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Global existence and growth of solutions to coupled degenerately damped Klein-Gordon equations

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ABSTRACT

In this work, we study a system of nonlinear Klein-Gordon equations with viscoelastic and degenerate damping in a bounded domain. By an appropriate auxiliary function (which is a small perturbation of the total energy), we prove the global existence and exponential growth of solutions for equation (1) with strong nonlinear function f_1 and f_2 satisfying appropriate conditions and the initial energy satisfying $E(0) < 0$.

1.Introduction

In this work, we investigate the global existence and exponential growth of solutions for the following system of nonlinear Klein-Gordon equations with viscoelastic and degenerate damping terms:

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + \int_0^t \mu_1(t-s)\Delta u(s)ds + (|u|^k + |v|^l)|u_t|^{\eta-1}u_t \\ \qquad = f_1(u,v),\ (x,t) \in \Omega \times (0,T), \\ v_{tt} - \Delta v + m_2^2 v + \int_0^t \mu_2(t-s)\Delta v(s)ds + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{\upsilon-1}v_t \\ \qquad = f_2(u,v),\ (x,t) \in \Omega \times (0,T) \\ \qquad u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \\ \qquad v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain with smooth boundary in $R^n\ (n \geq 1), m_1, m_2 > 0,\ \eta, \upsilon \geq 0$, $k, l, \theta, \varrho > 1;\ \mu_i(.): R^+ \to R^+\ (i = 1,2)$ are positive relaxation functions.

By taking

$$f_1(u,v) = a|u+v|^{2(\kappa+1)}(u+v) + b|u|^\kappa u|v|^{\kappa+2},$$
$$f_2(u,v) = a|u+v|^{2(\kappa+1)}(u+v) + b|v|^\kappa v|u|^{\kappa+2},$$

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where $a > 0, b > 0$ and

$$\begin{cases} -1 < \kappa & if\ n = 1,2, \\ -1 < \kappa \leq \frac{3-n}{n-2} & if\ n \geq 3 \end{cases} \quad . \tag{2}$$

Multiplying $f_1(u, v)\ by\ u, f_2(u, v)\ by\ v$, we get

$$\begin{aligned} &uf_1(u, v) + vf_2(u, v) \\ &= u\left(a|u + v|^{2(\kappa+1)}(u + v) + b|u|^{\kappa}u|v|^{\kappa+2}\right) \\ &\quad +v\left(a|u + v|^{2(\kappa+1)}(u + v) + b|v|^{\kappa}v|u|^{\kappa+2}\right) \\ &= a|u + v|^{2(\kappa+1)}(u + v)^2 + 2b|u|^{\kappa+2}|v|^{\kappa+2} \\ &= a|u + v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2} \\ &= 2(\kappa + 2)F(u, v), \end{aligned} \tag{3}$$

where $\forall(u, v) \in R^2$ for

$$F(u, v) = \frac{1}{2(\kappa+2)}\left[a|u + v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2}\right]. \tag{4}$$

The system (1) is a generalization of the following Klein-Gordon system:

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + \alpha uv^2 = 0, \\ v_{tt} - \Delta v + m_2^2 v + \alpha u^2 v = 0, \end{cases}$$

where m_1, m_2, α are non-negative constants, which is considered in the study of the quantum field theory. The above system defines the motion of a charged meson in an electromagnetic field and was proposed by Segal [21].

The generalized system (1) was earlier investigated by Yazid et al. [24]. The authors considered the global nonexistence of solution with positive initial energy.

Pişkin [8] investigated on coupled equations of the form

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + |u_t|^{\eta-1}u_t = f_1(u, v), \\ v_{tt} - \Delta v + m_2^2 v + |v_t|^{\upsilon-1}v_t = f_2(u, v), \end{cases} \tag{5}$$

and the author considered the decay of solution by using Nakao's inequality and the blow up of solution with negative initial energy. Then, Pişkin [9] studied same problem and proved lower bounds for the time of blow up is derived if the solutions blow up. In [10], the author studied blow up of solutions with negative initial energy in case $\eta = \upsilon = 1$. Furthermore, Ye [25] considered the problem (5) with $\eta = \upsilon$ and studied the asymptotic stability and global existence of solutions: In addition, some other authors investigated problem (5) with $\eta = \upsilon = 1$ see ([6,7,22]).

The effect of the degenerate damping terms often appear in many applications and practical problems and turns a lot of systems into different problems worth studying. Now, we state some present results in the literature: Firstly, we mention the pioneer work of Rammaha and Sakuntasathien [19] who focus on coupled equations of the form

$$\begin{cases} u_{tt} - \Delta u + (|u|^k + |v|^l)|u_t|^{\eta-1}u_t = f_1(u, v), \\ v_{tt} - \Delta v + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{\upsilon-1}v_t = f_2(u, v). \end{cases} \tag{6}$$

They investigated the global well posedness of the solution under some restriction on the parameters. In [2,26], authors studied the same problem treated in [19], and they studied the growth and blow up properties. For more depth, here are some papers that focused on the study of degenerate damping [3-5, 11, 13-17, 23, 27].

