psi))$ is an $L_1-space$.

Corollary4.14: If $(S,lc(\psi))$ has no Lindelöf -dense subset, then $(S,lc(\psi))$ is T_1.

Proof. Since $(S,lc(\psi))$ has dense - no Lindelöf subset, therefore $(S,lc(\psi))$is an L_3 (Corollary2.31and Theorem 2.14(i)), hence $(S,lc(\psi))$is T_1 (Theorem 2.14(v)).

Theorem4.15: For a locally compact $Q-set$ space $(S,\psi)$the following are equivalent:

(a) $(S,lc(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof:$\Rightarrow$: If $(S,lc(\psi))$ is a $P-space$,so $(S,\psi)$is an L_1(Theorem 2.17). Since (S,ψ) is a $Q-set$, therefore (S,ψ) is an LC (Corollary 2.18) and $\psi = lc(\psi)$ .Since $(S,\psi)$is a locally compact . Hence $(S,lc(\psi))$ is a Hausdorff by Theorem 2.22.

$\Leftarrow$: Obvious (Corollary 3.5(i)).

Theorem4.16: For a R_1 $Q-set$ space $(S,\psi)$the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is a Hausdorff

Proof. $\Rightarrow$:If $(S,l(\psi))$is a $P-space$,so $(S,\psi)$is an L_1(Theorem2.17). Since $(S,\psi)$is a $Q-set$, therefore $(S,\psi)$is an LC (Corollary 2.18)and $\psi = lc(\psi)$.Since$(S,\psi)$is a R_1 .Hence$(S,lc(\psi))$ is a Hausdorff (Corollary2.23(i)).

$\Leftarrow$: Obvious (Corollary 3.5(i)).

Theorem4.17: For a regular $Q-set$ space $(S,\psi)$the following are equivalent:

(a) $(S,l(\psi))$ is a $P-space$. (b) $(S,lc(\psi))$ is a Hausdorff.

Proof.$\Rightarrow$: If $(S,l(\psi))$is a $P-space$,so $(S,\psi)$is an L_1(Theorem 2.17). Since$(S,\psi)$is a $Q-set$, therefore $(S,\psi)$is an LC (Corollary 2.18)and $\psi = lc(\psi)$.Since$(S,\psi)$is a regular. Hence$(S,lc(\psi))$is a Hausdorff (Corollary 2.23(i)). $\Leftarrow$: Obvious (Corollary 8.1.20(i)).

5. CoLC and WALC

Corollary5.1: (i) If $(S,\psi)$is an$WALC-space$, then $(S,lc(\psi))$ is a $T_1-space$.

(ii) If (S,ψ) is an $ALC-space$, then $(S,lc(\psi))$ is a $T_1-space$.

Proof. Obvious) Definition 2.9and Proposition 2.36(.

Theorem5.2: Let (S,ψ) be a regular topological space. If $(S,lc(\psi))$ has no Lindelöf -dense subset, then (S,ψ) and $(S,lc(\psi))$ are $WALC-spaces$.

Proof. Since $(S,lc(\psi))$ has dense -no Lindelöf subset, so (S,ψ) is an LC) Corollary 2.31 and Theorem 2.32(.Since (S,ψ) is a regular, therefore (S,ψ) is an $WALC$)Theorem 2.28(, Since $\psi=lc(\psi)$ then $(S,lc(\psi))$ is an $WALC$.

Corollary5.3: Let (S,ψ) be a regular topological space. If $(S,lc(\psi))$ is a Hausdorff, then (S,ψ) is an $WALC-space$.

Proof. Since $(S,lc(\psi))$ is a Hausdorff, so (S,ψ) is an LC) Proposition 2.33(.Since (S,ψ) is a regular, then (S,ψ) is an $WALC-space$) Theorem 2.28(.

Theorem5.4:

If (S,ψ) is an $WALC-space$, then $(S,lc(\psi))$ is an $WALC-space$ and an $LC-space$.

Proof. Since (S,ψ) is an $WALC$, so (S,ψ) is an LC ,$\psi=lc(\psi)$. Hence $(S,lc(\psi))$ is an $WALC$ and an LC .

Theorem5.5: If $(S,lc(\psi))$ is $T_3-space$, then (S,ψ) is an $WALC-space$.

Proof. Let $w,z\in S$ and $w\neq z$. Since $(S,lc(\psi))$ is Hausdorff, $\exists O,V\in lc(\psi)\ni w\in O, z\in V$ and $O\cap V=\phi$. Thus $S=(S-O)\cup(S-V)$ and since $S-O$, $S-V$ are two closed, $LC-subspaces$ of (S,ψ). Therefore $(S,lc(\psi))$ is an LC (Theorem 2.29).Hence $(S,lc(\psi))$ is an $WALC$ (Theorem 2.28).

Theorem5.6: For a regular space (S,ψ) the following are equivalent:

(a) (S,ψ) is an $WALC-space$. (b) $(S,lc(\psi))$ is an $LC-space$.

Proof. $\Rightarrow$: Obvious (Theorem 5.4). $\Leftarrow$: If $(S,lc(\psi))$ is an LC,

so (S,ψ) is an LC (Theorem 2.32). Since (S,ψ) is a regular, hence (S,ψ) is an $WALC$ (Theorem 2.28).

