Sets And Functions In Terms Of Local Function

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ABSTRACT

In this paper, a collection of sets is investigated in such a way that the collection splits in the collection of preopen sets and the collection of b-open sets and also splits in the collection of semi-open sets and the collection of b-open sets. Functions in terms of this sets is a part of this paper and it will be considered that these collection remains invariant under homeomorphic image.

1. Introduction

Initially the author Levine [24] in 1963 introduced the generalization of open set in topological spaces. After that a good number of researchers have studied the generalization and introduced more generalized open sets. Some of the generalized open sets are: preopen sets of Mashhour et al [25], semi-open sets of Levine [24], semi-preopen sets of Andrijević [5,6] and α-sets of Njåstad [32]. In the field of generalization, the study of b-open set [3, 2, 6, 4, 8, 9, 11, 12, 17, 30, 35] is interesting because its collection contains both the collection of semi-open sets and preopen set, although the collection of semi-open sets and the collection of pre-open sets are not related to each other. It is also interesting that the collection of b-open sets contains the union of the collection semi-open sets and the collection of pre-open sets.

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For obtaining the generalized open sets, the study of ideal on the topological space is an admissible part. This study was introduced by Kuratowski [23] and Vaidyanathswamy [34]. The authors like Hashimoto [14], Hayashi [16], Hamlett and Janković [13, 18, 19], Modak [26, 27], Noiri [20], Al-Omari [1] gave new dimension on the study of ideal on topological spaces.

Through this paper we study sets, continuity, homeomorphism, lattice in terms of ideal of the topological space.

As preliminaries, we consider following:

Let (X, τ) be a topological space. Suppose $I \subseteq \wp(X)$ (the power set of X). Then I is called an ideal on X, if (i) $A \subseteq B \in I$ implies $A \in I$ and (ii) $A, B \in I$ implies $A \cup B \in I$. The study of local function like closure operator in terms of ideal on topological spaces was introduced by Kuratowski [23]. However, the mathematicians like Hamlett, Janković, Modak, Bandyopadhyay, Noiri, Popa, Al-Omari, Hayashi, Hashimoto, Hatir, Vaidyanathswamy (see [1, 13, 15, 14, 16, 18, 27, 34]) have used the modern notation of this local function and its formal definition is given below:

Definition 1.1. [23, 34] Let I be an ideal on a topological space (X, τ) and $A \subseteq X$. The local function $()^* : \wp(X) \to \wp(X)$ of A with respect to I and τ is

$$A^*(I, \tau) = \{x \in X \mid U_x \cap A \notin I, U_x \in \tau(x)\},$$ where $A \in \wp(X)$ and $\tau(x) = \{U \in \tau \mid x \in U\}$.

Simple notation of the local function of A is A^*.

Using this local function, Natkaniec has defined a set operator Ψ [31] by the equation,

$\Psi(A) = X \setminus (X \setminus A)^*$, where $A \in \wp(X)$.

The researchers like Hamlett and Janković [13, 18], Noiri [1], Bandyopadhyay and Modak [7, 27] have studied it in details.

Now we consider following tools for the study of this paper.

Definition 1.2. Let I be an ideal on a topological space (X, τ) and $A \subseteq X$. Then A is said to be a

(i) b-open [6], if $A \subseteq \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$.
(ii) Ψ-set [7], if $A \subseteq \text{Int}(\text{Cl}(\Psi(A)))$.
(iii) I -open [19], if $A \subseteq \text{Int}(A^*)$.
(iv) α-I -open [15], if $A \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A)))$.
(v) Ψ-C set [27], if $A \subseteq \text{Cl}(\Psi(A)))$.
(vi) Ψ^*-set [26], if $A \subseteq (\Psi(A))^*$.
(vii) f_I -set [20], if $A \subseteq (\text{Int}(A))^*$.
(viii) Δ-set [29], if $A \subseteq (\Psi(A))^* \cup \Psi(A^*)$.
(ix) regular-I -closed set [21], if $A = (\text{Int}(A))^*$.

The collection of all b-open sets (resp. Ψ-sets, I -open sets, Ψ-C sets, α-I -open sets, Ψ^*-sets, f_I -set, Δ-set, regular-I -closed set) in (X, τ, I) is denoted by $BO(X, \tau)$ (resp. τ^Ψ, $I O(X, \tau)$, $\Psi(X, \tau)$, $\alpha I O(X, \tau)$, $\Psi^*(X, \tau)$, $f_I (X, \tau)$, $\Delta(X, \tau)$, $R_I C(X, \tau)$).

Theorem 1.3. Let I be an ideal on a topological space (X, τ). Then the following properties hold:

(i) $f_I (X, \tau) \subseteq \Psi^*(X, \tau)$.

(ii) $f_I (X, \tau) \subseteq \Psi(X, \tau)$.

Proof. (i) Let $A \in f_I (X, \tau)$. Then $A \subseteq (Int(A))^* \subseteq (\Psi(Int(A)))^* \subseteq (\Psi(A))^*$. Therefore

$A \in \Psi^*(X, \tau)$.

(ii) Let $A \in f_I (X, \tau)$. Then $A \subseteq (Int(A))^* \subseteq (\Psi(Int(A)))^* \subseteq (\Psi(A))^* \subseteq Cl(\Psi(A))$. Therefore A $\in \Psi(X, \tau)$.