It is well known fact that "Exponential Growth" phenomenon is one of the most important phenomena of asymptotic behavior but many authors omit it. It presentations us very considerable information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view.

In this work, we investigate how we can apply the degenerate damping term for knowing the behavior of growth of solutions for a coupled nonlinear Klein-Gordon system with viscoelastic and source terms. The rest paper is organized as follows: In the next section, we present necessary assumptions that will be used later. Then, in Section 3, we establish the global existence of problem. The exponential growth of solution is presented in Section 4.

2. Preliminaries

$W^{m,p}$ is denote the Sobolev space and

$$\begin{cases} W^{0,p}(\Omega) = L^p(\Omega) \; if \; m = 0, \\ W^{m,2}(\Omega) = H^m(\Omega) \; if \; p = 2. \end{cases}$$

Also, let denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm by $\|.\|_p = \|.\|_{L^p(\Omega)}$ for details see ([1,18]).

Now, we make the following assumptions:

(H1) Regarding $\mu_i(.): R^+ \to R^+,\ (i = 1,2)$ are C^1-nonincreasing functions satisfying

$$\mu_i(\alpha) > 0,\quad \mu_i{}'(\alpha) \le 0,\ 1 - \int_0^\infty \mu_i(\alpha)d\alpha = l_i > 0,\ \alpha \ge 0.$$

(H2) $$\begin{cases} 1 \le \eta, \upsilon \qquad if \; n = 1, 2, \\ 1 \le \eta, \upsilon \le \frac{n+2}{n-2} \; if \; n \ge 3. \end{cases}$$

Also, we use the following notation:

$$(\mu_i \diamond \nabla w)(t) = \int_0^t \mu_i(t-s)\|\nabla w(t) - \nabla w(s)\|^2\, ds.$$

Now, we define the energy function

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{1}{2}[(\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) + m_1^2\|u\|^2 + m_2^2\|v\|^2]$$
$$+\frac{1}{2}\left[\left(1 - \int_0^t \mu_1(s)ds\right)\|\nabla u(t)\|^2 + \left(1 - \int_0^t \mu_2(s)ds\right)\|\nabla v(t)\|^2\right] - \int_\Omega F(u,v)dx. \tag{7}$$

By computation, we have

$$\frac{d}{dt}E(t) \le \frac{1}{2}[(\mu_1{}' \diamond \nabla u)(t) + (\mu_2{}' \diamond \nabla v)(t)]$$
$$-\frac{1}{2}(\mu_1(t)\|\Delta u\|^2 + \mu_2(t)\|\Delta v\|^2)$$
$$-\int_\Omega (|u|^k + |v|^l)|u_t|^{\eta+1}dx - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{\upsilon+1}\, dx$$
$$\le 0. \tag{8}$$

3. Global existence

In this part, we prove the global existence of solution for problem (1). For this aim we set

$$I(t) = m_1^2\|u\|^2 + m_2^2\|v\|^2 + (\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) - 2(\kappa+2)\int_\Omega F(u,v)dx$$
$$+\left(1 - \int_0^t \mu_1(s)ds\right)\|\nabla u(t)\|^2 + \left(1 - \int_0^t \mu_2(s)ds\right)\|\nabla v(t)\|^2$$

and

$$J(t) = \frac{1}{2}[(\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) + m_1^2\|u\|^2 + m_2^2\|v\|^2] - \int_\Omega F(u,v)dx$$
$$+\frac{1}{2}\left[\left(1 - \int_0^t \mu_1(s)ds\right)\|\nabla u(t)\|^2 + \left(1 - \int_0^t \mu_2(s)ds\right)\|\nabla v(t)\|^2\right].$$

Lemma 3.1. Assume that (4) holds. Then there exist $\rho > 0$ such that for any $u, v \in H_0^1(\Omega)$, we get

$$\|u+v\|_{2(\kappa+2)}^{2(\kappa+2)} + 2\|uv\|_{\kappa+2}^{\kappa+2} \le \rho(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2)^{\kappa+2}$$

is satisfied. [20]

Lemma 3.2. Assume that (H1) and (H2) hold. Let $u_0, v_0 \in H_0^1(\Omega), u_1, v_1 \in H_0^1(\Omega)$. If

$$I(0) > 0 \; and \; \chi = \rho \left(\frac{2(\kappa+2)}{\kappa+1} E(0)\right)^{\kappa+1} < 1, \tag{12}$$

then

$$I(t) > 0, \qquad \forall t > 0.$$

Proof. We have $I(0) > 0$, then by continuity of $I(t)$ about t, there exist a maximal time $t_m > 0$ such that

$$I(t) \geq 0 \; on \; t \in (0, t_m).$$

Let t_0 be as follows

$$\{I(t_0) = 0 \; and \; I(t) > 0 \; for \; all \; 0 \leq t < t_0\}. \tag{13}$$

By using

$$\begin{aligned} J(t) &= \frac{1}{2(\kappa+2)} I(t) + \frac{\kappa+1}{2(\kappa+2)} [(\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) + m_1^2 \|u\|^2 + m_2^2 \|v\|^2] \\ &+ \frac{\kappa+1}{2(\kappa+2)} \left[\left(1 - \int_0^t \mu_1(s) ds\right) \|\nabla u(t)\|^2 + \left(1 - \int_0^t \mu_2(s) ds\right) \|\nabla v(t)\|^2\right] \\ &\geq \frac{\kappa+1}{2(\kappa+2)} [l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 + (\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) + m_1^2 \|u\|^2 + m_2^2 \|v\|^2]. \end{aligned} \tag{14}$$

