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The Numerical Solution For Solving nth Order Integro-Differential Equations via Boubaker Scaling Functions

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ABSTRACT

In this paper, the continuous Boubaker scaling functions were constructed with the presentation on the interval [0,1], which obtained depending on Boubaker polynomials. In this study the Boubaker scaling polynomial has been applied for solving the n^{th} order integro–differential equations (IDE's).The collocation method with the aid of Boubaker scaling functions together were utilized to transform the higher order integro–differential equations into a linear system algebraic equations. Some numerical examples were added to show the simplicity and accuracy of the proposed technique. The results have been compared with the exact solution using MATLAB and illustrated by graphs.

1. Introduction

The linear or quasi-linear *n*th order volterra integro-differential equation of the second kind is

$$U^{(n)}(\tau) = F(\tau) + \int_0^\tau K(\tau, x)U^{(s)}(x)dx, \qquad n \geq s \ , 0 \leq \tau \leq 1, \qquad \dots \ (1)$$

where $K(\tau, x)$ and $F(\tau)$ are known functions, and $u(\tau)$ is an unknown function.

with initial conditions

$$U^{(n-i)}(0) = \gamma_i\,, \qquad i = 1,2,3,\dots,n \qquad \dots \ (2)$$

Obviously, in recent decades there is a large interest in scaling functions and wavelets for solving linear and nonlinear problems in physics and engineering, many researchers worked in this field. David C. and Heil C. (1994), obtained a characterization of all dilation equations that have contains compactly supported solution [4]. Ghader pandah S. and Klasa S. presented a new

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discrete method for real polynomial scaling [9]. Yousefi, (2007), applied Legendre scaling function with their properties presented [15]. In [7, 8], [11-14], the researchers presented other methods for solving linear and nonlinear IDE's.

In this study, a scaling function for Boubaker polynomial was deduced. Then, it has been applied for solving integro-differential equations, presenting a numerical technique for solving *n*th order Voltera IDE's. Then we listed some numerical examples comparing their results with the exact solutions. The paper is arranged as follows, the next section gives a fundamental idea about Boubaker polynomials, the third section is concerned with the scaling of Boubaker polynomials, the fourth section introduced the approximate solution of *n*th order Volterra integro-differential equations with the present method. In the end utilized some numerical examples to illustrate the efficiency of the proposed technique.

2 -Boubaker Polynomials

Boubaker polynomial is first utilized for solving heat equation in physical applications then many researchers concerning this polynomial have taken place in different proceedings [2, 5, 10, 17].

Boubaker polynomial $Bo_m(\tau)$ is presented as in the following equation

$$Bo_m(\tau) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(m-4r)}{(m-r)} \binom{m-r}{r} (-1)^r \tau^{m-2r}, \quad m=0,1,2,\ldots \qquad \ldots(3)$$

The first three terms of Boubaker polynomial are as follows

$Bo_0(\tau) = 1$,

$Bo_1(\tau) = \tau$,

$Bo_2(\tau) = \tau^2 + 2$,

And the recurrence relation is $Bo_m(\tau) = \tau Bo_{m-1}(\tau) - Bo_{m-2}(\tau)$, $m > 2$, $0 \le \tau \le 1$

3- Boubaker Scaling Functions

The continuous Boubaker scaling function $BS_{nm}(\tau)$ is defined by the following, see [6, 15]

$$BS_{nm}(\tau) = \begin{cases} 2^{\frac{k}{2}} Bo_m(2^{k+1}\tau - 2n - 1), & \text{for } \frac{2n-1}{2^{k+1}} \le \tau \le \frac{2n}{2^{k+1}} \\ 0 & O.W. \end{cases} \qquad \ldots(4)$$

where k is a non-negative integer number and $m = 0,\ldots, M-1$,

$n = 0,1,\ldots,2^k - 1$, $m = M-1$.

Here, $Bo\ \ (\tau)$ were well-known Boubaker's polynomials of order m.