The converse relations of the above theorem are not true and it is followed from the following example:

Example 1.4. Let X = $\{o_1, o_2, o_3\}$, $\tau = \{\emptyset, \{o_1, o_3\}, X\}$ and $I = \{\emptyset, \{o_3\}\}$. Let A = $\{o_1\}$. Then Int(A) = $\emptyset$, so $(Int(A))^* = \emptyset^* = \emptyset$. So A $\notin f_I (X, \tau)$. Again $(\Psi(A))^* = \{o_1, o_3\}^* = X$. Therefore A $\in \Psi^*(X, \tau)$. Also $Cl(\Psi(A)) = Cl(\{o_1, o_3\}) = X$. Therefore A $\in \Psi(X, \tau)$.

Before starting the main section we recall followings:

Lemma 1.5. [33] Let $f : X \to Y$ be a bijection function. If I is a proper ideal on X, then f (I) = $\{f (I) \mid I \in I \}$ is a proper ideal on Y.

Lemma 1.6. [33] Let $f : X \to Y$ be a surjective function. If I is a proper ideal on Y, then $f^{-1}(I) = \{f^{-1}(I) \mid I \in I \}$ is a proper ideal on X.

2. Sets via local function

Definition 2.1. Let I be an ideal on a topological space (X, τ) and $A \subseteq X$. Then A is said to be a b∗-set, if $A \subseteq Int(A^*) \cup (Int(A))^*$.

If $I = \{\emptyset\}$, the b∗-set and b-open set coincide. Again for $I = \wp(X)$, $A^* = \emptyset$ (where $A \subseteq X$). In this case, no non-empty subset of X is a b∗-set. Thus, the study of b∗-set will be interesting, when the ideal is non-trivial proper ideal.

The collection of all b∗-sets is denoted by b∗(X, τ).

Examples of a b∗-set and a non b∗-set:

Example 2.2. Let $\mathbb{R}$ be the set of reals and $\mathbb{R}_u$ be the usual topology on $\mathbb{R}$. Let $I = \{\emptyset\}$. Take A=□ . Then Int(A)=$\emptyset$. This implies $(Int(A))^* = \emptyset$. Also $A^* = Cl(A) = \mathbb{R}$ implies $Int(A^*) = \mathbb{R}$. Therefore $A \subseteq Int(A^*) \cup (Int(A))^*$. Thus A is a b∗-set.

Example 2.3. Let X= $\mathbb{R}$, $\tau = \{\emptyset, \mathbb{R}, □ \}$ and $I = \wp(□)$. Take A=□ . Then Int(A)=□ . This implies $(Int(A))^* = \emptyset$. Also $A^* = \emptyset$ implies $Int(A^*) = \emptyset$. Therefore $A \nsubseteq Int(A^*) \cup (Int(A))^*$. Thus A not is a b∗-set.

∗-topology τ^* on X: for an ideal I on a topological space (X, τ), one consider a new topology finer than τ by the basis $\beta(I , \tau) = \{V \setminus I : V \in \tau, I \in I \}$ and the topology is denoted as τ^* and it is called ∗-topology [14]. In this content, more finer topologies have been obtained in [7]. We have on ∗-topology,

$b_*(X, \tau) \subseteq b_*(X, \tau^*)$, since $A^*(X, \tau) = A^*(X, \tau^*)$ (see [18]) and $Int(A) \subseteq Int^*(A)$. Further, for any topology τ' finer than τ (except τ^*) on X, we have $b_*(X, \tau) \not\subseteq b_*(X, \tau')$ and $b_*(X, \tau') \not\subseteq b_*(X, \tau)$.

Example 2.4. Let $X = \{o_1, o_2\}$, $\tau = \{\emptyset, X\}$, $\tau' = \{\emptyset, \{o_1\}, X\}$ and $I = \{\emptyset, \{o_1\}\}$. Then $b_*(X, \tau) = \{\emptyset, \{o_2\}, X\}$ and $b_*(X, \tau') = \{\emptyset\}$. Thus $b_*(X, \tau) \not\subseteq b_*(X, \tau')$.

Example 2.5. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, \{o_1\}, X\}$, $\tau' = \{\emptyset, \{o_1\}, \{o_2, o_3\}, X\}$, and $I = \{\emptyset, \{o_1\}\}$. Thus $b_*(X, \tau') \not\subseteq b_*(X, \tau)$.

Proposition 2.6. Let I be an ideal on a topological space (X, τ) and A, B and $A_i \in \wp(X)$, $i \in \Lambda$. Then followings hold:

1. $\emptyset \in b_*(X, \tau)$.

