[Al-Qadisiyah Journal of Pure Science](https://qjps.researchcommons.org/home)

Volume 27 | Number 1 Article 8

4-7-2022

About Some Numerical SemigroupsWith Embedding Dimension Three

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Recommended Citation

Çelik, Ahmet; Alakuş, Mehmet Sait; and İlhan, Sedat (2022) "About Some Numerical SemigroupsWith Embedding Dimension Three," *Al-Qadisiyah Journal of Pure Science*: Vol. 27: No. 1, Article 8.
DOI: 10.29350/qjps.2022.27.1.1468
Available at: https://qjps.researchcommons.org/home/vol27/iss1/8

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Article History

Received on: 13/2/2022
Revised on: 21/2/022
Accepted on: 28/2/2022

Keywords: Telescopic numerical semigroups, Arf closure, Embedding dimension.

DOI: https://doi.org/10.29350/ jops. 2022.27. 1.1468

ABSTRACT

We will examine a family of numerical semigroups embedding dimension three, such that $S_u = \langle 8, 8u+2, 8u+9 \rangle$ where $u \geq 1, u \in \square$. Also, we will give some relations between these semigroups and their Arf closure.

1.Introduction

Let ¥ be nonnegative integers set. The subset S of ¥ is a numerical semigroup if S is submonoid of ¥ under addition and $Card(¥ \backslash S)$ is finite. If $S = < c_1, c_2, ..., c_r >$ such that $c_1 < c_2 < ... < c_r$ and $(c_1, c_2, ..., c_r) = 1$ then $M = \{c_1, c_2, ..., c_r\}$ is called generator set of S. In thıs case, the number r is called embedding dimension of S, and denote by $d(S)$. Let S be a numerical semigroup S, the first positive element which it is non zero of S is called multiplicity of S, and denote by $m(S)$. For a numerical semigroup S, it is known that $m(S)$ £ $d(S)$. Thus, a numerical semigroup S has maximum

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embedding dimension if $m(S)=d(S)$. Also, $f(S)=\max(x\in \mathbb{Z}: x\notin S)$ and $d(S)=Card(\{0,1,2,\ldots,f(S)\}\cap S)$ are called Frobenius number and determine number of S, respectively. ([4], [6]). The numerical semigroup S is symmetric if $f(S)-t\in S$, for all $t\in \mathbb{Z}\backslash S$. On the other hand, it is known that the every numerical semigroup generated by two elements is symmetric, and if S is a symmetric numerical semigroup then $d(S)=\dfrac{f(S)+1}{2}$ ([5], [6]). If S is a numerical semigroup such that $S=<c_1,c_2,\ldots,c_n>$, then we can express

$$S=<c_1,c_2,\ldots,c_n>=\{s_0=0,s_{,}s_2,\ldots,s_{n-1},s_n=f(S)+1,\rightarrow\ldots\}$$

where $s_j<s_{j+1}$, $n=d(S)$, and the arrow means all $x\in S$, such that $x>f(S)+1$, for $j=1,2,\ldots,n=d(S)$. The element w is said gap of S if $w\in \mathbb{N}$ but $w\notin S$. We denote the set of gaps of S, by $G(S)$, i.e, $G(S)=\{w\in \mathbb{N}: w\notin S\}$. So, the $g(S)=Card(G(S))$ is called the genus of S, and it is well known $f(S)+1=g(S)+d(S)$, and if S is a symmetric then $d(S)=g(S)$. $S=<c_1,c_2,c_3>$ is a telescopic numerical semigroup if $c_3\in<\dfrac{c_1}{x},\dfrac{c_2}{x}>$, where $x=(c_1c_2)$. Let $S=<c_1,c_2,\ldots,c_n>$ be a numerical semigroup, then

$$L(S)=L=\langle c_1,c_2-c_1,c_3-c_1,\ldots,c_n-c_1\rangle$$

is called Lipman numerical semigroup of S, and it is known that

$$L_0(S)=S\subseteq L_1(S)=L(L_0(S))\subseteq L_2=L(L_1(S))\subseteq\ldots\subseteq L_p=L(L_{p-1}(S))\subseteq\ldots\subseteq\mathbb{N}$$

([7], [8], [1], [3]) .