From (7) and (8), we have

$$\begin{aligned} l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 &\leq \frac{2(\kappa+2)}{\kappa+1} J(t) \\ &\leq \frac{2(\kappa+2)}{\kappa+1} E(t) \\ &\leq \frac{2(\kappa+2)}{\kappa+1} E(0), \quad \forall t \in [0, t_0]. \end{aligned} \tag{15}$$

By (11) and (12), we reach at

$$\begin{aligned} 2(\kappa+2) \int_\Omega F\big(u(t_0), v(t_0)\big) dx &\leq \rho (l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2)^{\kappa+2} \\ &\leq \rho \left(\frac{2(\kappa+2)}{\kappa+1} E(0)\right)^{\kappa+1} (l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2) \\ &\leq l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2 \\ &\leq \left(1 - \int_0^t \mu_1(s) ds\right) \|\nabla u(t)\|^2 + \left(1 - \int_0^t \mu_2(s) ds\right) \|\nabla v(t)\|^2. \end{aligned}$$

Then, since (9) we have

$$I(t_0) > 0$$

which contradicts to (13). So, $I(t) > 0 \; on \; [0, T]$.

Teorem 3.3. Assume that the conditions of Lemma 3.2 hold, then the solutions (1) is bounded and global in infinite time.

Proof. It suffices to show that

$$\|(u, v)\|_H \coloneqq \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2$$

is bounded independently of t (time). For this purpose, we apply (7), (8), (10) and (15) to get

$$\begin{aligned} E(0) &\geq E(t) = J(t) + \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2) \\ &\geq \frac{\kappa+1}{2(\kappa+2)} [l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 + (\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t) \\ &+ \frac{\kappa+1}{2(\kappa+2)} [m_1^2 \|u\|^2 + m_2^2 \|v\|^2] + \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2). \end{aligned} \tag{16}$$

Thus,

$$\|(u,v)\|_H \le CE(0),$$

where positive constant C, which depends only on κ, l_1, l_2.

4. Growth

In this part, our purpose to show that the energy grow up as an exponential function as time goes to infinity.

Theorem 4.1. Assume that

$$2(\kappa+2) > max\{k+\eta+1, l+\eta+1, \theta+\upsilon+1, \varrho+\upsilon+1\},$$

and the initial energy $E(0) < 0$. Then, the solution of the system (1) grows exponentially.

Proof. We set

$$H(t) = -E(t),$$

from assumption $E(0) < 0$ and (8) gives $H(t) \ge H(0) > 0$.

Then, define $\Phi(t)$ by

$$\Phi(t) = H(t) + \varepsilon\left(\int_\Omega u_t u dx + \int_\Omega v_t v dx\right) \tag{17}$$

where $0 < \varepsilon \le 1$.

By differentiating (17) and using Eq.(1), we have

$$\begin{aligned}\Phi'(t) = {} & H'(t) + \varepsilon(\|u_t\|^2 + \|v_t\|^2) - \varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2) \\ & +2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx - \varepsilon(m_1^2\|u\|^2 + m_2^2\|v\|^2) \\ & +\varepsilon\int_\Omega\int_0^t \mu_1(t-s)\nabla u(s)\nabla u(t)ds\,dx + \varepsilon\int_\Omega\int_0^t \mu_2(t-s)\nabla v(s)\nabla v(t)ds\,dx \\ & -\varepsilon\left(\int_\Omega u(|u|^k + |v|^l)u_t|u_t|^{\eta-1}dx - \int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t|v_t|^{\upsilon-1}dx\right).\end{aligned} \tag{18}$$

We would like to estimate the last two terms right hand side in (18) by using the following Young's inequality

$$AB \le \frac{\delta^\alpha A^\alpha}{\alpha} + \frac{\delta^{-\beta}B^\beta}{\beta},$$

where $A, B \ge 0$, $\delta > 0$, $\alpha, \beta \in R^+$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Therefore, we have for all $\delta_1 > 0$

$$|uu_t|u_t|^{\eta-1}| \le \frac{\delta_1{}^{\eta+1}}{\eta+1}|u|^{\eta+1} + \frac{\eta\delta_1{}^{-\frac{\eta+1}{\eta}}}{\eta+1}|u_t|^{\eta+1},$$

and therefore

$$\begin{aligned}\int_\Omega (|u|^k + |v|^l)|uu_t|u_t|^{\eta-1}|\,dx \le {} & \frac{\delta_1{}^{\eta+1}}{\eta+1}\int_\Omega (|u|^k + |v|^l)|u|^{\eta+1}\,dx \\ & + \frac{\eta\delta_1{}^{-\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega (|u|^k + |v|^l)|u_t|^{\eta+1}\,dx.\end{aligned} \tag{19}$$