2. If $A, B \in b_*(X, \tau)$, then $A \cup B \in b_*(X, \tau)$.

3. If $\{A_i : i \in \Lambda\} \subseteq b_*(X, \tau)$, then $\bigcup_i A_i \in b_*(X, \tau)$, where Λ is an index set.

Proof. 2. Let $A, B \in b_*(X, \tau)$. Then $A \subseteq Int(A^*) \cup (Int(A))^*$ and $B \subseteq Int(B^*) \cup (Int(B))^*$. Then $A \cup B \subseteq [Int(A^*) \cup (Int(A))^*] \cup [Int(B^*) \cup (Int(B))^*] = [Int(A^*) \cup [Int(B^*)] \cup [(Int(A))^* \cup (Int(B))^*] \subseteq Int(A^* \cup B^*) \cup (Int(A) \cup Int(B))^* \subseteq Int((A \cup B)^*) \cup (Int(A \cup B))^*$. Therefore $A \cup B \in b_*(X, \tau)$.

Given that arbitrary union of b_*-sets is a b_*-set, although the operator $*$ is not distributive over arbitrary union.

Example 2.7. Let $\mathbb{R}$ be the set of reals. Consider the usual topological space $(\mathbb{R}, \mathbb{R}_u)$. Let $I = \{\emptyset\}$. Then for the set $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$, each $\{\frac{1}{n}\}^* = \{\frac{1}{n}\}$ but $(\bigcup_n \{\frac{1}{n}\})^* = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$.

However, intersection of two b_*-sets may not be a b_*-set.

Example 2.8. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, X\}$, $I = \{\emptyset, \{o_1\}\}$. Let $A = \{o_1, o_2\}$, $B = \{o_1, o_3\}$. Then $A \cap B = \{o_1\}$. We have $A^* = B^* = X$. Then $Int(A^*) \cup (Int(A))^* = X$ and $Int(B^*) \cup (Int(B))^* = X$. Therefore $A, B \in b_*(X, \tau)$. Now $A \cap B = \{o_1\}$ implies $(A \cap B)^* = \emptyset$. It follows that $Int((A \cap B)^*) \cup (Int(A \cap B))^* = \emptyset$. Therefore $A \cap B \not\in b_*(X, \tau)$.

However, following result holds:

Proposition 2.9. Let I be an ideal on a topological space (X, τ). Then for $B \in b_*(X, \tau)$ and $U \in \tau$, $B \cap U \subseteq U \cap (Int(B))^* \cup Int(B^*)$.

Proof. Given $B \subseteq (Int(B))^* \cup Int(B^*)$. Then $U \cap B \subseteq U \cap (Int(B))^* \cup Int(B^*) = [U \cap (Int(B))^*] \cup [U \cap Int(B^*)] = [U \cap (Int(B))^*] \cup Int(U \cap B^*) = (U \cap Int(B))^* \cup Int(U \cap B^*)$ [18]$= (Int(U \cap B))^* \cup Int((U \cap B)^*)$.

Due to Njåstad [32], we have following corollary:

Corollary 2.10. For the ideal I, on a topological space (X, τ),

$\tau(b_*(X, \tau)) = \{W \subseteq X : W \cap B \in b_*(X, \tau), \text{ whenever } B \in b_*(X, \tau)\}$

is a topology on X with $\tau \subseteq \tau$ (b∗(X, τ)).

Theorem 2.11. Let I be an ideal on a topological space (X, τ). Then the following properties hold:

(i) f_I (X, τ) $\subseteq$ b∗(X, τ).
(ii) I O(X, τ) $\subseteq$ b∗(X, τ).
(iii) R_I C(X, τ) $\subseteq$ b∗(X, τ).

Proof. (*i*) Follows from the fact that $A \subseteq (Int(A))^* \subseteq Int(A^*) \cup (Int(A))^*$.

(ii) Follows from the fact that $A \subseteq Int(A^*) \subseteq Int(A^*) \cup (Int(A))^*$.
(iii) Follows from the fact that $A = Int(A))^* \subseteq Int(A^*) \cup (Int(A))^*$.

Converse relations of the above theorem are not true in general:

Example 2.12. *In the Example* 1.4.*, the set A is a b∗-set but it is not a* f_I *-set.*

More example in the infinite space is given below:

Example 2.13. Let $\mathbb{R}$ be the set of reals. Consider the usual topological space ($\mathbb{R}$, $\mathbb{R}_u$). Let I = {∅}. Consider A=□ . Then Int(A)=∅. This implies (Int(A))*=∅. Also A*= Cl(A)= $\mathbb{R}$. This implies A⊆ Int(A*) ∪ (Int(A))*. Therefore A is a b∗-set. But □ ⊄ (Int(□))*. So A is not f_I -set.

Example 2.14. *Let X = {o_1, o_2, o_3, o_4}, τ = { ∅, {o_1}, {o_2}, {o_1, o_2}, X} and I = { ∅, {o_3}}. Let A ={o_1, o_3}. Then A*={o_1, o_3, o_4} implies Int(A*) = {o_1}. So A ⊈ Int(A*). Therefore A ⊄̸ I O(X, τ). Now (Int(A))* ={o_1}* = {o_1, o_3, o_4}. So A ∈ b ∗(X, τ).*

Example 2.15. *In Example 2.14., the set A is a b∗-set, but it is not a regular- I -closed set, since A ≠ (Int(A))*.*

More example in the infinite space is given below:

Example 2.16. Let $\mathbb{R}$ be the set of reals. Consider the usual topological space ($\mathbb{R}$, $\mathbb{R}_u$). Let I = {∅}. Consider A=□ . Then Int(A)= ∅. This implies (Int(A))*= ∅. Also A*= Cl(A)= $\mathbb{R}$. This implies A ⊆ Int(A*) ∪ (Int(A))*. Therefore A is a b∗-set. Again since (Int(A))*= ∅. Therefore A ≠ (Int(A))*. So A is not a regular- I -closed set.

