[Al-Qadisiyah Journal of Pure Science](https://qjps.researchcommons.org/home)

Volume 27 | Number 1 Article 10

4-7-2022

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Al-Qadisiyah Journal of Pure Science Vol.(27) Issue (1) (2022)math pp.103-111

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Article History

Received on:5 / 3/2022
Revised on: 8 / 5 /2022
Accepted on: 21 /5 /2022

Keywords: Random dynamical system, minimal random set, Random periodic point, Almost random periodic point.

DOI: https://doi.org/10.29350/jops. 2022.27. 1.1489

ABSTRACT

The aim of this paper is to study of almost periodic points in random dynamical systems. Some of their characteristics were studied and theirs highlighted relationship with some concepts in random dynamical systems such as periodic points as well as minimal set.

1.Introduction

Random dynamical systems (RDSs) are the most widely used in the modeling of many phenomena in physics, climatology economics, biology, etc. The random effects frequently reproduce essential properties of these phenomena before just to reward for the faults in deterministic models. The history of the study of RDSs energies backbone to Ulam and von Neumann in 1945 [7] and succeeded since the1980s due to the detection that the solutions of stochastic ordinary differential equations profit a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. For deterministic dynamical system on metric spaces, the concept of almost periodic point is established well for details see [4]. Throughout this paper, X =metric space, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, X^{Ω} the set of all measurable functions from Ω to X, N the set of all nbds. In section 2, we present some definitions and the results [1], [2],[4],[5],[6]. In section 3, the concept of almost random periodic point is introduced and some essential properties of such set are proved.

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2.Preliminary

Definition 2.1[1]: A subset $E \subseteq \mathbb{R}$ is said to be relatively dense in $\mathbb{R}$,if there exists a positive number $l \in \mathbb{R}$ such that $E \cap [a, a+l] \neq \emptyset$ for all $a \in \mathbb{R}$ .Where $[a, a+l] \coloneqq \{t \in \mathbb{R}: a \leq t \leq a+l\}$**.**

Definition 2.2[6]: The 5-tuple $(\mathbb{R}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a **metric dynamical system**(Shortly MDS) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and

(i) $\theta: \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ −measurable ,

(ii) $\theta(0, \omega) = \omega$, $\forall \omega \in \Omega$,

(iii) $\theta(t+s, \omega) = \theta(t, \theta(s, \omega))$ $\forall$, $t, s \in \mathbb{R}, \omega \in \Omega$

(iv) $\mathbb{P}(\theta_t(F)) = \mathbb{P}(F)$, for every $F \in \mathcal{F}$ and every $t \in \mathbb{R}$.

Definition 2.3[6]: A topological RDS on the topological space X over the MDS $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a measurable RDS which satisfies in addition the following property: For each $\omega \in \Omega$ the function $\varphi(\cdot, \omega, \cdot): \mathbb{R} \times X \to X$, $(t, x) \mapsto \varphi(t, \omega, x)$, is continuous**.**

Definition 2.4[6]: A measurable random dynamical system on the measurable space$(X, \mathcal{B}(X))$ over (or covering, or extending) an MDS $(\mathbb{R}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ with time is a mapping $\varphi: \mathbb{R} \times \Omega \times X \to X$, with the following properties:

Measurability, φ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$ − measurable.

Cocycle property: The mappings $\varphi(t, \omega) \coloneqq \varphi(t, \omega, \cdot): X \to X$ form a cocycle over $\theta(\cdot)$, i. e. they satisfy

$\varphi(0, \omega, x) = x$ for all $\omega \in \Omega, x \in X$. (1)

$\varphi(t+s, \omega, x) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega, x)$ for all $s, t \in \mathbb{R}, \omega \in \Omega, x \in X$. (2)

If there is no ambiguity the RDS is denoted by (θ, φ) rather than $(\mathbb{R}, \Omega, X, \theta, \varphi)$.

Definition 2.5[6]: Let $(\Omega, \mathcal{F})$ be a measurable space and (X, d) be a metric space which is considered a measurable space with Borel σ- algebra $\mathcal{B}(X)$. The set-valued function $A: \Omega \to \mathcal{B}(X), \omega \longmapsto A(\omega)$, is said to be random set if for each $x \in$ X the function $\omega \longmapsto d(x, A(\omega))$is measurable. If $A(\omega)$is connected closed (compact) for all $\omega \in \Omega$, it is called a random connected closed(compact) set.