Now choose k=1 and m=5, the five terms Boubaker scaling $BS_m(\tau)$ were found by using equation (4) as follows

$BS_0(\tau) = \sqrt{2}$,

$BS_1(\tau) = \sqrt{2}\,(4\tau - 1)$,

$BS_2(\tau) = \sqrt{2}\,(16\tau^2 - 8\tau + 3)$,

$BS_3(\tau) = \sqrt{2}\,(64\tau^3 - 48\tau^2 + 16\tau - 2)$,

$BS_4(\tau) = \sqrt{2}\,(256\tau^4 - 256\tau^3 + 96\tau^2 - 16\tau - 1)$,

$\vdots$

Now, the Boubaker scaling $BS_m(\tau)$ was used for solving the integro-differential equation for varying order, the method will be introduced as in the following:

3-The proposed method for solving *n*th order integro-differential equation

The Boubaker Scaling will be applied to solve Equation (1) the integro-differential equation *n*-th order,

$$U_i^{(n)}(\tau) = F_i(\tau) + \int_0^\tau K_{i,j}(\tau,x)U_i^{(s)}(x)dx, \qquad n \geq s \qquad ...(5)$$

with the following initial conditions $U_i^{(s)}(0) = \gamma_i$, i =1, …, s = 0,1,2,…n-1.

A function $U_i^{(n)}(\tau)$ is defined on the interval $\tau \in [0,1]$ can be expanded into the Boubaker Scaling functions

$$U_i^{(n)}(\tau) = \sum_{i=0}^{M-1} a_i BS_i(\tau) . \qquad ... (6)$$

Where a_i are Boubaker scaling coefficients, integrate Equation (6) n- times yields

$$U(\tau) = \sum_{i=0}^{M-1} a_i \int_0^\tau ... \int_0^\tau BS_i(x)dx^n + \sum_{j=0}^{M-1} \frac{\tau^j}{j!}\gamma_{M-j}. \qquad ... (7)$$

Using the following formula

$$\int_0^\tau ... \int_0^\tau BS_i(x)dx^n = \frac{1}{(n-1)!}\int_0^\tau (\tau - x)^{n-1} BS_i(x)dx,$$

Therefore Equation (7) becomes

$$U(\tau) = \sum_{i=0}^{M-1} a_i \frac{1}{(n-1)!}\int_0^\tau (\tau - x)^{n-1} BS_i(x)dx + \sum_{j=0}^{n-1} \frac{\tau^j}{j!}\gamma_{n-j}.$$

let $K_n(\tau, x) = \frac{(\tau - x)^{n-1}}{(n-1)!}$,

and $B_i^n = \int_0^\tau K_n(\tau,x) BS_i(x)dx$, i=0, …,M.

$$U(\tau) = \sum_{i=0}^{M} a_i B_i^n + \sum_{j=0}^{n-1} \frac{\tau^j}{j!}\gamma_{n-j}. \qquad ... (8)$$

in the same way we can get $U_i^{(s)}(\tau)$

$$U(\tau) = \sum_{i=0}^{M} a_i\, B_i^{n-s} + \sum_{j=0}^{n-s-1} \frac{\tau^j}{j!} \gamma_{n-s-j}\ . \qquad \dots (9)$$

Substituting Equation(8) and (9) in Equation (5), we get

$$\sum_{i=0}^{M} a_i\, B_i(\tau) + \sum_{j=0}^{n-1} \frac{\tau^j}{j!} \gamma_{n-j} = F_i(\tau) + \int_0^{\tau} K_{i,j}(\tau,x) \left[\sum_{i=0}^{M} a_i B_i^{n-s} + \sum_{j=0}^{n-s-1} \frac{\tau^j}{j!} a_{n-s-j}\right] dx\ . \qquad \dots (10)$$

Next, the interval $\tau \in [0,1]$ is divided into $h = \frac{1}{n}$ and $n = 0,1, \dots$,

and $U(0) = U_0$(initial condition) then $\tau_i = ih$, i=0,1,2,…,n

Then, using the collocation method with different value of τ in Equation (10) the results are algebraic equations. Solve the system utilizing the Gaussian elimination, the unknown coefficients will be found of $U(\tau)$.

