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Blow up of Solutions for Viscoelastic Wave Equations of Kirchhoff Type with Variable Exponents

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ABSTRACT

In this paper, we consider a viscoelastic wave equation of Kirchhoff type with variable exponents. We show that under suitable conditions on the initial data and initial energy, the energy of solutions blow up in finite time with negative initial energy.

1.Introduction

In this paper, we study the following viscoelastic wave equation of Kirchhoff type:

$$u_{tt} - M(\|\nabla u\|^2 \Delta u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{p(x)-2}\, u_t = |u|^{q(x)-2}u, (x,t) \in \Omega \times (0,T), \quad (1)$$

with the initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (2)$$

and boundary condition

$$u(x,t) = 0,\ x \in \partial\Omega, \quad (3)$$

where Ω is a bounded domain in R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, and $M(s) = \alpha + \beta s^\gamma$, $\alpha, \beta \geq 0, \gamma \geq 1$. Without loss of generality, we can assume that $\alpha = \beta = 1$.

The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$2 \leq p^- \leq p(x) \leq p^+ \leq q^- \leq q(x) \leq q^+ \leq q^* \quad (4)$$

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here

$$p^- = essinf_{x\in\Omega}p(x),\ \ p^+ = esssup_{x\in\Omega}p(x),$$
$$q^- = essinf_{x\in\Omega}q(x),\ \ q^+ = esssup_{x\in\Omega}q(x),$$

and

$$q^* = \begin{cases} \infty, n = 1,2, \\ \dfrac{2n}{n-2}, n \geq 3. \end{cases}$$

We state some assumptions on g:

(A1) Let $g \in C^1$ function satisfying

$$1 - \int_0^\infty g(\tau)d\tau = l > 0.$$

(A2) $g(\tau)d\tau \geq 0$, $g'(\tau)d\tau \leq 0$ and

$$\int_0^\infty g(\tau)d\tau < \frac{q^-(1-\xi)-2}{q^-(1-\xi)-2+\frac{1}{4q^-(1-\xi)}}, \qquad 0 < \xi < 1.$$

The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, image processing, electrorheological fluids [2, 3, 15].

When $p(x)$ and $q(x)$ are constants, (1) become the following form

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{p-2}u_t = |u|^{q-2}u. \tag{5}$$

The problem (5) has been discussed by many authors and several results concerning blow up have been established for $p = 2$. See in this case, [6, 8].

When $M(\|\nabla u\|^2) \equiv 1$, (5) become the following form

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{p-2}u_t = |u|^{q-2}u. \tag{6}$$

In [9], the author proved blow up and global existence of solutions, for the equation (6). In [10], the same author extended this result in the case of positive initial energy.

When $M(\|\nabla u\|^2) \equiv 1$ and $g \equiv 0$ the problem (1) reduces to the following form

$$u_{tt} - \Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u. \tag{7}$$

Messaoudi et al. [11] studied the local existence and blow up of the solutions of the equation (7). Recently, Pişkin [12] investigated (1) and established the blow up result with negative initial energy for $g \equiv 0$. In [13] the same author investigated (1) and proved the blow up result for $M(\|\nabla u\|^2) \equiv 1$.

Motivated by the above results, in this work, we prove the blow up result of solutions (1) under some conditions. To the best our knowledge, this is the first paper that deals with blow up of solutions to problems involving a viscoelastic wave equations of Kirchhoff type with variable exponents.

The outline of this work is as follows. In part 2, we recall the definitions of the the variable exponent $L^{p(x)}(\Omega)$ Lebesgue space and $W^{1,p(x)}(\Omega)$ Sobolev space. In part 3, we state and prove the blow up results.

2. Preliminaries

In this part, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (see [3, 4, 7, 14]).

Let $p:\Omega \to [1,\infty)$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u:\Omega \to R, u \text{ is measurable and } \int_\Omega |u|^{p(x)}dx < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(x)} = inf\left\{\lambda > 0: \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

where $L^{p(x)}(\Omega)$ is a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

Variable exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Let the variable exponent $p(.)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \le \frac{A}{log\left|\frac{1}{x-y}\right|}, \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{8}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.1: [3] (*Poincare inequality*) Let Ω be a bounded domain of R^n and $p(.)$ satisfies log-Hölder condition, then

$$\|u\|_{p(.)} \le c\|\nabla u\|_{p(.)}, \qquad \text{for all } u \in W_0^{1,p(.)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2: [3] (*Embedding*) Let $p(.) \in C(\overline{\Omega})$ and $q:\Omega \to [1,\infty)$ be a measurable and satisfy

$$ess_{x\in\bar{\Omega}} inf\left(p^*(x) - q(x)\right) > 0.$$

Then the Sobolev embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \dfrac{np(x)}{ess_{x\in\bar{\Omega}} sup\left(n - p(x)\right)}, & if\ p^- < n, \\ \infty, & if\ p^- \geq n. \end{cases}$$

The local existence and uniqueness of solutions for the problem (1) which can be established by combining arguments of [1, 5, 11].