Theorem5.7: For a regular space (S,ψ) the following are equivalent:

(a) (S,ψ) is an $WALC-space$. (b) $(S,lc(\psi))$ is an $WALC-space$

Proof. $\Rightarrow$: Obvious (Theorem 5.4). $\Leftarrow$: If $(S,lc(\psi))$ is an LC, so (S,ψ) is an LC (Theorem2.32). Since (S,ψ) is a regular, hence (S,ψ) is an $WALC$ (Theorem 2.28).

Corollary5.8: (i) If (S,ψ) is an $WALC-space$, then $l(\psi)\subseteq lc(\psi)$.

(ii) If (S,ψ) is an $ALC-space$, then $l(\psi)\subseteq lc(\psi)$.

Proof. (i) and (ii) Obvious (Definition 2.9and Proposition 2.37(i)).

Theorem5.9: Let (S,ψ) be a regular space. Then $(S,lc(\psi))$ is an $WALC-space$ if it can be writtenas the union of two closed sets which are not equal to S.

Proof.

Let $S=K_1 \cup K_2$ and let $K_1\neq S, K_2\neq S$ be closed in $(S,lc(\psi))$. Then K_1 and K_2 are $LC-subspace$ of (S,ψ) by Definition CoLC . Therefore (S,ψ) is an $LC-space$ (Theorem2.29),since K_1 and K_2 are closed in (S,ψ). So $(S,lc(\psi))$ is an $LC-space$ (Theorem 2.32) and $\psi=lc(\psi)$. Since (S,ψ) is a regular space, then $(S,lc(\psi))$ is an $WALC-space$ (Theorem 2.28).

Corollary5.10: Let (S,ψ) be a regular topological space. If $(S,lc(\psi))$ is disconnected, then $(S,l(\psi))$ is an $WALC-space$.

Proof. Obvious (Theorem 5.9).

Theorem5.11: For a regular $Q-set$ space (S,ψ) the following are equivalent:

(a) (S,ψ) is an $LC-space$. (b) (S,ψ) is an $WALC-space$.

(c) $(S,l(\psi))$ is a $P-space$. (d) $(S,lc(\psi))$ is a Hausdorff and an $LC-space$.

Proof.

(a) $\Rightarrow$(b): Obvious (Theorem 2.28).

(b) $\Rightarrow$(a): Obvious (Definition2.9).

(b) $\Rightarrow$(c): Let (S,ψ) be an $WALC-space$, so (S,ψ) is an $LC-space$ (Theorem 2.28). Therefore $(S,l(\psi))$ is a $P-space$ (Theorem 2.15).

(c) $\Rightarrow$(b): If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an L_1 (Theorem 2.17). Since (S,ψ) is a $Q-set$, therefore (S,ψ) is an LC (Corollary 2.18)and $\psi=lc(\psi)$.Since (S,ψ) is a regular. Hence $(S,lc(\psi))$ is an $WALC$ (Theorem 2.28).

(c) $\Rightarrow$(d): If $(S,l(\psi))$ is a $P-space$,so (S,ψ) is an L_1 (Theorem2.17). Since (S,ψ) is a $Q-set$, therefore (S,ψ) isan $LC-space$ (Corollary 2.18) and $\psi=lc(\psi)$. Since (S,ψ) is a regular . Hence $(S,lc(\psi))$ is a Hausdorff (Corollary 2.23(ii)).

(d) $\Rightarrow$(c): Obvious (Corollary 3.5).

<u>**Theorem5.12:**</u> For a regular $P-space$ S the following are equivalent:

(a) (S,ψ) is $T_1-space$.(b) (S,ψ) is a Hausdorff space.(c) (S,ψ) is an $WALC-space$

(d) (S,ψ) is an $LC-space$.(e) (S,ψ) is a Locally $LC-space$.(f) (S,ψ) is a $KC-space$.

Proof.

(a) $\Rightarrow$(b):If S is a T_1, since S is a regular, hence S is a Hausdorff.

(b) $\Rightarrow$(a): Obvious. (b) $\Rightarrow$(c): Obvious (Theorem 2.42).

(c) $\Rightarrow$(b): If S is an $WALC$, so S is an T_1,since S is a regular, therefore S is a Hausdorff.

(c)$\Rightarrow$(d): Obvious (Definition2.9).

(d)$\Rightarrow$(c): Obvious (Theorem2.28). (d)$\Leftrightarrow$(e): Obvious (Definition 2.10).

(e) $\Rightarrow$(f): Let S be a Locally LC , since S is a regular, so S is an LC (Definition 2.10),therefore S is a KC .

(f) $\Rightarrow$(e): Let S be a KC, then S is an T_1, since S is a regular , therefore S is a Hausdorff , so S is an LC (Theorem 2.24), therefore S is a Locally LC .

Corollary5.13: For a regular $P-space$ (S,ψ) the following are equivalent:

(a) (S,ψ) is $T_1-space$.(b) $(S,\psi)$is a Hausdorff space.(c) (S,ψ) is an $WALC-space$

(d) $(S,lc(\psi))$is an $LC-space$ and $T_1-space$. (e) $(S,\psi)$is an $LC-space$

.(f) $(S,\psi)$is a Locally $LC-space$. (g) (S,ψ) is a $KC-space$.

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