4-Numerical Examples:

In this section**,** we give some numerical examples to illustrate the method for solving *n*-th order integro-differential equations with Boubaker scaling functions. The exact solution was used to compare with our approximate solution, we used Matlab to solve the examples.

Example1: Consider the following second order Volterra integro-differential equation [3]

$$u''(\tau) = e^{2\tau} - \int_0^{\tau} e^{2(\tau-x)} u'(x)dx,$$

with $u(0) = 0\ ,\ u'(0) = 0.$

The exact solution of equation is $u(\tau) = \tau e^{\tau} - e^{\tau} + 1.$

Solution:

Table 1 shows the numerical results for Example 1 with *M*=5 and *M*=7 compared with exact solution and graphically illustrated in Figure 1.

when *M*=5.

$$u_{appr.}(\tau) = (-0.0610560961)BS_0 + (0.0420583012)BS_1 + (0.0223207451)BS_2 + (0.0028266978)BS_3 + (0.0002904734)BS_4.$$

when *M*=7.

$$u_{appr.}(\tau) = (-0.0438230232)BS_0 + (0.0514840596)BS_1 + (0.0354706964)BS_2 + (0.0053510447)BS_3 + (0.0004865133)BS_4 + (0.000029400)BS_5 + (0.0000023027)BS_6.$$

τ	$u_{exact}(\tau)$	$u_{appr.}(\tau)$ $M=5$	Error	$u_{appr.}(\tau)$ $M=7$	Error
0	0	0	0	0	0
0.1	0.00534617	0.005292333	$5.3e^{-5}$	0.0053451220	$1.0e^{-6}$
0.2	0.022877793	0.022895291	$1.7e^{-5}$	0.0228781686	$3.7e^{-7}$
0.3	0.055098834	0.055145293	$4.6e^{-5}$	0.0550995061	$6.7e^{-7}$
0.4	0.104905181	0.104914179	$8.8e^{-6}$	0.1049054824	$3.0e^{-7}$
0.5	0.175639364	0.175609187	$3.0e^{-5}$	0.1756395190	$1.5e^{-7}$
0.6	0.271152479	0.271172954	$2.0e^{-5}$	0.2711528061	$3.2e^{-7}$
0.7	0.395874187	0.396083518	$2.0e^{-4}$	0.3958746029	$4.1e^{-7}$
0.8	0.554891814	0.555354317	$4.6e^{-4}$	0.5548921410	$3.2e^{-7}$
0.9	0.754039688	0.754534191	$4.9e^{-4}$	0.7540401317	$4.4e^{-7}$
1.0	1.000000000	0.999707380	$2.9e^{-4}$	0.9999998780	$1.2e^{-7}$

Table 1. Numerical results for Example1

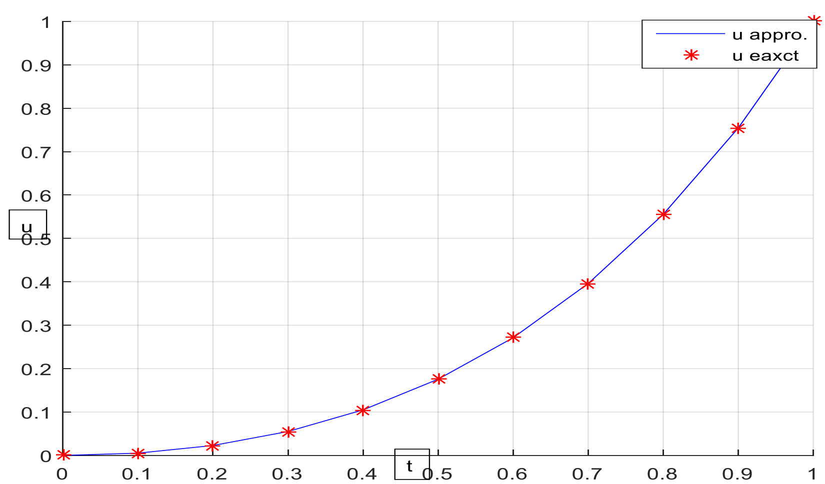

Figure1.