Theorem 2.17. *Let I be an ideal on a topological space (X, τ) and τ ∩ I = {∅}. Then τ ⊆ b∗(X, τ).*

Proof. Let $U \in \tau$. Since the space is Hayashi-Samuel, $U \subseteq U^*$. Then $U = Int(U) \subseteq Int(U^*) \subseteq$

$Int(U^*) \cup (Int(U))^*$. Hence $U \in$ b∗(X, τ).

The above theorem may not be true, if the space is not Hayashi-Samuel. For this, we consider the following example:

Example 2.18. *Let X = {o_1, o_2, o_3}, τ = { ∅, {o_1}, X} and I = { ∅, {o_1}}. Then the space is not Hayashi-Samuel. Consider A = {o_1}. Then A* = ∅ implies Int(A*) = ∅ and (Int(A))* = ∅. Therefore Int(A*) ∪ (Int(A))* = ∅. Therefore A ⊈ Int(A*) ∪ (Int(A))*. Thus A is not a b∗-set, but A is an open set.*

Theorem 2.19. *Let I be an ideal on a topological space (X, τ) and τ ∩ I = {∅}. Then b∗(X, τ)*

$\subseteq \Delta(X, \tau)$

Proof. Let $A \in b_*(X, \tau)$. Then $A \subseteq Int(A^*) \cup (Int(A))^* \subseteq \Psi(Int(A^*)) \cup (\Psi(Int(A)))^* \subseteq$

$\Psi(A^*) \cup (\Psi(A))^*$.

    The above theorem may be false if the space does not satisfy $\tau \cap I = \{\emptyset\}$ and it follows from the following example:

Example 2.20. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, \{o_3\}, \{o_1, o_3\}, \{o_2, o_3\}, X\}$ and $I = \{\emptyset, \{o_3\}\}$. Then the space is not Hayashi-Samuel. Let $A = \{o_1, o_2\}$. We see that $A^* = A = \{o_1, o_2\}$. Therefore $Int(A^*) = \emptyset$ and $(Int(A))^* = \emptyset$. Therefore $A \nsubseteq Int(A^*) \cup (Int(A))^*$. Therefore $A \notin b_*(X, \tau)$. Now $\Psi(A^*) = X$ and $(\Psi(A))^* = \{o_1, o_2\}$. Therefore $A \subseteq (\Psi(A))^* \cup \Psi(A^*)$. Hence $A \in \Delta(X, \tau)$.

Theorem 2.21. Let I be an ideal on a topological space (X, τ). Then for $U \in \tau$ and $A \in b_*(X, \tau)$, $U \cap A \in b_*(X, \tau)$.

Proof. Let U be open and $A \in b_*(X, \tau)$. Then $A \subseteq Int(A^*) \cup (Int(A))^*$. Then $U \cap A \subseteq U \cap [Int(A^*) \cup (Int(A))^*] \subseteq [U \cap Int(A^*)] \cup [U \cap (Int(A))^*] = [Int(U) \cap Int(A^*)] \cup [U \cap (Int(A))^*] \subseteq Int(U \cap A^*) \cup (U \cap Int(A))^* \subseteq Int((U \cap A)^*) \cup (Int(U \cap A))^*$. Hence $U \cap A \in b_*(X, \tau)$.

Proposition 2.22. [22] Let I be an ideal on a topological space (X, τ). Then for $U \in \alpha I O(X, \tau)$ and $A \in f_I (X, \tau)$, $U \cap A \in f_I (X, \tau)$.

Corollary 2.23. Let (X, τ, I) be an ideal topological space. Then for $U \in \alpha I O(X, \tau)$ and

$A \in f_I (X, \tau)$, $U \cap A \in b_*(X, \tau)$.

    Proof. Proof follows from Proposition 2.22. and Theorem 2.11. (i).

Proposition 2.24. Let I be an ideal on a topological space (X, τ) and $\tau \cap I = \{\emptyset\}$. Then $b_*(X, \tau) \subseteq BO(X, \tau)$.

Proof. Let $A \in b_*(X, \tau)$. Then $A \subseteq Int(A^*) \cup (Int(A))^* \subseteq Int(Cl(A)) \cup Cl(Int(A))$.

    The converse is not true and it follows from the following example:

Example 2.25. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, X\}$ and $I = \{\emptyset, \{o_1\}\}$. Let $A = \{o_1\}$. Then $A \in BO(X, \tau)$. Now $A^* = \emptyset$ and $Int(A) = \emptyset$. Therefore $Int(A^*) = \emptyset$ and $(Int(A))^* = \emptyset$. Therefore $A \notin b_*(X, \tau)$.

If we consider the collection $I_n (X, \tau)$ of all nowhere dense subsets of a topological space

(X, τ), then we have the following theorem:

Theorem 2.26. Let $I_n (X, \tau)$ be the ideal on a topological space (X, τ). Then $b_*(X, \tau) = BO(X, \tau)$.