Theorem 2.3: (*Local existence and uniqueness*). Suppose that (A1), (A2), (4) and (8) holds. Then for any initial data $(u_0, u_1) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times L^2(\Omega)$, problem (1) has a solution

$$u \in C\left([0,T); H_0^1(\Omega) \cap H^2(\Omega)\right),\ \ u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p(.)}\left(\Omega \times (0,T)\right),$$

for some $T > 0$.

3. Blow up

In this part, we will consider the blow up of the solution for problem (1). Firstly, we give following lemmas:

Lemma 3.1: [11] If $q: \Omega \to [1, \infty)$ is a measurable function and

$$2 \leq q^- \leq q(x) \leq q^+ < \tfrac{2n}{n-2}; n \geq 3 \tag{9}$$

holds. Then, we have following inequalities:

i)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(\|\nabla u\|^2 + \rho_{q(.)}(u)\right), \tag{10}$$

ii)

$$\|u\|_{q^-}^{s} \leq c\left(\|\nabla u\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{11}$$

iii)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(|H(t)| + \|u_t\|^2 + \rho_{q(.)}(u)\right), \tag{12}$$

iv)

$$\|u\|_{q^-}^{s} \leq c\left(|H(t)| + \|u_t\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{13}$$

v)

$$c\|u\|_{q^-}^{q^-} \leq \rho_{q(.)}(u) \tag{14}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq q^-$. Where $\rho_{q(.)}(u) = \int_\Omega |u|^{q(.)}\, dx$, and $c > 1$ a positive constant and

$$H(t) = -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t g(\tau)\, d\tau\right)\|\nabla u\|^2 - \frac{1}{2(\gamma+1)}\ \|\nabla u\|^{2(\gamma+1)}$$
$$-\frac{1}{2}(g \circ \nabla u)(t) + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx.$$

Lemma 3.2: Assume that (A1), (A2), (4) and (8) hold. Then

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\, d\tau\right)\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\ \|\nabla u\|^{2(\gamma+1)}$$
$$+\frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \tag{15}$$

is a nonincreasing function and

$$E'(t) = -\int_\Omega |u_t|^{p(x)}\, dx - \frac{1}{2}g(\tau)\int_\Omega |\nabla u(t)|^2\, dxd\tau$$
$$+\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega [\nabla u(\tau) - \nabla u(t)]^2\, dxd\tau.$$

Proof. We multiply (1) by u_t, and integrate over Ω, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx + \frac{1}{2(\gamma+1)}\ \|\nabla u\|^{2(\gamma+1)}\right]$$
$$-\int_0^t\int_\Omega g(t-\tau)\,\nabla u(\tau)\nabla u_t(t)dxd\tau = -\int_\Omega |u_t|^{p(x)}\, dx. \tag{16}$$

We estimate the last term in the left-hand side as follows

$$\int_0^t\int_\Omega g(t-\tau)\,\nabla u(\tau)\nabla u_t(t)dxd\tau$$
$$= \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t)[\nabla u(\tau) - \nabla u(t) + \nabla u(t)]\, dxd\tau$$
$$= \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t)[\nabla u(\tau) - \nabla u(t)]\, dxd\tau$$
$$+\int_0^t g(t-\tau)\int_\Omega \nabla u_t(t)\nabla u(t)\, dxd\tau$$
$$= -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\left[\int_\Omega [\nabla u(\tau) - \nabla u(t)]^2 dx\right]d\tau$$
$$+\frac{1}{2}\int_0^t g(\tau)\left[\frac{d}{dt}\int_\Omega [\nabla u(t)]^2 dx\right]d\tau$$
$$= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega [\nabla u(\tau) - \nabla u(t)]^2 dxd\tau\right]$$
$$+\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega [\nabla u(\tau) - \nabla u(t)]^2 dxd\tau$$

$$+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\int_\Omega [\nabla u(t)]^2 dxd\tau\right]-\frac{1}{2}g(\tau)\int_\Omega [\nabla u(t)]^2 dx. \quad (17)$$