Definition 2.6 [2]: Let $D: \omega \longmapsto D(\omega)$ be a multifunction. We call the multifunction $\omega \longmapsto \gamma_D^t(\omega) \coloneqq \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)$, the tail (from the moment t) of the pull back trajectories emanating from D. If

$D(\omega) = \{v(\omega)\}$ is a single valued function, then $\omega \longmapsto \gamma_v(\omega) \equiv \gamma_D^0(\omega)$ is said to be the **(pull back) trajectory** emanating from v. $\omega \longmapsto \gamma_v(\omega) := \bigcup_{\tau\geq 0} \varphi(\tau, \theta_{-\tau}\omega)v(\theta_{-\tau}\omega)$.

In the deterministic case Ω is a one-point set and $\varphi(t, \omega) = \varphi(t)$ is a semigroup of continuous mappings.

Therefore in this case the tail γ_D^t has the form:

$$\gamma_D^t = \bigcup_{\tau\geq t} \varphi(\tau)D = \bigcup_{\tau\geq 0} \varphi(\tau)(\varphi(t)D) = \gamma_{\varphi(t)D}^0.$$

Definition(Invariance Property) 2.7[6]: Let (θ, φ) be a measurable RDS and a multifunction $\omega \longmapsto D(\omega)$ is said to be invariant with respect to (θ, φ) if $\varphi(t, \omega)D(\omega) = D(\theta_t\omega)$ for all $t > 0$ and $\omega \in \Omega$.

Definition 2.8[8]: A random variable $v \in X^\Omega$ is said to be **random periodic point** of a RDS (θ, φ) if $\forall t \in \mathbb{R}, \exists\, \tau \neq 0$ such that $\varphi(\tau + t, \omega)x(\omega) = \varphi(t, \theta_\tau\omega)x(\theta_\tau\omega)$.

Definition 2.9[3]: Let (θ, φ) be a random dynamical system. A random subset $M(\omega)$ of X is called minimal whenever it is non-empty random invariant set and random closed set while no proper random subset of $M(\omega)$ has these properties.

Corollary 2.10[4]: Let X, Y and Z be topological spaces and $f: X \times Y \to Z$ be continuous mapping and let A and B be compact subset of X and Y respectively, then $\forall\, w \in N_{A\times B} \exists (u, v) \in N_A \times N_B$ such that $u \times v \subseteq w$, consequently$\forall\, w' \in N_{f(A)\times f(B)} \exists (u, v) \in N_{f(A)} \times N_{f(B)}$ such that $f(u \times v) \subseteq w'$.

Theorem2.11[5]: Let (X, τ) be any topological space, let $A \subseteq X$ be any open set ,and let $B \subseteq X$ be any other set (not necessarily open or closed). If $A \cap \bar{B} \neq \emptyset$ then $A \cap B \neq \emptyset$.

3.MainResults

In this section the concept of almost random periodic point is stated and some new properties of such concept are studied.

Theorem3.1: Let $M(\omega)$ be anon-empty random set in X the following properties of $M(\omega)$ are equivalent :

(i) $M(\omega)$ is minimal in X

(ii) $\overline{\gamma_v^t(\omega)} = M(\theta_\omega)$ for every $v \in M(\omega)$

(iii) $M(\omega)$ is closed and invariant and for every non- empty random open subset U of X either

$$M(\omega) \cap (\cup\, \{\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}) = \emptyset \text{ or } M(\omega) \subset (\cup\, \{\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\})$$

Proof: (**i**) → (**ii**) if $\boldsymbol{v} \in M(\omega)$ then $\gamma_v^t(\omega) \subset M(\theta_\omega)$ and $\overline{\gamma_v^t(\omega)} \subset M(\theta_\omega)$ because $M(\omega)$ is closed and invariant. Since $\overline{\gamma_v^t(\omega)}$ is closed and invariant set we must have $\overline{\gamma_v^t(\omega)} = M(\theta_\omega)$.i.e. $\overline{\gamma_v^t(\omega)}$ will be a non-empty proper subset of $M(\theta_\omega)$, a contradiction to minimality of $M(\omega)$. $\overline{\gamma_v^t(\omega)} \neq M(\theta_\omega)$

(**ii**) → (**iii**) if (ii) is holds then $M(\omega)$ is closed and invariant , suppose U is anon-empty random open set in X such that