Similarly, for all $\delta_2 > 0$

$$|vv_t|v_t|^{\upsilon-1}| \le \frac{\delta_2{}^{\upsilon+1}}{\upsilon+1}|v|^{\upsilon+1} + \frac{\upsilon\delta_2{}^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1}|v_t|^{\upsilon+1},$$

which gives

$$\begin{aligned}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|vv_t|v_t|^{\upsilon-1}|dx \le {} & \frac{\delta_2{}^{\upsilon+1}}{\upsilon+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v|^{\upsilon+1}dx \\ & + \frac{\upsilon\delta_2{}^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{\upsilon+1}dx.\end{aligned} \tag{20}$$

Inserting the estimates (19), (20) into (18), we have

$$\Phi'(t) \ge H'(t) + \varepsilon(\|u_t\|^2 + \|v_t\|^2) - \varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2)$$

$$+2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx-\varepsilon(m_1^2\|u\|^2+m_2^2\|v\|^2)$$
$$+\varepsilon\int_\Omega\int_0^t\mu_1(t-s)\nabla u(s)\nabla u(t)ds\,dx+\varepsilon\int_\Omega\int_0^t\mu_2(t-s)\nabla v(s)\nabla v(t)ds\,dx$$
$$-\varepsilon\frac{\delta_1{}^{\eta+1}}{\eta+1}\int_\Omega(|u|^k+|v|^l)|u|^{\eta+1}dx-\varepsilon\frac{\eta\delta_1{}^{-\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega(|u|^k+|v|^l)|u_t|^{\eta+1}dx$$
$$-\boldsymbol{\varepsilon}\frac{\delta_2{}^{\upsilon+1}}{\upsilon+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v|^{\upsilon+1}dx-\varepsilon\frac{\upsilon\delta_2{}^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v_t|^{\upsilon+1}dx. \tag{21}$$

Now, the ninth term in the right hand side of (21) can be estimated, as follows (see [12]):

$$\int_\Omega\nabla u(t)\int_0^t\mu_1(t-s)\nabla u(s)dsdx$$
$$\le\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\int_\Omega\left(\int_0^t\mu_1(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)\,ds\right)^2dx.$$

Thanks to Young's inequality and assumption (H1), we have, for any $\xi_1>0$,

$$\int_\Omega\nabla u(t)\int_0^t\mu_1(t-s)\nabla u(s)dsdx\le\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(1+\xi_1)\int_\Omega\left(\int_0^t\mu_1(t-s)\nabla u(s)ds\right)^2dx$$
$$+\frac{1}{2}\left(1+\frac{1}{\xi_1}\right)\int_\Omega\left(\int_0^t\mu_1(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2dx$$
$$\le\frac{1+(1+\xi_1)(1-l_1)^2}{2}\|\nabla u\|^2$$
$$+\frac{\left(1+\frac{1}{\xi_1}\right)(1-l_1)}{2}(\mu_1\diamond\nabla u)(t).$$

Similar calculations also yield, for any $\xi_2>0$,

$$\int_\Omega\nabla v(t)\int_0^t\mu_2(t-s)\nabla v(s)dsdx\le\frac{1+(1+\xi_2)(1-l_2)^2}{2}\|\nabla v\|^2$$
$$+\frac{\left(1+\frac{1}{\xi_2}\right)(1-l_2)}{2}(\mu_2\diamond\nabla v)(t).$$

Then, add $2H(t)$ to both side of (21), we have

$$\Phi'(t)\ge H'(t)+2\varepsilon(\|u_t\|^2+\|v_t\|^2)$$
$$+\varepsilon\left((1-l_1)+\frac{(1+\xi_1)(1-l_1)^2-1}{2}\right)\|\nabla u\|^2$$
$$+\varepsilon\left((1-l_2)+\frac{(1+\xi_2)(1-l_2)^2-1}{2}\right)\|\nabla v\|^2$$
$$+2\varepsilon(\kappa+1)\int_\Omega F(u,v)dx+2\varepsilon H(t)$$
$$+\varepsilon\left(1+\frac{\left(1+\frac{1}{\xi_1}\right)(1-l_1)}{2}\right)(\mu_1\diamond\nabla u)(t)$$
$$+\varepsilon\left(1+\frac{\left(1+\frac{1}{\xi_2}\right)(1-l_2)}{2}\right)(\mu_2\diamond\nabla v)(t)$$
$$-\varepsilon\frac{\delta_1{}^{\eta+1}}{\eta+1}\int_\Omega(|u|^k+|v|^l)|u|^{\eta+1}dx-\varepsilon\frac{\eta\delta_1{}^{-\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega(|u|^k+|v|^l)|u_t|^{\eta+1}dx$$
$$-\boldsymbol{\varepsilon}\frac{\delta_2{}^{\upsilon+1}}{\upsilon+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v|^{\upsilon+1}dx-\varepsilon\frac{\upsilon\delta_2{}^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v_t|^{\upsilon+1}dx. \tag{22}$$