Proof. Let $A \in BO(X, \tau)$. Then $A \subseteq Int(Cl(A)) \cup Cl(Int(A)) = Int(Cl(Int(Cl(A)))) \cup Cl(Int(Cl(Int(A)))) = Int(A^*) \cup (Int(A))^*$. Hence $A \in b_*(X, \tau)$.

Theorem 2.27. Let I be an ideal on a topological space (X, τ) and $\tau \cap I = \{\emptyset\}$. Then $X \in b_*(X, \tau)$.

Proof. We have $X = X^*$. Then $Int(X^*) = X$ and $(Int(X))^* = X^* = X$. Hence $X \subseteq Int(X^*) \cup (Int(X))^*$.

However, it is noted that the above theorem may not be true, if the space does not satisfy the condition $\tau \cap I = \{\emptyset\}$. For this, we give the following example:

Example 2.28. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, \{o_3\}, X\}$ and $I = \{\emptyset, \{o_3\}\}$. Then $\tau \cap I = \{\emptyset\}$. Now $Int(X^*) = \emptyset$ and $(Int(X))^* = X^* = \{o_1, o_2\}$. Therefore $X \not\subseteq Int(X^*) \cup (Int(X))^*$. Hence $X \not\in$ b∗(X, τ).

Theorem 2.29. Let I be an ideal on a topological space (X, τ) and $A \subseteq X$. If A is a I-open and f_I-set, then A is b∗-set.

The converse is not true and it is followed from the following example:

Example 2.30. In Example 2.13., A is a b∗-set but A is not f_I-set. Thus the converse of the above theorem is not true.

Characterization of b∗**-set:**

Since the interior, closure, ()∗ and Ψ operators obey the following relations: $Int(A) = X \setminus Cl(X \setminus A)$ and $A^* = X \setminus \Psi(X \setminus A)$, then characterization of b∗-set is:

Theorem 2.31. Let I be an ideal on a topological space (X, τ) and $A \subseteq X$. Then A is a b∗-set, if and only if $Cl(\Psi(X \setminus A)) \cap \Psi(Cl(X \setminus A)) \subseteq X \setminus A$.

Proof. Let $A \in$ b∗(X, τ). Then $A \subseteq Int(A^*) \cup (Int(A))^*$. Then $X \setminus [Int(A^*) \cup (Int(A))^*] \subseteq X \setminus A$. This implies $[X \setminus [Int(A^*)] \cap [X \setminus (Int(A))^*] \subseteq X \setminus A$. Then $Cl(X \setminus A^*) \cap \Psi(X \setminus Int(A)) \subseteq X \setminus A$. Thus $Cl(\Psi(X \setminus A)) \cap \Psi(Cl(X \setminus A)) \subseteq X \setminus A$.

If $I = I_n(X, \tau)$, then the characterization will be:
A subset A of X is a b∗-set, if and only if $Cl(Int(X \setminus A)) \cap Int(Cl(X \setminus A)) \subseteq X \setminus A$.

We get following consequence of this section:
(i) The collection b∗(X, τ) does not form a topology.
(ii) The collection b∗(X, τ) does not form a lattice under the operations 'union', 'intersection', and 'complement' and the special element $\emptyset$ and X. Because intersection of two b∗-sets is not a b∗-set and the complement of a b∗-set is not necessarily a b∗-set, it is followed by the following example.

Example 2.32. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, \{o_1\}, \{o_2, o_3\}, X\}$ and $I = \{\emptyset, \{o_1\}\}$. Take $A = \{o_2, o_3\}$. Then $Int(A^*) = (Int(A))^* = \{o_2, o_3\}$. Therefore $A \in$ b∗(X, τ). Now $A^c = \{o_1\}$. Then $Int((A^c)^*) = (Int(A^c))^* = \emptyset$. Thus $A^c \not\in$ b∗(X, τ).

Furthermore, the collection b∗(X, τ) does not form an ideal on X, because it is not necessary that $A \subseteq B \in$ b∗(X, τ) implies $A \in$ b∗(X, τ). It is followed by the following example:

Example 2.33. Let $X = \{o_1, o_2, o_3\}$, $\tau = \{\emptyset, X\}$ and $I = \{\emptyset, \{o_1\}\}$. Take $A = \{o_1, o_2\}$. Then $(Int(A))^* \cup Int(A^*) = X$ and so $A \in$ b∗(X, τ). Now $\{o_1\} \subseteq A$ but $\{o_1\} \not\in$ b∗(X, τ). Hence the collection b∗(X, τ) does not form an ideal on X.

However, following holds as well:

Lemma 2.34. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a homeomorphism and I be a proper ideal on (X, τ). Then

(i) $f(I) = \{f(I) : I \in I\}$ is an ideal on (Y, σ).

(ii) $f[A_*(I, \tau)] = (f(A))^*(f(I), \sigma)$.

Proof. (ii) For $A \in \wp(X)$, let $x \in X$ with $f(x) \notin f[A^*(I, \tau)]$. This implies that $x \notin A^*(I, \tau)$. Thus there exists $U_x \in \tau(x)$ such that $U_x \cap A \in I$. Therefore, $f(U_x \cap A) \in f(I)$, hence $f(U_x) \cap f(A) \in f(I)$. So, we have, $f(x) \notin [f(A)]^*(f(I), \sigma)$, since $f(U_x) \in \sigma(f(x))$. Therefore, $f[A^*(I, \tau)] \supseteq [f(A)]^*(f(I), \sigma)$.