Inserting (17) into (16), we obtain

$$\frac{d}{dt}\left[\begin{array}{c}\frac{1}{2}\|u_t\|^2+\frac{1}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2+\frac{1}{2}(g\circ\nabla u)(t)\\ -\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\end{array}\right]$$

$$=-\int_\Omega |u_t|^{p(x)}\,dx-\frac{1}{2}g(\tau)\int_\Omega |\nabla u(t)|^2\,dxd\tau$$

$$+\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega [\nabla u(\tau)-\nabla u(t)]^2 dxd\tau\le 0, \quad (18)$$

where

$$(g\circ\nabla u)(t)=\int_0^t g(t-\tau)\int_\Omega [\nabla u(\tau)-\nabla u(t)]^2 dxd\tau.$$

Now, we state and prove the blow up result:

Theorem 3.3: Suppose that the assumptions of Theorem 2.3 hold. If $E(0)<0$, then the solution (1) blows up in finite time T^*, and

$$T^*\le\frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Proof. We set

$$H(t)=-E(t).$$

By (18), we see that $H(t)\ge 0$. By the definition $H(t)$, we obtain

$$H(t)=-\frac{1}{2}\|u_t\|^2-\frac{1}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2-\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}$$

$$-\frac{1}{2}(g\circ\nabla u)(t)+\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\le\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx$$

$$\le\frac{1}{q^-}\rho_{q(.)}(u). \quad (19)$$

Let

$$\Psi(t)=H^{1-\sigma}(t)+\varepsilon\int_\Omega uu_t\,dx, \quad (20)$$

for ε small to be chosen later and

$$0<\sigma\le min\left\{\frac{q^--p^+}{(p^+-1)q^-},\frac{q^--2}{2q^-}\right\}. \quad (21)$$

By differentiating (20) and then using (1), we get

$$\Psi'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_{\Omega} (u_t^2 + uu_{tt})dx$$
$$= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2$$

$$-\varepsilon\|\nabla u\|^{2(\gamma+1)} + \varepsilon \int_0^t g(t-\tau)\int_{\Omega} \nabla u(t)\nabla u(\tau)\,dxd\tau$$
$$+\varepsilon \int_{\Omega} |u|^{q(.)}\,dx - \varepsilon \int_{\Omega} uu_t\,|u_t|^{p(.)-2}dx$$
$$= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)}$$
$$+\varepsilon \int_{\Omega} |u|^{q(.)}\,dx - \varepsilon \int_{\Omega} uu_t\,|u_t|^{p(.)-2}dx$$
$$+\varepsilon \int_0^t g(\tau)d\tau\,\|\nabla u\|^2 + \varepsilon \int_0^t g(t-\tau)\int_{\Omega} \nabla u(t)[\nabla u(\tau) - \nabla u(t)]dxd\tau. \quad (22)$$

To estimate the last term of (22), we use Cauchy-Schwarz and Young's inequalities

$$\int_0^t g(t-\tau)\int_{\Omega} \nabla u(t)[\nabla u(\tau) - \nabla u(t)]dxd\tau$$
$$\leq \int_0^t g(t-\tau)\,\|\nabla u(t)\|\|\nabla u(\tau) - \nabla u(t)\|d\tau$$
$$\leq \lambda(g\circ\nabla u)(t) + \frac{1}{4\lambda}\int_0^t g(\tau)d\tau\|\nabla u\|^2, \;\; \lambda > 0. \quad (23)$$

Substituting (23) into (22), we have

$$\Psi'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)}$$
$$+\varepsilon \int_{\Omega} |u|^{q(.)}\,dx - \varepsilon \int_{\Omega} uu_t\,|u_t|^{p(.)-2}dx + \varepsilon\int_0^t g(\tau)d\tau\,\|\nabla u\|^2$$
$$-\varepsilon\lambda(g\circ\nabla u)(t) - \frac{\varepsilon}{4\lambda}\int_0^t g(\tau)d\tau\,\|\nabla u\|^2.$$

By using the definition of the $H(t)$, it follows that

$$-\varepsilon q^-(1-\xi)H(t) = \frac{\varepsilon q^-(1-\xi)}{2}\|u_t\|^2 + \frac{\varepsilon q^-(1-\xi)}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2$$
$$+\frac{\varepsilon q^-(1-\xi)}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{\varepsilon q^-(1-\xi)}{2}(g\circ\nabla u)(t)$$
$$-\varepsilon q^-(1-\xi)\int_{\Omega} \frac{1}{q(x)}|u|^{q(.)}dx, \quad (24)$$

where $0 < \xi < 1$.