Then, by using Young's inequality, we get

$$\int_\Omega(|u|^k+|v|^l)|u|^{\eta+1}dx\le\int_\Omega|u|^{k+\eta+1}dx+\int_\Omega|v|^l|u|^{\eta+1}dx$$
$$\le\int_\Omega|u|^{k+\eta+1}dx+\frac{l}{l+\eta+1}\gamma_1{}^{\frac{l+\eta+1}{l}}\int_\Omega|v|^{l+\eta+1}dx$$

$$+\frac{\eta+1}{l+\eta+1}\gamma_1^{-\frac{l+\eta+1}{\eta+1}}\int_\Omega |u|^{l+\eta+1}dx$$
$$=\|u\|_{k+\eta+1}^{k+\eta+1}+\frac{l}{l+\eta+1}\gamma_1^{\frac{l+\eta+1}{l}}\|v\|_{l+\eta+1}^{l+\eta+1}$$
$$+\frac{\eta+1}{l+\eta+1}\gamma_1^{-\frac{l+\eta+1}{\eta+1}}\|u\|_{l+\eta+1}^{l+\eta+1}$$

and

$$\int_\Omega \left(|v|^\theta+|u|^\varrho\right)|v|^{\upsilon+1}dx \le \int_\Omega |v|^{\theta+\upsilon+1}dx+\int_\Omega |u|^\varrho|v|^{\upsilon+1}dx$$
$$\le \int_\Omega |v|^{\theta+\upsilon+1}dx+\frac{\varrho}{\varrho+\upsilon+1}\gamma_2^{\frac{\varrho+\upsilon+1}{\varrho}}\int_\Omega |u|^{\varrho+\upsilon+1}dx$$
$$+\frac{\upsilon+1}{\varrho+\upsilon+1}\gamma_2^{-\frac{\varrho+\upsilon+1}{\upsilon+1}}\int_\Omega v^{\varrho+\upsilon+1}dx$$
$$=\|v\|_{\theta+\upsilon+1}^{\theta+\upsilon+1}+\frac{\varrho}{\varrho+\upsilon+1}\gamma_2^{\frac{\varrho+\upsilon+1}{\varrho}}\|u\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1}$$
$$+\frac{\upsilon+1}{\varrho+\upsilon+1}\gamma_2^{-\frac{\varrho+\upsilon+1}{\upsilon+1}}\|\boldsymbol{v}\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1}.$$

Then, (22) deduce to

$$\Phi'(t)\ge H'(t)+2\varepsilon(\|u_t\|^2+\|v_t\|^2)+2\varepsilon H(t)$$
$$+\varepsilon\left((1-l_1)+\frac{(1+\xi_1)(1-l_1)^2-1}{2}\right)\|\nabla u\|^2$$
$$+\varepsilon\left((1-l_2)+\frac{(1+\xi_2)(1-l_2)^2-1}{2}\right)\|\nabla v\|^2$$
$$+2\varepsilon(\kappa+1)\left[\|u\|_{2(\kappa+2)}^{2(\kappa+2)}+\|v\|_{2(\kappa+2)}^{2(\kappa+2)}\right]$$
$$+\varepsilon\left(1+\frac{\left(1+\frac{1}{\xi_1}\right)(1-l_1)}{2}\right)(\mu_1\diamond\nabla u)(t)$$
$$+\varepsilon\left(1+\frac{\left(1+\frac{1}{\xi_2}\right)(1-l_2)}{2}\right)(\mu_2\diamond\nabla v)(t)$$
$$-\varepsilon\frac{{\delta_1}^{\eta+1}}{\eta+1}\left(\|u\|_{k+\eta+1}^{k+\eta+1}+\frac{l}{l+\eta+1}\gamma_1^{\frac{l+\eta+1}{l}}\|v\|_{l+\eta+1}^{l+\eta+1}+\frac{\eta+1}{l+\eta+1}\gamma_1^{-\frac{l+\eta+1}{\eta+1}}\|u\|_{l+\eta+1}^{l+\eta+1}\right)$$
$$-\boldsymbol{\varepsilon}\frac{{\delta_2}^{\upsilon+1}}{\upsilon+1}\left[\|v\|_{\theta+\upsilon+1}^{\theta+\upsilon+1}+\frac{\varrho}{\varrho+\upsilon+1}\gamma_2^{\frac{\varrho+\upsilon+1}{\varrho}}\|u\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1}+\frac{\upsilon+1}{\varrho+\upsilon+1}\gamma_2^{-\frac{\varrho+\upsilon+1}{\upsilon+1}}\|\boldsymbol{v}\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1}\right]$$
$$-\varepsilon\frac{\eta\delta_1^{-\frac{\eta+1}{\eta}}}{\eta+1}\int_\Omega(|u|^k+|v|^l)|u_t|^{\eta+1}dx-\varepsilon\frac{\upsilon\delta_2^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1}\int_\Omega\left(|v|^\theta+|u|^\varrho\right)|v_t|^{\upsilon+1}dx. \quad (23)$$