Converse part:
Let $t \in X$ with $f(t) \notin (f(A))^*(f(I), \sigma)$. Then there exists $U_{f(t)} \in \sigma(f(t))$ such that $U_{f(t)} \cap f(A) \in$

$f(I)$. Thus, $f^{-1}[U_{f(t)} \cap f(A)] \in f^{-1}(f(I)) = I$, implies $t \notin A^*(I, \tau)$, since $f^{-1}(U_{f(t)}) \in \tau(t)$. Thus, $f(t) \notin f(A^*(I, \tau))$. Hence we have, $[f(A)]^*(f(I), \sigma) \supseteq f[A^*(I, \tau)]$.

Corollary 2.35. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a homeomorphism and I be a proper ideal on (X, τ). Then for $A \in b_*(X, \tau)$, $f(A) \in b_*(Y, \sigma)$.

3. Functions in terms of local function

In this section, the various types of continuity have been discussed in terms of ideal on a topological space.

Definition 3.1. Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ). A function $f : X \rightarrow Y$ is said to be a

  (i) continuous, if $f^{-1}(U) \in \tau$, for every $U \in \sigma$.
  (ii) b-continuous [10], if $f^{-1}(U) \in BO(X, \tau)$, for every $U \in \sigma$.
  (iii) b∗-continuous, if $f^{-1}(U) \in b_*(X, \tau)$, for every $U \in \sigma$.
  (iv) Ψ∗-function [28], if $f^{-1}(U) \in \Psi^*(X, \tau)$, for every $U \in \sigma$.
  (v) I -continuous [31], if $f^{-1}(U) \in f_I(X, \tau)$, for every $U \in \sigma$.
  (vi) f_I -continuous [20], if $f^{-1}(U) \in f_I(X, \tau)$, for every $U \in \sigma$.
  (vii) Ψ−C-function, if $f^{-1}(U) \in \Psi(X, \tau)$, for every $U \in \sigma$.
  (viii) R_I C-continuous [20], if $f^{-1}(U) \in R_I C(X, \tau)$, for every $U \in \sigma$.
  (ix) Δ-continuous, if $f^{-1}(U) \in \Delta(X, \tau)$, for every $U \in \sigma$.

Theorem 3.2. Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ). Then a f_I -continuous function $f : X \rightarrow Y$ is a Ψ∗-function.

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a f_I -continuous function, therefore $f^{-1}(U) \in f_I(X, \tau) \subseteq \Psi^*(X, \tau)$. Hence f is a Ψ∗-function.
    The converse is not true:

Example 3.3. Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_2\}\}$ and $\tau = \{\emptyset, \{o_1, o_2\}, X\}$. Define a function $f : X \rightarrow X$ by $f(o_1) = o_2$, $f(o_2) = o_3$ and $f(o_3) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_2\}) = \{o_1, o_3\}$. Since $\{o_1, o_3\} \notin f_I(X, \tau)$, f is not f_I -continuous, but $\emptyset, X, \{o_1, o_3\} \in \Psi(X, \tau)$. Hence f is a Ψ∗-function.

Theorem 3.4. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ). Then a f_I -continuous function $f : X \rightarrow Y$ is a Ψ−C-function.*

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a f_I -continuous function, therefore $f^{-1}(U) \in f_I(X, \tau) \subseteq \Psi(X, \tau)$. Hence f is a Ψ−C-function.

For the converse of the above theorem, we consider the following example:

Example 3.5. *Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_2\}\}$ and $\tau = \{\emptyset, \{o_1, o_2\}, X\}$. Define a function f : $X \rightarrow X$ by $f(o_1) = o_2$, $f(o_2) = o_3$ and $f(o_3) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_2\}) = \{o_1, o_3\}$. Since $\{o_1, o_3\} \not\subseteq f_I$ (X, τ), f is not f_I -continuous, but $\emptyset, X, \{o_1, o_3\} \in \Psi(X, \tau)$. Hence f is a $\Psi - C$-function.*

Theorem 3.6. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ). Then every f_I -continuous function $f : X \rightarrow Y$ is a b∗-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a f_I -continuous function, therefore $f^{-1}(U) \in f_I$ $(X, \tau) \subseteq$ b∗(X, τ). Hence f is a b∗-continuous function.

About the converse of the above theorem, we discuss following example:

Example 3.7. *Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_2\}\}$ and $\tau = \{\emptyset, \{o_1, o_2\}, X\}$. Define a function $f : X \rightarrow X$ by $f(o_1) = o_2$, $f(o_2) = o_3$ and $f(o_3) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_2\}) = \{o_1, o_3\}$. Since $\{o_1, o_3\} \not\subseteq f_I$ (X, τ), f is not f_I -continuous, but $\emptyset, X, \{o_1, o_3\} \in$ b∗(X, τ). Hence f is a b∗-continuous function.*

Theorem 3.8. *Let (X, τ) and (Y, σ) be two topological space and I be an ideal on (X, τ). Then every R_I C-continuous function $f : X \rightarrow Y$ is a b∗-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a R_I C-continuous function, therefore $f^{-1}(U) \in R_I$ $C(X, \tau) \subseteq$ b∗(X, τ). Hence f is a b∗-continuous function.