Adding and subtracting (24) on the right hand side of (22), we have

$$\Psi'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon q^{-}(1-\xi)H(t)$$
$$+\varepsilon\left(\frac{q^{-}(1-\xi)}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q^{-}(1-\xi)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)}$$
$$+\varepsilon\left[\frac{q^{-}(1-\xi)}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)-1+\left(1-\frac{1}{4\lambda}\right)\int_0^t g(\tau)\,d\tau\right]\|\nabla u\|^2$$
$$+\varepsilon\left(\frac{q^{-}(1-\xi)}{2}-\lambda\right)(g\circ\nabla u)(t)$$
$$+\varepsilon\xi\int_\Omega |u|^{q(.)}dx - \varepsilon\int_\Omega uu_t\,|u_t|^{p(.)-2}dx. \tag{25}$$

Then, for ξ small enough, we obtain

$$\Psi'(t) \geq \varepsilon\beta\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)+\rho_{q(.)}(u)\right]$$
$$+(1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\int_\Omega uu_t\,|u_t|^{p(.)-2}dx, \tag{26}$$

here

$$\beta = min\left\{\begin{matrix} q^{-}(1-\xi), \dfrac{q^{-}(1-\xi)}{2}+1, \dfrac{q^{-}(1-\xi)}{2}-\lambda, \xi, \dfrac{q^{-}(1-\xi)}{2(\gamma+1)}-1, \\ \dfrac{q^{-}(1-\xi)}{2}\left(1-\displaystyle\int_0^t g(\tau)\,d\tau\right)-1+\left(1-\dfrac{1}{4\lambda}\right)\displaystyle\int_0^t g(\tau)\,d\tau \end{matrix}\right\} > 0$$

and

$$\rho_{q(.)}(u) = \int_\Omega |u|^{q(.)}\,dx.$$

By the following Young's inequality, we have

$$XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^{-l}Y^l}{l},$$

where $X,Y \geq 0,\ \delta > 0, k,l \in R^+$ such that $\frac{1}{k}+\frac{1}{l}=1$. Consequently, applying the previous, we obtain

$$\int_\Omega u\,|u_t|^{p(.)-1}dx \leq \int_\Omega \frac{1}{p(x)}\delta^{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{p(x)-1}{p(x)}\delta^{-\frac{p(x)}{p(x)-1}}|u_t|^{p(x)}dx$$
$$\leq \frac{1}{p^-}\int_\Omega \delta^{p(x)}|u|^{p(x)}dx + \frac{p^+-1}{p^+}\int_\Omega \delta^{-\frac{p(x)}{p(x)-1}}|u_t|^{p(x)}dx, \tag{27}$$

where δ is constant depending on the time t and specified later. Inserting (27) into (26), we have

$$\Psi'(t) \geq \varepsilon\beta\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)+\rho_{q(.)}(u)\right]$$
$$+(1-\sigma)H^{-\sigma}(t)H'(t)$$
$$-\varepsilon\frac{1}{p^-}\int_\Omega \delta^{p(x)}\,|u|^{p(x)}dx - \varepsilon\frac{p^+-1}{p^+}\int_\Omega \delta^{-\frac{p(x)}{p(x)-1}}\,|u_t|^{p(x)}dx. \tag{28}$$

Let us choose δ, so that $\delta^{-\frac{p(x)}{p(x)-1}} = k_1 H^{-\sigma}(t)$, where $k_1 > 0$ is specified later, we get

$$\Psi'(t) \geq \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \rho_{q(.)}(u)\right]$$
$$+(1-\sigma)H^{-\sigma}(t)H'(t)$$
$$-\varepsilon\frac{1}{p^-}\int_\Omega k_1^{1-p(x)}H^{\sigma(p(x)-1)}(t)\,|u|^{p(x)}dx - \varepsilon\frac{p^+-1}{p^+}\int_\Omega kH^{-\sigma}(t)|u_t|^{p(x)}\,dx$$
$$\geq \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \rho_{q(.)}(u)\right]$$
$$+(1-\sigma)H^{-\sigma}(t)H'(t)$$
$$-\varepsilon\frac{k_1^{1-p^-}}{p^-}H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx - \varepsilon\left(\frac{p^+-1}{p^+}\right)kH^{-\sigma}(t)\int_\Omega |u_t|^{p(x)}\,dx$$
$$\geq \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \rho_{q(.)}(u)\right]$$
$$\left[(1-\sigma) - \varepsilon\left(\frac{p^+-1}{p^+}\right)k\right]H^{-\sigma}(t)H'(t) - \varepsilon\frac{k_1^{1-p^-}}{p^-}H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx.$$