By using $2(\kappa+2)>max\{k+\eta+1,l+\eta+1,\theta+\upsilon+1,\varrho+\upsilon+1\}$ assumption and the following algebraic inequality

$$x^\sigma\le x+1\le\left(1+\frac{1}{a}\right)(x+a),\ \forall x\ge 0, 0<\sigma\le 1,\ a\ge 0, \quad (24)$$

we obtain for all $t\ge 0$

$$\|v\|_{\theta+\upsilon+1}^{\theta+\upsilon+1}\le c_1\|v\|_{2(\kappa+2)}^{\theta+\upsilon+1}\le d\left(\|v\|_{2(\kappa+2)}^{2(\kappa+2)}+H(t)\right),$$

where $d=1+\frac{1}{H(0)}$. In the same way, we obtain

$$\|u\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1}\le c_2\|u\|_{2(\kappa+2)}^{\varrho+\upsilon+1}\le d\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)}+H(t)\right),$$
$$\|v\|_{l+\eta+1}^{l+\eta+1}\le c_3\|v\|_{2(\kappa+2)}^{l+\eta+1}\le d\left(\|v\|_{2(\kappa+2)}^{2(\kappa+2)}+H(t)\right),$$
$$\|u\|_{l+\eta+1}^{l+\eta+1}\le c_4\|u\|_{2(\kappa+2)}^{l+\eta+1}\le d\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)}+H(t)\right),$$

$$\|u\|_{k+\eta+1}^{k+\eta+1} \le c_5\|u\|_{2(\kappa+2)}^{k+\eta+1} \le d\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)} + H(t)\right),$$

and

$$\|v\|_{\varrho+\upsilon+1}^{\varrho+\upsilon+1} \le c_6\|v\|_{2(\kappa+2)}^{\varrho+\upsilon+1} \le d\left(\|v\|_{2(\kappa+2)}^{2(\kappa+2)} + H(t)\right).$$

Selecting C_1, C_2, C_3, C_4, C_5 as follows

$$C_1 = \frac{\eta\delta_1^{-\frac{\eta+1}{\eta}}}{\eta+1}, \quad C_2 = \frac{\upsilon\delta_2^{-\frac{\upsilon+1}{\upsilon}}}{\upsilon+1},$$

$$C_3 = \frac{\delta_1^{\eta+1}}{\eta+1}\left(1 + \frac{\eta+1}{l+\eta+1}\gamma_1^{-\frac{l+\eta+1}{\eta+1}}\right) + \frac{\delta_2^{\upsilon+1}}{\upsilon+1}\frac{\varrho}{\varrho+\upsilon+1}\gamma_2^{\frac{\varrho+\upsilon+1}{\varrho}},$$

$$C_4 = \frac{\delta_2^{\upsilon+1}}{\upsilon+1}\left(1 + \frac{\upsilon+1}{\varrho+\upsilon+1}\gamma_2^{-\frac{\varrho+\upsilon+1}{\upsilon+1}}\right) + \frac{\delta_1^{\eta+1}}{\eta+1}\frac{l}{l+\eta+1}\gamma_1^{\frac{l+\eta+1}{l}},$$

and

$$C_5 = \frac{\delta_1^{\eta+1}}{\eta+1}\left(1 + \frac{l}{l+\eta+1}\gamma_1^{\frac{l+\eta+1}{l}} + \frac{\eta+1}{l+\eta+1}\gamma_1^{-\frac{l+\eta+1}{\eta+1}}\right) + \frac{\delta_2^{\upsilon+1}}{\upsilon+1}\left(1 + \frac{\upsilon+1}{\varrho+\upsilon+1}\gamma_2^{-\frac{\varrho+\upsilon+1}{\upsilon+1}} + \frac{\varrho}{\varrho+\upsilon+1}\gamma_2^{\frac{\varrho+\upsilon+1}{\varrho}}\right).$$

where we pick $\delta_1, \delta_2, \gamma_1$ and γ_2 to find small enough $C_1,\ C_2,\ C_3,\ C_4$ and C_5.
This implies