S is called Arf numerical semigroup if $c_1+c_2-c_3\in S$, for all $c_1,c_2,c_3\in S$, where $c_1\geq c_2\geq c_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and it is denoted by $Arf(S)$ ([2]).

In this paper, we will examine some results on a family of numerical semigroups embedding dimension three, such that $S_u=\langle 8,8u+2,8u+9\rangle$ where $u\geq 1, u\in\mathbb{N}$. Also, we will give some relations between these semigroups and their Arf closure.

2. Main Results

Proposition 1. ([2]) Let S be a numerical semigroup and let $m_k = m(L_k)$ where L_k is the k th term of the Lipman sequence of semigroups of S for each $k \geq 0$. Let $f(Arf(S)) = f^{(a)}$, $n(Arf(S) = n^{(a)}$. Then $n^{(a)} = l(S) = l$ and we have

$$Arf(S) = \{s_0^{(a)} = 0, s_1^{(a)}, s_2^{(a)}, ..., s_{l-1}^{(a)}, s_l^{(a)} = f^{(a)} + 1, \to ...\}$$

where

$$s_1^{(a)} = m_0 = m(S),$$
$$s_2^{(a)} = m_0 + m_1,$$
$$...$$
$$s_{l-1}^{(a)} = m_0 + m_1 + m_2 + ... + m_{l-2},$$
$$s_l^{(a)} = m_0 + m_1 + m_2 + ... + m_{l-2} + m_{l-1}$$

and

$$f^{(a)} = m_0 + m_1 + m_2 + ... + m_{l-2} + m_{l-1} - 1.$$

Proposition 2. ([8]) The numerical semigroup $S_u = \langle 8, 8u+2, y \rangle$ is telescopic such that $u \geq 1$, $u \in \mathbb{Z}$, $i \neq 0, 2, 4, ..., 2(u-1)$ and $y = 8u + 2 + (2i+1)$ is odd integer number.

Proposition 3. ([3]) Let $S = < c_1, c_2, ..., c_r >$ be a numerical semigroup and $a = (c_1, c_2, ..., c_{r-1})$. If $T = < \frac{c_1}{a}, \frac{c_2}{a}, ..., \frac{c_{r-1}}{a} >$ numerical semigroup then

(a) $f(S) = a.f(T) + (a-1).c_r$

(b) $g(S) = a.g(T) + \frac{(a-1)(c_r - 1)}{2}$.

Proposition 4. We have following equalities for the telescopic numerical semigroup $S_u = \langle 8, 8u+2, 8u+9 \rangle$, where $u \geq 1$, $u \in \mathbb{Z}$:

(a) $f(S_u) = 32u + 7$

(b) $n(S_u) = 16u + 4$

(c) $g(S_u) = 16u + 4$.

Proof. (a) $f(T) = 16u + 4 - 4u - 4 - 1 = 12u - 1$ since $a = (8, 8u + 2) = 2$, and

$$T = < \frac{8}{2}, \frac{8u+2}{2} > = < 4, 4u + 1 >,$$

where u ³ 1, u Î ¢ . So, we find $f(S_u) = 2(12u - 1) + (8u + 9) = 32u + 7$ by Proposition 3/(a).

(b)-(c) It is clear that

$$d(S_u) = g(S_u) = \frac{f(S_u) + 1}{2} = \frac{32u + 8}{2} = 16u + 4$$

since S_u is symmetric numerical semigroup.

Theorem 1. If $S_u = \langle 8, 8u + 2, 8u + 9 \rangle$ then

$$Arf(S_u) = \{0, 8, 16, 24, ..., 8u, 8u + 2, 8u + 4, 8u + 6, 8u + 8, ® ...\},$$

where u ³ 1, u Î ¢ .