$M(\omega) \cap \{\cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}\} \neq \emptyset$ if $\boldsymbol{v} \in M(\omega)$ then by (ii)

$\overline{\gamma_v^t(\omega)} \cap \{\cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}\} \neq \emptyset$, since $\varphi(\tau,\theta_{-\tau}\omega): X \to X$ is homeomorphism $\forall \tau \in \mathbb{R}$ and $\forall \omega \in \Omega$, we get $\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega)$ is open $\forall \tau \in \mathbb{R}$ and $\forall \omega \in \Omega$,since $\overline{\gamma_v^t(\omega)} \cap \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega) \neq \emptyset$ $\forall \tau \in \mathbb{R}$ and $\forall \omega \in \Omega$, then by **Theorem2.11** we get $\gamma_v^t(\omega) \cap \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega) \neq \emptyset$ then there exist $y \in \gamma_v^t(\omega)$ *and* $y \in \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega)$ $\forall \tau \in \mathbb{R}$ and $\forall \omega \in \Omega$ .Since $y \in \gamma_v^t(\omega)$ then there exist $\tau_\circ \geq \tau \ni y = \varphi(\tau_\circ,\theta_{-\tau_\circ}\omega)v$ and $\tau_1 \geq \tau \ni y \in \varphi(\tau_1,\theta_{-\tau_1}\omega)U(\theta_{-\tau_1}\omega)$ then $\varphi(\tau_\circ,\theta_{-\tau_\circ}\omega)v \in \varphi(\tau_1,\theta_{-\tau_1}\omega)U(\theta_{-\tau_1}\omega)$, since $\varphi(\tau_1,\theta_{-\tau_1}\omega)U(\theta_{-\tau_1}\omega) \subset \cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}$,

Then $\varphi(\tau_\circ,\theta_{-\tau_\circ}\omega)v \in \cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}$

$$\Longrightarrow v \in \cup\left\{\varphi(\tau_\circ,\theta_{-\tau_\circ}\omega)^{-1} \circ \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\right\}$$

$$\in \cup\{\varphi(-\tau_\circ,\omega) \circ \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\} \ni \omega := \theta_{-\tau_\circ}\omega$$

$$\in \cup\{\varphi(\tau-\tau_\circ,\acute{\omega}) \circ \varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\} \ni \acute{\omega} := \theta_{-\tau}\omega$$

$$\Longrightarrow v \in \cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}$$

$$\Longrightarrow M(\omega) \subset \cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}.$$

(**iii**) → (**i**) let $N(\omega)$ be closed random invariant subset of $N(\omega)$,$N(\omega) \neq M(\omega)$. Then $U(\omega) := X / N(\omega)$ is anon-empty random set Then by (iii) either $M(\omega) \cap (\cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}) = \emptyset$

Or $M(\omega) \subset \cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in R\}$.

Now if $M(\omega) \cap (\cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}) = \emptyset$

$M(\omega) \cap (\cup\{\varphi(\tau,\theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}) = \emptyset$

$M(\omega) \cap (\cup \{U(\omega): \tau \in \mathbb{R}\}) = \emptyset \Longrightarrow M(\omega) \cap U(\omega) = \emptyset$

$M(\omega) \cap \ X \,/\, N(\omega) = \emptyset \Longrightarrow M(\omega) = N(\omega)$ or if

$M(\omega) \subset \cup \{\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\}$ then

$M(\omega) \subset \cup (\{\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in \mathbb{R}\})$

$\Longrightarrow M(\omega) \subset (\cup \{U(\omega): \tau \in \mathbb{R}\})$

$\Longrightarrow M(\omega) \subset U(\omega)$

$\Longrightarrow M(\omega) \subset X \,/\, N(\omega)$ .Then $N(\omega) \subset M(\omega) \subset X \,/\, N(\omega)$ i.e

$N(\omega) = \emptyset$. Thus $M(\omega)$ is minimal ■

Theorem 3. 2: $\overline{\gamma_v^t(\omega)}$ is minimal if and only if $\forall y \in X$ such that then $v \in \overline{\gamma_y^t(\theta_{-t}\omega)}$.