$$\begin{aligned}\Phi'(t) \ge{}& H'(t) + 2\varepsilon(\|u_t\|^2 + \|v_t\|^2) + \varepsilon(2 - dC_5)H(t) \\ &+\varepsilon\omega_1\|\nabla u\|^2 + \varepsilon\omega_2\|\nabla v\|^2 \\ &+\varepsilon\theta_1(\mu_1 \diamond \nabla u)(t) + \varepsilon\theta_2(\mu_2 \diamond \nabla v)(t) \\ &+(1 - \varepsilon C_1)\int_\Omega (|u|^k + |v|^l)|u_t|^{\eta+1}\,dx \\ &+(1 - \varepsilon C_2)\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{\upsilon+1}dx \\ &+\varepsilon(2(\kappa+1) - dC_3)\|u\|_{2(\kappa+2)}^{2(\kappa+2)} + \varepsilon(2(\kappa+1) - dC_4)\|v\|_{2(\kappa+2)}^{2(\kappa+2)}\end{aligned} \tag{25}$$

where $\omega_i = \left((1 - l_i) + \frac{(1+\xi_i)(1-l_i)^2 - 1}{2}\right) > 0$ and $\theta_i = \left(1 + \frac{\left(1+\frac{1}{\xi_i}\right)(1-l_i)}{2}\right) > 0$ $(i = 1,2)$ for choosing $\xi_i = \frac{l_i}{1-l_i}$.

We can find positive constants K_1, K_2, K_3 and C_6 such that

$$\begin{aligned}\Phi'(t) \ge{}& (1 - \varepsilon C_6)H'(t) + 2\varepsilon(\|u_t\|^2 + \|v_t\|^2) + \varepsilon K_1 H(t) \\ &+\varepsilon\omega_1\|\nabla u\|^2 + \varepsilon\omega_2\|\nabla v\|^2 + \varepsilon K_2\|u\|_{2(\kappa+2)}^{2(\kappa+2)} + \varepsilon K_3\|v\|_{2(\kappa+2)}^{2(\kappa+2)}.\end{aligned} \tag{26}$$

We pick ε small enough such that $(1 - \varepsilon C_6) \ge 0$ and

$$\Phi(0) = H(0) + \varepsilon\left(\int_\Omega u_t u_0 dx + \int_\Omega v_t v_0 dx\right) > 0.$$

As a result, there exists $M > 0$ such that (26) deduce to

$$\Phi'(t) \ge \varepsilon M\left(H(t) + \|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(\kappa+2)}^{2(\kappa+2)} + \|v\|_{2(\kappa+2)}^{2(\kappa+2)}\right). \tag{27}$$

Thus, $\Phi(t)$ is strictly positive and increasing for all $t \ge 0$.

Now, by applying Holder's and Young's inequalities, we have

$$\begin{aligned}\left|\int_\Omega u_t u dx\right| &\le \|u_t\|\|u\| \\ &\le C\left(\|u_t\|\|u\|_{2(\kappa+2)}\right) \\ &\le \frac{C}{2}\left(\|u_t\|^2 + \|u\|_{2(\kappa+2)}^2\right)\end{aligned}$$

$$\leq \frac{C}{2}\left(\|u_t\|^2 + \left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)}\right)^{\frac{1}{\kappa+2}}\right).$$

Applying (24) for $\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)}\right)^{\frac{1}{\kappa+2}}$, we get

$$\left|\int_\Omega u_t u dx\right| \leq \frac{C}{2}\left(\|u_t\|^2 + \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)} + H(t)\right)\right).$$

Likewise, we get

$$\left|\int_\Omega v_t v dx\right| \leq \frac{C}{2}\left(\|v_t\|^2 + \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(\kappa+2)}^{2(\kappa+2)} + H(t)\right)\right).$$

Then, we have

$$\Phi(t) \leq C\left(H(t) + \|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(\kappa+2)}^{2(\kappa+2)} + \|v\|_{2(\kappa+2)}^{2(\kappa+2)}\right) \tag{28}$$

and from (27) and (28), we reach

$$\frac{d\Phi(t)}{dt} \geq \Gamma\Phi(t), \forall t \geq 0 \tag{29}$$

where Γ is a positive constant.

Integration of (29) over $(0,t)$, we obtain

$$\Phi(t) \geq \Phi(0)exp(\Gamma t)$$

and this completes the proof.

Conclusion

In this work, we are interested in the exponential growth of solutions for coupled of Klein-Gordon equations with degenerate damping and viscoelastic term. This kind of problem is mostly found in some mathematical models in applied sciences. What interests us in this current work is the combination of Klein-Gordon system with these terms of damping (viscoelastic term, degenerate damping and source terms), which dictates the emergence of these terms in the system.

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References

References should be printed together at the end of the manuscript using EndNote with numbered styles. References that illustrated are referred with numbers within square brackets and the references are listed in a numbered reference list after the text. References are numbered in the order of Alphabetical as follows:

[1] R.A. ADAMS, J.J.F. FOURNIER, SOBOLEV SPACES, ACADEMIC PRESS, NEW YORK, 2003.

[2] A. Benaissa, D. Ouchenane, K. Zennir, Blow up of positive initial energy solutions to system of nonlinear wave equations with degenerate damping and source terms. Nonlinear Studies, 19(4) (2012), 523-535.

[3] F. Ekinci, E. Pişkin, Blow up and exponential growth to a Petrovsky equation with degenerate damping. Universal Journal of Mathematics and Applications, 4(2) (2021), 82-87.