The converse is not true:

Example 3.9. *Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_2\}\}$ and $\tau = \{\emptyset, \{o_1, o_2\}, X\}$. Define a function $f : X \rightarrow X$ by $f(o_1) = o_2$, $f(o_2) = o_3$ and $f(o_3) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_2\}) = \{o_1, o_3\}$. Since $\emptyset, X, \{o_1, o_3\} \in$ b∗(X, τ), f is a b∗-continuous function, but f is not a R_I C-continuous function, since $\{o_1, o_3\} \neq (Int(\{o_1, o_3\}))^*$.*

Theorem 3.10. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ). Then every I-continuous function $f : X \rightarrow Y$ is a b∗-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a I -continuous function, therefore $f^{-1}(U) \in$ I $O(X, \tau) \subseteq$ b∗(X, τ). Hence f is a b∗-continuous function.

Against the converse of the above theorem, we discuss the following example:

Example 3.11. *Let $X = \{o_1, o_2, o_3, o_4\}$, $I = \{\emptyset, \{o_3\}\}$, $\tau_1 = \{\emptyset, \{o_1\}, \{o_2\}, \{o_1, o_2\}, X\}$ and $\tau_2 = \{\emptyset, \{o_1, o_3\}, X\}$. Define a function $f : (X, \tau_1, I) \rightarrow (X, \tau_2)$ by $f(o_1) = o_3$, $f(o_2) = o_2$, $f(o_3) = o_1$ and $f(o_4) = o_4$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_3\}) = \{o_1, o_3\}$. Since $\emptyset, X, \{o_1, o_3\} \in$ b∗(X, τ), f is a b∗-continuous function, but f is not a I -continuous function, since $\{o_1, o_3\} \not\subseteq Int(\{o_1, o_3\}^*)$.*

Theorem 3.12. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ) with $\tau \cap I = \{\emptyset\}$. Then every continuous function $f : X \rightarrow Y$ is a b∗-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \rightarrow Y$ is a continuous function, therefore $f^{-1}(U) \in \tau \subseteq$ b∗(X, τ). Hence f is a b∗-continuous function.

Converse of the above theorem may not be true:

Example 3.13. *Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_2\}\}$ and $\tau = \{\emptyset, \{o_1, o_2\}, X\}$. Define a function $f : X \to X$ by $f(o_1) = o_2$, $f(o_2) = o_3$ and $f(o_3) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1, o_2\}) = \{o_1, o_3\}$. Since $\{o_1, o_3\} \not\subseteq \tau$, f is not continuous, but $\emptyset, X, \{o_1, o_3\} \in$ b∗(X, τ). Hence f is a b∗-continuous function.*

Theorem 3.14. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ) with $\tau \cap I = \{\emptyset\}$. Then every b∗-continuous function $f : X \to Y$ is a Δ-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \to Y$ is a b∗-continuous function, therefore $f^{-1}(U) \in$ b∗(X, τ)⊆ Δ(X, τ). Hence f is a Δ-continuous function.

For the converse of the above theorem, we discuss following example:

Example 3.15. *Let $X = \{o_1, o_2\}$, $I = \{\emptyset, \{o_1\}\}$ and $\tau = \{\emptyset, \{o_1\}, X\}$. Define a function $f : X \to X$ by $f(o_1) = o_2$, $f(o_2) = o_1$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1\}) = \{o_2\}$. Then $\Delta(\{o_2\}) = (\Psi(\{o_2\}))^* \cup \Psi((\{o_2\})^*) = X$, so $\emptyset, \{o_2\}, X \in \Delta(X, \tau)$. Hence f is Δ-continuous. Since $\{o_2\} \not\subseteq$ b∗(X, τ), hence f is not b∗-continuous.*

Theorem 3.16. *Let (X, τ) and (Y, σ) be two topological spaces and I be an ideal on (X, τ) with $\tau \cap I = \{\emptyset\}$. Then every b∗-continuous function $f : X \to Y$ is a b-continuous function.*

Proof. Let $U \in \tau$. Since $f : X \to Y$ is a b∗-continuous function, therefore $f^{-1}(U) \in$ b∗(X, τ)⊆ BO(X, τ). Hence f is a b-continuous function.

Example 3.17. *Let $X = \{o_1, o_2, o_3\}$, $I = \{\emptyset, \{o_1\}\}$, $\tau_1 = \{\emptyset, X\}$ and $\tau_2 = \{\emptyset, \{o_1\}, X\}$. Define a function $f : (X, \tau_1, I) \to (X, \tau_2)$ by $f(x) = x$, for all $x \in X$. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$ and $f^{-1}(\{o_1\}) = \{o_1\}$. Since $\emptyset, X, \{o_1\} \in BO(X, \tau_1)$, f is a b-continuous function, but f is not a b∗-continuous function, since $\{o_1\} \not\subseteq$ b∗(X, τ).*

Theorem 3.18. *For the ideal $I_n(X, \tau)$ on the topological space (X, τ), a function is b∗-continuous function if and only if it is a b-continuous function.*

Proof. Proof follows from the fact that, in (X, τ, I_n), BO(X, τ) = b∗(X, τ).