Proof. It is evident $m_0 = 8$ since $L_0(S_u) = S_u$, and $L_1(S_u) = < 8, 8u - 6, 8u + 1 >$. Thus,

(1) If $8u - 6 < 8$ (if $u = 1$) then $S_1 = \langle 8, 10, 17 \rangle$ and we obtain

$L_1(S_1) = < 8, 2, 9 > = < 2, 9 >$, $m_1 = 2$,

$L_3(S_1) = < 2, 5 >$, $m_3 = 2$,

$L_4(S_1) = < 2, 3 >$, $m_4 = 2$,

and

$L_5(S_1) = < 1 > =$ ¥ , $m_5 = 1$.

So, we write $Arf(S_1) = \{0, 8, 10, 12, 14, 16, ® ...\}$.

(2) If $8u - 6 > 8$ (if u ³ 2) then

(a) If $u = 2$ then $S_2 = \langle 8, 18, 25 \rangle$ and we find

$L_0(S_2) = S_2$, $m_0 = 8$.

$$L_1(S_2) = <8,10,17>, \; m_1 = 8.$$

$$L_2(S_2) = <8,2,9> = <2,9>, \; m_2 = 2.$$

$$L_3(S_2) = <2,7>, \; m_3 = 2.$$

$$L_4(S_2) = <2,5>, \; m_4 = 2.$$

$$L_5(S_2) = <2,3>, \; m_5 = 2$$

and

$$L_6(S_2) = <1> = ¥, \; m_6 = 1.$$

Thus, we obtain that

$$Arf(S_2) = \{0,8,16,18,20,22,24,® \;...\}.$$

(b) if $u = 3$ then $S_3 = \langle 8,26,33 \rangle$ and we obtain $L_0(S_3) = S_3$, $m_0 = 8$.

(c) $L_1(S_3) = <8,18,25>, \; m_1 = 8,$

$$L_2(S_3) = <8,10,17>, \; m_2 = 8,$$

$$L_3(S_3) = <8,2,9> = <2,9>, \; m_3 = 2,$$

$$L_4(S_3) = <2,7>, \; m_4 = 2,$$

$$L_5(S_3) = <2,5>, \; m_5 = 2,$$

$$L_6(S_3) = <2,3>, \; m_6 = 2$$

and

$$L_7(S_3) = <1> = ¥, \; m_7 = 1.$$

So, we write

$$Arf(S_3) = \{0,8,16,24,26,28,30,32,® \;...\}.$$

If we make same operations for $u > 3$ then we obtain that $Arf(S_u)$ as follows

$$Arf(S_u) = \{0, 8, 16, 24, \ldots, 8u, 8u+2, 8u+4, 8u+6, 8u+8, \rightarrow \ldots\}.$$

So, the proof is completed.

Corollary 1 . We write following results for Arf closure of $S_u = \langle 8, 8u+2, 8u+9 \rangle$ telescopic numerical semigroup, for $u \geq 1, u \in \mathbb{Z}$:

(a) $f(Arf(S_u)) = 8u + 7$

(b) $d(Arf(S_u)) = u + 4$

(c) $g(Arf(S_u)) = 7u + 4$.

Proof.

(a) It is trivial.

(b) Let $p = Card(\{8, 16, 24, \ldots, 8u\})$ and $q = Card(\{8u+2, 8u+4, 8u+6, 8u+8\})$. In this case, we have $p = \frac{8u-8}{8} + 1 = u$ and $q = 4$. Thus, we obtain $d(Arf(S_u)) = p + q = u + 4$.