Example 2: Consider the following fifth order Volterra integro-differential equation [16]

$$u^{(5)}(\tau) = -2\sin(\tau) + 2\tau cos(\tau) - \tau + \int_0^\tau (\tau + x)u^{(3)}(x)dx,$$

$$u(0) = 1, u'(0) = 0, u''(0) = -1, u^{(3)}(0) = 0, u^{(4)}(0) = 1.$$

The exact solution of equation is : $u(\tau) = \cos(\tau)$.

Solution:

Table 2 shows the numerical results for Example 2 compared with exact solution and graphically illustrated in Figure 2.

when M=5.

$$u_{appr.}(\tau) = (0.7283302899)BS_0 + (-0.0442551091)BS_1 + (-0.0214573875)BS_2 + (0.0005067314)BS_3 + (0.0000929923)BS_4.$$

when M=7.

$$u_{appr.}(\tau) = (0.7281691943)BS_0 + (-0.0441939664)BS_1 + (-0.0214109421)BS_2 + (0.0004544063)BS_3 + (0.0001109284)BS_4 + (-0.0000014154)BS_5 + (-0.0000002172)BS_6.$$

τ	$u_{exact}(\tau)$	$u_{appr.}(\tau)$ M=5	Error	$u_{appr.}(\tau)$ M=7	Error
0	1.0000000000	1.0000000000	0	1.0000000000	0
0.1	0.9950041652	0.9951205585	$1.1e^{-4}$	0.9950047095	$5.4e^{-7}$
0.2	0.9800665778	0.9802154689	$1.4e^{-4}$	0.9800672548	$6.7e^{-7}$
0.3	0.9553364891	0.9554791154	$1.4e^{-4}$	0.9553371380	$6.4e^{-7}$
0.4	0.9210609940	0.9211866824	$1.2e^{-4}$	0.9210615895	$5.9e^{-7}$
0.5	0.8775825618	0.8776941549	$1.1e^{-4}$	0.8775831076	$5.4e^{-7}$
0.6	0.8253356149	0.8254383184	$1.0e^{-4}$	0.8253360936	$4.7e^{-7}$
0.7	0.7648421872	0.7649367583	$9.4e^{-5}$	0.7648425794	$3.9e^{-7}$
0.8	0.6967067093	0.6967878610	$8.1e^{-5}$	0.6967070509	$3.4e^{-7}$
0.9	0.6216099682	0.6216708127	$6.0e^{-5}$	0.6216103637	$3.9e^{-7}$
1.0	0.5403023058	0.5403456003	$4.3e^{-5}$	0.5403027544	$4.4e^{-7}$

Table2. Numerical results for Example2

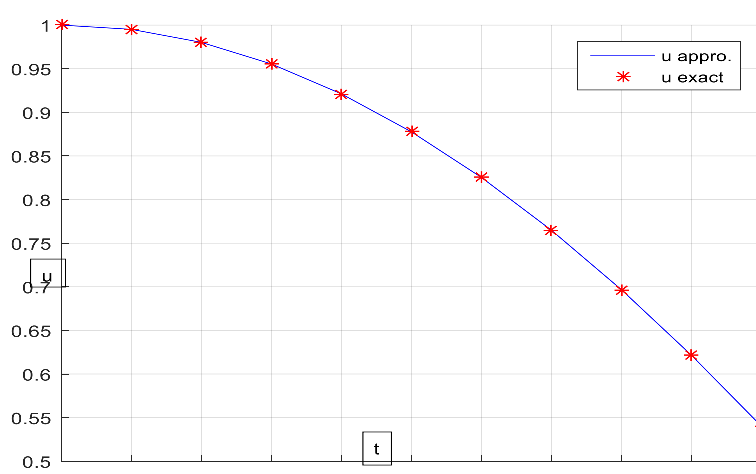

Figure 2.