References

[1]   A. Al-Omari and T. Noiri, Local closure functions in ideal topological spaces, Novi Sad J. Math., 43(2)(2013), 139-149.

[2]   A. Al-Omari and M. Noorani, Decomposition of continuity via b-open set, Bol. Soc. Paran. Mat., 26(1-2)(2008), 53-64.

[3]   A. Al-Omari and M. Noorani, On generalized b-closed sets, Bull. Malays. Math. Sci. Soc., 32(1)(2009), 19-30.

[4]   Z. A. Ameen, On types of generalized closed sets, J. Taibah Univ. Sci., 12(3)(2018), 290-293.

[5]   D. Andrijević, Semi-preopen sets, Mat. Vesnik, 38(1986), 24-32.

[6]   D. Andrijević, On b-open sets, Mat. Vesnik, 48(1996), 59-64.

[7]   C. Bandyopadhyay and S. Modak, A new topology via Ψ-operator, Proc. Nat. Acad. Sci. India, 76(A)(IV)(2006), 317-320.

[8]   M. Caldas, S. Jafari and R. M. Latif, Applications of b-open sets and (b, s)-continuous funcions, King Fahd University of Petroleum and Min-erals. Dept. of Math. Sci. Technical Report Series, TR 400(2008).

[9]   C. Carpintero, N. Rajesh and E. Rosas, Operation-b-open sets in topological spaces, Fasc. Math., 48(2012), 13-21.

[10]   E. Ekici and M. Caldas, Slightly γ-continuous functions, Bol. Soc. Paran. Math., 22(2)(2004), 63-74.

[11]   M. E. El-Monsef, A. A. A. El-Atik and M. M. El-Sharkasy, Some topologies induced by b- open sets, Kyungpook Math. J., 45(2005), 539-547.

[12]   M. Ganster and M. Steiner, On some questions about b-open sets, Questions Answers Gen. Topology, 25(1)(2007), 45-52.

[13]   T. R. Hamlett and D. Janković, Ideals in Topological Spaces and the Set Operator Ψ, Boll. U.M.I., 7(4-B)(1990), 863-874.

[14]   H. Hashimoto, On the $*$-topology and its applications, Fund. Math., 91(1976), 5-10.

[15]   E. Hatir and T. Noiri, On decomposition of continuity via idealization, Acta Math. Hungar, 96(2002), 341-349.

[16]   E. Hayashi, Topologies defined by local properties, Math. Ann., 156(1964), 205-215.

[17]   H. Z. Ibrahim, Bc-open sets in topological spaces, Advances in Pure Math., 3(2013), 34-40.

[18]   D. Janković and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(1990), 295-310.

[19]   D. Janković and T. R. Hamlett, Compatible extensions of ideals, Boll. U.M.I., 7(6-B)(1992), 453-465.

[20]   A. Keskin, T. Noiri and S. Yuksel, f_I -sets and decomposition of R_I C-continuity, Acta Math. Hungar., 104(2004), 307-313.

[21]   A. Keskin, T. Noiri and S. Yuksel, Idealization of decomposition theorem, Acta Math. Hungar., 102(2004), 269-277.

[22]   A. Keskin, T. Noiri and S. Yuksel, A note on f_I -sets and f_I -continuous functions, Acta Math. Hungar., 107(4)(2005), 287-293.

[23]   K. Kuratowski, Topology, Vol. I., Academic Press, New York (1966).

[24] N. Levine, Semi-Open Sets and Semi-Continuity in Topological Spaces, Amer. Math. Monthly, 70(1)(1963), 36-41.

[25] A. S. Mashhour, M. E. A. El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. [Egypt], 53(1982), 47-53.

[26] S. Modak, Some new topologies on ideal topological spaces, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci, 82(3)(2012), 233-243.

[27] S. Modak and C. Bandyopadhyay, A note on Ψ- operator, Bull. Malyas. Math. Sci. Soc., 30(1)(2007), 43-48.

[28] S. Modak and Md. M. Islam, On $*$ and Ψ operators in topological spaces with ideals, Trans. A. Razmadze Math. Inst., 172(2018), 372-377.

[29] S. Modak and Md. M. Islam, Generalized open sets vis-a-vis Δ-sets, Konuralp J. Math., 8(1)(2020), 50-56.

[30] A. A. Nasef, On b-locally closed sets and related topics, Chaos, Solitons and Fractals., 12(2001), 1909-1915.

[31] T. Natkaniec, On I -continuity and I -semicontinuity points, Math. Slovaca., 36(3)(1986), 297-312.

[32] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(3)(1965), 961-970.

[33] Sk. Selim, T. Noiri and S. Modak, Ideals and associated filters on topological spaces, Eurasian Bull. Math., 2(3)(2019), 80-85.

[34] R. Vaidyanathswamy, The localisation theory in set topology, Proc. India Acad. Sci., 20(1945), 51-61.

[35] I. Zorlutuna, On b-closed spaces and θ -b-continuous functions, The Arab. J. Sci. Eng., 34(2A)(2009), 205-216.