Using (14) and (19), we obtain

$$H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx \leq H^{\sigma(p^+-1)}(t)\left[\int_{\Omega_-}|u|^{p^-}dx + \int_{\Omega_+}|u|^{p^+}dx\right]$$
$$\leq H^{\sigma(p^+-1)}(t)c\left[\left(\int_{\Omega_-}|u|^{q^-}dx\right)^{\frac{p^-}{q^-}} + \left(\int_{\Omega_+}|u|^{q^-}dx\right)^{\frac{p^+}{q^-}}\right]$$
$$= H^{\sigma(p^+-1)}(t)c\left[\|u\|_{q^-}^{p^-} + \|u\|_{q^-}^{p^+}\right]$$
$$\leq c\left(\frac{1}{q^-}\rho_{q(.)}(u)\right)^{\sigma(p^+-1)}\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^-}{q^-}} + \left(\rho_{q(.)}(u)\right)^{\frac{p^+}{q^-}}\right]$$
$$= c_1\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^-}{q^-}+\sigma(p^+-1)} + \left(\rho_{q(.)}(u)\right)^{\frac{p^+}{q^-}+\sigma(p^+-1)}\right] \tag{29}$$

where $c_1 = c\left(\frac{1}{q^-}\right)^{\sigma(p^+-1)}$, $\Omega_- = \{x \in \Omega : |u| < 1\}$ and $\Omega_+ = \{x \in \Omega : |u| \geq 1\}$.

By Lemma 3.1 and (21), for

$$s = p^- + \sigma q^-(p^+-1) \leq q^-$$

or

$$s = p^+ + \sigma q^-(p^+-1) \leq q^-,$$

to deduce, from (29),

$$H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}\,dx \leq c_1\left[\|\nabla u\|^2 + \rho_{q(.)}(u)\right]. \tag{30}$$

Thus, inserting estimate (30) into (26), we obtain

$$\Psi'(t) \geq \varepsilon\left(\beta - \frac{k^{1-p^-}}{p^-}c_1\right)\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \rho_{q(.)}(u)\right]$$

$$+\left[(1-\sigma) - \varepsilon\left(\frac{p^+-1}{p^+}\right)k\right]H^{-\sigma}(t)H'(t). \tag{31}$$

Let us choose k large enough so that $\gamma = \beta - \frac{k^{1-p^-}}{p^-}c_1 > 0$, and picking ε small enough such that

$$(1-\sigma) - \varepsilon\left(\frac{p^+-1}{p^+}\right)k \geq 0$$

and

$$\Psi(t) \geq \Psi(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega u_0 u_1 dx > 0, \forall t \geq 0. \tag{32}$$

Consequently, (31) yields

$$\Psi'(t) \geq \varepsilon\gamma\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \rho_{q(.)}(u)\right]$$

$$\geq \varepsilon\gamma\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \|u\|_{q^-}^{q^-}\right], \tag{33}$$

due to (14). Therefore we get

$$\Psi(t) \geq \Psi(0) > 0, for\ all\ t \geq 0.$$

On the other hand, applying Hölder's inequality, we have

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}$$

$$\leq C\left(\|u\|_{q^-}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}\right).$$

Using Young's inequality gives

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_{q^-}^{\frac{\mu}{1-\sigma}} + \|u_t\|^{\frac{\theta}{1-\sigma}}\right), \tag{34}$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\sigma)$, to obtain $\frac{\mu}{1-\sigma} = \frac{2}{1-2\sigma} \leq q^-$ by (21). Therefore, (34) becomes

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-}\right).$$

Thus,

$$\Psi^{\frac{1}{1-\sigma}}(t) = \left[H^{1-\sigma}(t) + \varepsilon\int uu_t\, dx\right]^{\frac{1}{1-\sigma}}$$

$$\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}}\right)$$

$$\leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)\right)$$

$$\leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g \circ \nabla u)(t) + \|u\|_{q^-}^{q^-}\right) \quad (35)$$

where

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

is used. By combining of (33) and (35), we have

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \quad (36)$$

where ξ is a positive constant.

A simple integration of (36) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi \sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi \sigma \Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

This completes the proof of the theorem.

Conclusion

In this work, we considered a viscoelastic wave equations of Kirchhoff type with variable exponents. Under the suitable conditions, we showed the blow up of solutions in a finite time with negative initial energy.

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