Proof: Suppose that $\overline{\gamma_v^t(\omega)}$ is minimal .Let $y \in X$, if $y \in \overline{\gamma_v^t(\theta_{-t}\omega)}$ then by **Theorem3.1** $\overline{\gamma_v^t(\theta_{-t}\omega)} = \overline{\gamma_y^t(\omega)}$ so $v \in \overline{\gamma_y^t(\omega)}$, conversely suppose that $\forall y \in X: y \in \overline{\gamma_v^t(\theta_{-t}\omega)}$ then $v \in \overline{\gamma_y^t(\theta_{-t}\omega)}$.We show that $\overline{\gamma_v^t(\omega)}$ is minimal , and is non-empty closed invariant random set. Suppose that $M(\omega)$ be anon-empty closed invariant subset of $\overline{\gamma_v^t(\omega)}$. If $y \in M(\omega)$ then $y \in \overline{\gamma_v^t(\omega)}$ and by hypothesis $v \in \overline{\gamma_y^t(\theta_{-t}\omega)}$.Since $M(\omega)$ is closed and invariant then $\overline{\gamma_y^t(\omega)} \subset M(\theta_t\omega)$ so $\overline{\gamma_y^t(\theta_{-t}\omega)} \subset M(\omega)$.Then $v \in M(\omega)$ by a similar argument we have $\overline{\gamma_v^t(\theta_{-t}\omega)} \subset M(\omega)$ but $\overline{\gamma_v^t(\omega)} \subset M(\omega) \equiv \overline{\gamma_v^t(\theta_{-t}\omega)} \subset M(\omega)$, thus we have $\overline{\gamma_v^t(\omega)} = M(\omega)$. This means that $\overline{\gamma_v^t(\omega)}$ is Minimal ■

Definition 3.3: A random variable ν is said to be random almost periodic point if for each open random set $U(\omega)$ with $\nu(\omega) \in U(\omega)$, $\forall\omega \in \Omega$, the set $D(\nu, U) \coloneqq \{t \in \mathbb{R}: \varphi(t, \theta_t\omega)\nu(\omega) \in U(\omega)\}$ is relatively dense.

Theorem3.4: If $\nu \in X$ is a random almost periodic point then $\varphi(s, \omega, \nu)$ is also random almost periodic point.

Proof: let $U(\omega)$ be an open random set with $\nu(\omega) \in U(\omega)$, $\forall\omega \in \Omega$ one has

: $D(\varphi(s, \theta_t\omega)\nu(\omega), U) \coloneqq \{t + s \in \mathbb{R}: \varphi(t + s, \omega)\nu(\omega) \in U(\omega)\}$

$$= \{t + s \in \mathbb{R}: \varphi(t, \theta_t\omega, \varphi(s, \omega))\nu(\omega) \in U(\omega)\}$$

$$= \{t + s \in \mathbb{R}: \varphi(t, \theta_t \omega) \circ \varphi(s, \omega)\nu(\omega) \in U(\omega)\}$$

$$= \{t + s \in \mathbb{R}: \varphi(t, \theta_t \omega)\nu(\omega) \in \varphi(-s, \theta_t \omega)U(\omega)\}$$

$$= D\big(\nu, \varphi(-s, \theta_t \omega)U(\omega)\big)$$

Since $D(\nu, U)$ is relatively dense, then $D\big(\nu, \varphi(-s, \theta_t \omega)U(\omega)\big)$ is also relatively dense. Hence $\varphi(s, \omega, \nu)$ is random almost periodic point ■

Theorem 3.5: The set of all random almost periodic point in RDS (θ, φ) is a random invariant set.

Proof: Let M be the set of all random almost periodic points define $M: \Omega \longrightarrow 2^X$to show that $\omega \longrightarrow M(\omega) \neq \emptyset$ is measurable, let $x \in X$ and $\delta > 0$ then $\big\{\omega: d\big(x, M(\omega)\big) < \delta\big\} = \{\omega: M(\omega) \cap B(x, \delta) \neq \emptyset\}$. Set $U \coloneqq B(x, \delta)$ be an open set implies $\{\omega: M(\omega) \cap U \neq \emptyset\}$ is measurable and hence $\omega \longrightarrow d\big(x, M(\omega)\big)$ is measurable and hence M is random set. Now since $\nu \in M$ then ν is random almost periodic points i.e. $\forall$ open random set $U(\omega)$ with $\nu(\omega) \in U(\omega)$, $\forall \omega \in \Omega$ the set $D(\nu, U) \coloneqq \{t \in \mathbb{R}: \varphi(t, \theta_t \omega)\nu(\omega) \in U(\omega)\}$ is relatively dense, then $\varphi(t, \theta_t \omega)\nu(\omega) \in M$. Hence M is random invariant set ■

Theorem 3.6: If ν is random periodic point in X, then ν is a random almost periodic point.