Example 3: Consider the following integro-differential equation [13]

$$u'(\tau) = 1 + \sin(\tau) + \int_0^\tau u(x)dx,$$

with initial condition $u(0) = -1$.

The exact solution is $u(\tau) = \frac{1}{4}e^{\tau} - \frac{3}{4}e^{-\tau} - \frac{1}{2}\cos(\tau)$.

Solution:

Table 3 shows the numerical results for Example 3 compared with exact solution and graphically illustrated in Figure 3.

when *M*=5.

$u_{appr.}(\tau) =$

$(-0.5389105260)BS_0 + (0.1805365508)BS_1 + (0.0049808924)BS_2 + (0.0013205608)BS_3 + (-0.0000387400)BS_4$.

when *M*=7.

$u_{appr.}(\tau) =$

$(-0.538553824)BS_0 + (0.1804422361)BS_1 + (0.004891838)BS_2 + (0.001444374)BS_3 + (-0.0000855137)BS_4 + (0.0000057381)BS_5 + (0.000001066)BS_6$.

τ	$u_{exact}(\tau)$	$u_{appr.}(\tau)$ M=5	Error	$u_{appr.}(\tau)$ M=7	Error
0	-1.0000000000	-1.0000000000	0.0	-1.0000000000	0.0
0.1	-0.8998374166	-0.9001224091	$2.8e^{-4}$	-0.8998384865	$1.0e^{-6}$
0.2	-0.7987306641	-0.7991071168	$3.7e^{-4}$	-0.7987319440	$1.2e^{-6}$
0.3	-0.6958172081	-0.6962033170	$3.8e^{-4}$	-0.6958185192	$1.3e^{-6}$
0.4	-0.5903143571	-0.5906938654	$3.7e^{-4}$	-0.5903157023	$1.3e^{-6}$
0.5	-0.4815089580	-0.4818952798	$3.8e^{-4}$	-0.4815103556	$1.3e^{-6}$
0.6	-0.3687468344	-0.3691577395	$4.1e^{-4}$	-0.3687482972	$1.4e^{-6}$
0.7	-0.2514218946	-0.2518650859	$4.4e^{-4}$	-0.2514234398	$1.5e^{-6}$
0.8	-0.1289648456	-0.1294348221	$4.6e^{-4}$	-0.1289664851	$1.6e^{-6}$
0.9	-0.0008314511	-0.0013181132	$4.8e^{-4}$	-0.0008331733	$1.7e^{-6}$
1.0	0.13350972330	0.1330002138	$5.0e^{-4}$	0.1335079119	$1.8e^{-6}$

Table 3. Numerical results for Example3

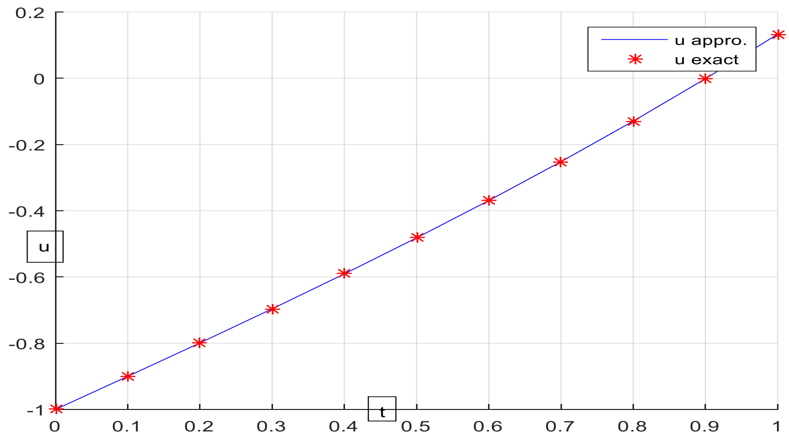

Figure 3.

Example 4: Consider the following first order integro-differential equation [1]

$$u'(\tau) - u(\tau) = e^{\tau} - e + \int_0^1 u(x)dx - \int_0^{\tau} u(x)dx\,,$$

with conditions $u(0) = 1,\ u(0) + \int_0^1 u(x)dx = e.$

and $u(\tau)_{exact} = e^{\tau}.$

Solution:

Table 4 shows the numerical results for Example 4 compared with exact solution and graphically illustrated in Figure 4.