(c) We can easily see that

$$g(Arf(S_u)) = f(Arf(S_u)) + 1 - d(Arf(S_u)) = 8u + 7 + 1 - (u+4) = 7u + 4.$$

Corollary 2. There exists relations between $S_u = \langle 8, 8u+2, 8u+9 \rangle$ and $Arf(S_u)$ as follows, where $u \geq 1, u \in \mathbb{Z}$:

(a) $f(S_u) = f(Arf(S_u)) + 24u$

(b) $d(S_u) = d(Arf(S_u)) + 15u$

(c) $g(S_u) = g(Arf(S_u)) + 9u$

Corollary 3. It satisfies following equalities for the numerical semigroup $S_u = \langle 8, 8u+2, 8u+9 \rangle$, where $u \geq 1, u \in \mathbb{Z}$:

(a) $f(S_{u+1}) = f(S_u) + 32$

(b) $n(S_{u+1}) = n(S_u) + 16$

(c) $g(S_{u+1}) = g(S_u) + 16$.

Corollary 4. For the numerical semigroup $S_u = \langle 8, 8u+2, 8u+9 \rangle$, the following are available, where $u \geq 1, u \in \mathbb{Z}$:

(a) $f(Arf(S_{u+1})) = f(Arf(S_u)) + 8$

(b) $d(Arf(S_{u+1})) = d(Arf(S_u)) + 1$

(c) $g(Arf(S_{u+1})) = g(Arf(S_u)) + 7$.

Example. We take $u = 3$ in $S_u = \langle 8, 8u+2, 8u+9 \rangle$ telescopic numerical semigroup. Then, we write

$$S_3 = <8, 26, 33> = \left\{ \begin{array}{l} 0,8,16,24,26,32,33,34,40,41,42,48,49,50,52,56,57,58,59,60, \\ 64,65,66,67,68,72,73,74,75,76,78,80,81,82,83,84,85,86,88, \\ 89,90,91,92,93,94,96,97,98,99,100,101,102,104,\rightarrow \ldots \end{array} \right\}.$$

Also, we obtain

$f(S_3) = 103,\ d(S_3) = 52,\ g(S_3) = Card(G(S_3)) = 52,\ Arf(S_3) = \{0,8,16,24,26,28,30,32,\rightarrow \ldots\}$,
$f(Arf(S_3)) = 31,\ d(Arf(S_3)) = 7$

$G(Arf(S_3)) = \{1,2,3,4,5,6,7,9,10,11,12,13,14,15,17,18,19,20,21,22,23,25,27,29,31\}$

and

$g(Arf(S_3)) = Card(G(ArfS_3)) = 25$.

On the other hand, if we write $u = 4$ in $S_u = \langle 8, 8u+2, 8u+9 \rangle$ then we have

$$S_4 = <8, 34, 41> = \{0,8,16,24,32,34,40,41,42,48,49,\ldots,133,134,136,\rightarrow \ldots\}$$

and

$f(S_4) = 135,\ d(S_4) = 68,\ g(S_4) = 68,\ Arf(S_4) = \{0,8,16,18,24,32,34,36,38,40,\circledR \ldots\}$,

$f(Arf(S_4)) = 39,\ d(Arf(S_4)) = 8$ and $g(Arf(S_4)) = 32$.

So, we find

$$f(Arf(S_3)) + 24.3 = 31 + 72 = 103 = f(S_3),$$

$$d(Arf(S_3)) + 15.3 = 7 + 45 = 52 = d(S_3),$$

$$g(Arf(S_3)) + 9.3 = 25 + 27 = 52 = g(S_3),$$

$$f(S_3) + 32 = 103 + 32 = 135 = f(S_4),$$

$$d(S_3) + 16 = 52 + 16 = 68 = d(S_4),$$

$$g(S_3) + 16 = 52 + 16 = 68 = g(S_4)$$

and

$$f(Arf(S_3)) + 8 = 31 + 8 = 39 = f(Arf(S_4)),$$

$$d(Arf(S_3)) + 1 = 7 + 1 = 8 = d(Arf(S_4)),$$

$$g(Arf(S_3)) + 7 = 25 + 7 = 32 = g(Arf(S_4)).$$

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