Proof: Let ν be a random periodic point and let $U(\omega)$ be a random open set with $\nu(\omega) \in U(\omega)$ and $\varphi(t, \theta_t \omega)\nu(\omega) \in U(\omega)$ and let $l \in \mathbb{R}$ be a positive number and $s \in [0, l]$. Now

$\varphi(s, \omega) \circ \varphi(t, \theta_t \omega)\nu(\omega) \in \varphi(s, \omega)U(\omega)$, $t \in \mathbb{R}$, implies$\varphi(t + s, \omega)\nu(\omega) \in \varphi(s, \omega)U(\omega)$

$$\Longrightarrow \varphi(t, \theta_s \omega)\nu(\theta_s \omega) \in \varphi(s, \omega)U(\omega)$$
$$\Longrightarrow \varphi(-s, \theta_t \omega) \circ \varphi(t, \theta_s \omega)\nu(\theta_s \omega)U(\omega)$$
$$\Longrightarrow \varphi(t - s, \theta_{t+s} \omega)\nu(\theta_s \omega) \in U(\omega)$$

Set $t \coloneqq t + s$

$$\Longrightarrow \varphi(t - s, \theta_t \omega)\nu(\theta_s \omega) \in U(\omega)$$
$$\Longrightarrow \{t - s \in \mathbb{R}: \varphi(t - s, \theta_t \omega)\nu(\theta_s \omega) \in U(\omega)\}$$
$$\Longrightarrow t - s \in D(\nu, U) \text{ implies that } [t - l, t] \cap D(\nu, U) \neq \emptyset,$$

so $D(\nu, U)$ is relatively dense and hence ν is random almost periodic point ■

Theorem3.7: Let (θ, φ) be a continuous RDS , $A \subseteq \mathbb{R}$ and $B(\omega) \subseteq X$ .If A is compact and $B(\omega)$ is closed then

$$C(\omega) \coloneqq \left\{\{\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega): t \in A, \forall\, \omega \in \Omega\}\right\}$$

is closed random set.

Proof: Let $x \in X/C(\omega)$, so $x \notin C(\omega)$. Then

$$x \notin \varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega) \text{ , for some } t \in A\,, \omega \in \Omega.$$

So

$$\varphi(t,\omega)x \notin \varphi(t,\omega) \circ \varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega) = B(\omega) \ ,$$

for some $t \in A\,, \omega \in \Omega$. This implies that

$$\{\varphi(s,\theta_{-s}\omega)x : s \in -A, \omega \in \Omega\} \subset X/B(\omega)$$

As $-A$ is compact subset of $\mathbb{R}$ ($\because\ \gamma: R \to R \ni \gamma(x) = -x$ is continuous and A is compact in $\mathbb{R}$ then $\gamma(A) = -A$ is compact in $\mathbb{R}$) and $X/B(\omega)$
is open in X it follows from the **Corollary2.10** then there exists anon-empty $u(\omega)$ open set such that $\{\varphi(s,\theta_{-s}\omega)u(\theta_{-s}\omega): s \in -A, \forall\, \omega \in \Omega\} \subset X/B(\omega)$

$\Longrightarrow \varphi(s,\theta_{-s}\omega)u(\theta_{-s}\omega) \subset X/B(\omega)$ for all $s \in -A$ and all $\omega \in \Omega$

$\Longrightarrow \varphi(s,\omega) \circ \varphi(s,\theta_{-s}\omega)u(\theta_{-s}\omega) \subset \varphi(t,\omega)(X/B(\omega))$

$\Longrightarrow u(\theta_{-s}\omega) \subset \left\{\varphi(s,\omega)\big(X/B(\omega)\big): s \in -A, \omega \in \Omega\right\}$

$= \{\varphi(s,\omega)(X)/\varphi(s,\omega)B(\omega): s \in -A, \omega \in \Omega\}$

$= X/C(\omega)$

Then $U(\omega) \cap C(\omega) = \emptyset$.Thus each point of $X/C(\omega)$ is interior point of $X/C(\omega)$then $X/C(\omega)$ is random closed set ■

Theorem 3.8: Let (θ,φ) be a continuous RDS with X is regular Hausdorff space and let $v \in X$.

i. If v is a random almost periodic point, then $\overline{\gamma_v^t(\omega)}$ is a minimal random subset of X.
ii. If $\overline{\gamma_v^t(\omega)}$ is compact and minimal, then v is a random almost periodic point.
iii. If X is locally compact space, then v is a random almost periodic point iff $\overline{\gamma_v^t(\omega)}$ is a compact minimal random set .