[4] F. Ekinci, E. Pişkin, S. M. Boulaaras, I. Mekawy, Global existence and general decay of solutions for a quasilinear system with degenerate damping terms. Journal of Function Spaces, 2021 (2021), Article ID 4316238. Doi: 10.1155/2021/4316238

[5] F. Ekinci, E. Pişkin, K. Zennir, Existence, blow up and growth of solutions for a coupled quasi-linear viscoelastic Petrovsky equations with degenerate damping terms. Journal of Information & Optimization Sciences,1-30. DOI: 10.1080/02522667.2021.1972619

[6] M. O. Korpusov, Blow up the solution of a nonlinear system of equations with positive energy. Theoretical and Mathematical Physics, 171(3) (2012), 725-738.

[7] M. M. Miranda, L. A. Medeiros, On the existence of global solutions of a coupled nonlinear Klein-Gordon equations. Funkcialaj Ekvacioj, 30 (1987), 147-161.

[8] E. Pişkin, Uniform decay and blow-up of solutions for coupled nonlinear Klein-Gordon equations with nonlinear damping terms. Mathematical Methods in the Applied Sciences, 37(18) (2014), 3036-3047.

[9] E. Pişkin, Lower bounds for blow-up time of coupled nonlinear Klein-Gordon equations. Gulf Journal of Mathematics, 5(2) (2017), 56-61.

[10] E. Pişkin, Blow-up of solutions for coupled nonlinear Klein-Gordon equations with weak damping terms. Mathematical Sciences Letters, 3(3) (2014), 189-191.

[11] E. Pişkin, Blow up of positive initial-energy solutions for coupled nonlinear wave equations with degenerate damping and source terms. Boundary Value Problems, 43 (2015), 1-11.

[12] E. Pişkin, F. Ekinci, General decay and blow up of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms. Mathematical Methods in the Applied Sciences, 42(16) (2019), 5468-5488.

[13] E. Pişkin, F. Ekinci, Blow up of solutions for a coupled Kirchhoff-type equations with degenerate damping terms. Applications & Applied Mathematics., 14(2) (2019), 942-956.

[14] E. Pişkin, F. Ekinci, Global existence of solutions for a coupled viscoelastic plate equation with degenerate damping terms. Tbilisi Mathematical Journal, 14(3) (2021), 195-206

[15] E. Pişkin, F. Ekinci, Nonexistence of global solutions for coupled Kirchhoff-type equations with degenerate damping terms, Journal of Nonlinear Functional Analysis 2018 (2018), Article ID 48.

[16] E. Pişkin, F. Ekinci, Local existence and blow up of solutions for a coupled viscoelastic Kirchhoff-type equation with degenerate. Miskolc Mathematical Notes, 22(2) (2021), 861–874.

[17] E. Pişkin, F. Ekinci, H. Zhang, Blow Up, lower Bounds and exponential growth to a coupled quasilinear wave equations with degenerate damping terms, Dynamics of Continuous, Discrete and Impulsive Systems, 1-23 (in press).

[18] E. Pişkin, B. Okutmuştur, An Introduction to Sobolev Spaces, Bentham Science, 2021.

[19] M. A. Rammaha, S. Sakuntasathien, Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms. Nonlinear Analysis: Theory, Methods & Applications, 72 (2010), 2658-2683.

[20] B. Said-Houari, S.A. Messaoudi, A. Guesmia, General decay of solutions of a nonlinear system of viscoelastic wave equations. Nonlinear Differential Equations and Applications NoDEA, 18 (2011), 659-684.

[21] I. Segal, Nonlinear partial differential equations in quantum field theory. Proc. Symp. Appl. Math. AMS. 17 (1965), 210-226.

[22] S. T. Wu, Blow-up results for system of nonlinear Klein-Gordon equations with arbitrary positive initial energy. Electronic Journal of Differential Equations, 2012 (2012), 1-13.

[23] ST. Wu, General decay of solutions for a nonlinear sysem of viscoelastic wave equations with degenerate damping and source terms. Journal of Mathematical Analysis and Applications, 406 (2013), 34-48.

[24] F. Yazid, D. Ouchenane, K. Zennir, Global nonexistence of solutions to system of Klein-Gordon equations with degenerate damping and strong source terms in viscoelasticity. Studia Universitatis Babeş-Bolyai Mathematica, (2020), 1-17.

[25] Y. Ye, Global existence and asymptotic stability for coupled nonlinear Klein–Gordon equations with nonlinear damping terms. Dynamical Systems, 28(2) (2013), 287-298.

[26] K. Zennir, Growth of solutions to system of nonlinear wave equations with degenerate damping and strong sources. Nonlinear Analysis and Application, 2013 (2013), 1-11.

[27] K. Zennir, Growth of solutions with positive initial energy to system of degeneratly damped wave equations with memory. Lobachevskii Journal of Mathematics, 35(2) (2014), 147-156.