$u_{appr.}(\tau) =$
$(0.8507923172)BS_0 + (0.224671857)BS_1 + (0.0285193826)\,BS_2 + (0.0021770614)BS_3 + (0.0002177037)BS_4$.

when M=7.

$$u_{appr.}(\tau) = (0.85148860)BS_0 + (0.2246735129)BS_1 + (0.0283746961)BS_2 + (0.0023705916)BS_3 + (0.0001494896)BS_4 + (0.0000069165)BS_5 + (0.0000004319)BS_6.$$

τ	$u_{exact}(\tau)$	$u_{appr.}(\tau)$ M=5	Error	$u_{appr.}(\tau)$ M=7	Error
0	1.0000000000	1.0000000000	0	1.0000000000	0
0.1	1.1051709180	1.1045607812	$6.1e^{-4}$	1.1051676194	$3.2e^{-6}$
0.2	1.2214027581	1.2204789529	$9.2e^{-4}$	1.2213984894	$4.2e^{-6}$
0.3	1.3498588075	1.3487411235	$1.1e^{-3}$	1.3498540696	$4.7e^{-6}$
0.4	1.4918246976	1.4905234716	$1.3e^{-3}$	1.4918194751	$5.2e^{-6}$
0.5	1.6487212707	1.6471917457	$1.5e^{-3}$	1.6487154960	$5.7e^{-6}$
0.6	1.8221188003	1.8203012644	$1.8e^{-3}$	1.8221124190	$6.3e^{-6}$
0.7	2.0137527074	2.0115969164	$2.1e^{-3}$	2.0137456496	$7.0e^{-6}$
0.8	2.2255409284	2.2230131605	$2.5e^{-3}$	2.2255331376	$7.7e^{-6}$
0.9	2.4596031111	2.4566740256	$2.9e^{-3}$	2.4595946031	$8.5e^{-6}$
1.0	2.7182818284	2.7148931105	$3.3e^{-3}$	2.7182725646	$9.2e^{-6}$

Table 4. Numerical results for Example4

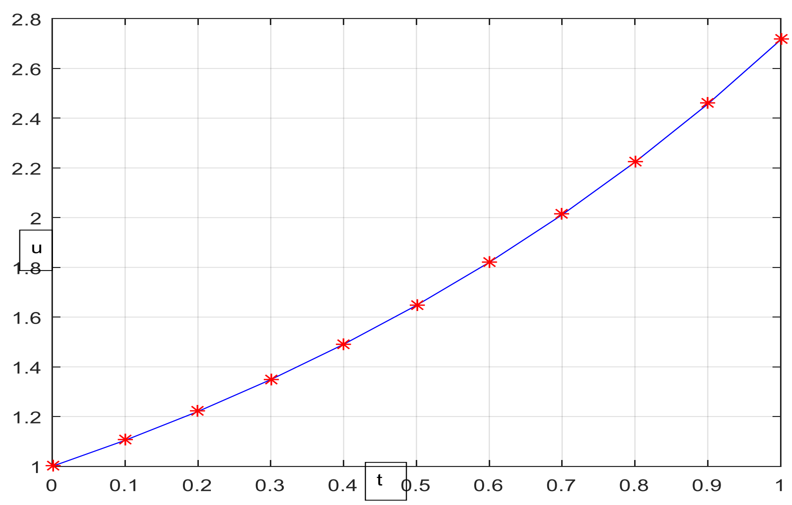

Figure 4.

5- Conclusions

The main purpose of this paper to use an efficient and accurate method to solve different order of integro-differential equation with the initial conditions by using collocation method together with Boubaker Scaling functions. A new approach method was used to reduce the problem into the solution of algebraic equations. Illustrated examples are included to show the validity and powerful method with good approximate results and rapid convergence are achieved.

Author Contributions: All authors contributed equally in writing this article. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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