Proof: (i) By **Theorem 3.2.** let $y \in \overline{\gamma_v^t(\omega)}$,in order to prove that $v \in \overline{\gamma_y^t(\omega)}$, consider an open random set $U(\omega)$ with $v \in U(\omega)$.As X is regular, $U(\omega)$ may be assumed to be closed in X .Since v is a random almost periodic point, we have $\mathbb{R} = K + D(v, U)$ for some compact subset K of $\mathbb{R}$ then

$$\gamma_v^t(\omega) = \{\varphi(\tau, \omega)v(\omega): \tau \in \mathbb{R}\}$$

$$= \{\varphi(\tau, \omega)v(\omega): \tau \in K + D(v, U)\}$$

$$\because t \in K + D(v, U) \Longrightarrow \exists k \in K, d \in D \ni \tau = k + d$$

$$\Longrightarrow \gamma_v^t(\omega) = \{\varphi(k + d, \omega)v(\omega): k \in K, d \in D\}$$

$$= \{\varphi(k, \theta_d\omega) \circ \varphi(d, \omega)v(\omega): k \in K, d \in D\} \subseteq \varphi(k, \omega)U.$$

Now **Theorem 3. 7** impels that $\varphi(k, \omega)U$ is a closed subset of X hence $\overline{\gamma_v^t(\omega)} \subseteq \varphi(k, \omega)U$,consequently,$y \in \varphi(k, \omega)U$ when $\gamma_y^t(\omega) \cap U(\omega) \neq \emptyset$. This shows that $v \in \overline{\gamma_y^t(\omega)}$ ■

(ii) let $U(\omega)$ be an open random set with $v \in U(\omega)\ \forall \omega \in \Omega$. By assumption that $\overline{\gamma_v^t(\omega)}$ is a compact and minimal random there is a finite subset K of $\mathbb{R}$ such that $\overline{\gamma_v^t(\omega)} \subset \cup \{\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega): \tau \in K\}$. Now if $s \in \mathbb{R}$,then $\varphi(s, \theta_{-\tau}\omega)v(\omega) \in \overline{\gamma_v^t(\omega)}$, so $\varphi(s, \theta_{-\tau}\omega)v(\omega) \in \varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega)$ for some $\tau \in K$ that is $\varphi(-\tau + s, \theta_{-\tau}\omega)v(\omega) \in U(\theta_{-\tau}\omega) \Longrightarrow \varphi(-\tau, \theta_{-s}\omega) \circ \varphi(s, \omega)v(\omega)) \in U(\theta_{-\tau}\omega)$, hence $s \in \tau + D(v, U) \subseteq K + D(v, U)$. This show that $D(v, U)$ is relatively dense in $\mathbb{R}$ ■

(iii) Since X is locally compact space .Then $\forall\ x \in X\ \exists$ neigherborhood $\bar{V}$ to x such that $\bar{V}$ is compact set in X.Recall that $\forall\ v \in X\ \exists$ an open random set $U(\omega)$ with $v \in U(\omega)$ such that closure of $U(\omega)$ is compact in X.Now if v is random almost periodic point then $\overline{\gamma_v^t(\omega)}$ is minimal random set by **(i)** Since $\gamma_v^t(\omega) = \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega)v(\theta_{-\tau}\omega)$ thus $\gamma_v^t(\omega)$ is a compact minimal random set. Conversely, if $\gamma_v^t(\omega)$ is compact and minimal random set then by **(ii)** we get v is random almost periodic point ■

Corollary 3.9: If X is compact and $x \in X$,then x is a random almost periodic point iff $\overline{\gamma_x^t(\omega)}$ is a minimal random set.

Proof: Since X is compact and $x \in X$ then X is locally compact then by **Theorem 3.8 (iii)** we get x is a random almost periodic point iff $\overline{\gamma_v^t(\omega)}$ is a minimal random set ■

Conclusion

The main objective of this work is to study the almost periodic point in random dynamical systems, where some of its characteristics have been studied as well as its relationship with certain concepts in random dynamical systems such as periodic points as well as minimal sets.

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