Bayesian Estimation For Cox Ingersoll Ross Process

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ABSTRACT

The Cox Ingersoll Ross (CIR) stochastic process is consider one of the most important models in finance to study the term structure of interest rates, as introduced in 1985, then it has been the object for many even recent studies and extensions. This article aims to estimate the parameters of CIR process by using two methods which are Maximum Likelihood Estimation (MLE) method and Bayesian Estimation method. It is Implement in the R programing.

1. Introduction:

Models of term structure of interest rates are useful applications for grasp interest rate dynamics. We consider one of the most popular model is called one-factor Cox Ingersoll Ross (CIR) process. It has been applied in a now widely used in finance as modelling interest rates and stochastic volatility of are common applications. It was presented by John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross in 1989. Our aim is to estimate the parameters of CIR process using Maximum Likelihood Estimation method and Bayesian method to show the best estimation for the CIR parameters.

In literature review, (Cox, J, et al. ,1985) [5] study the term structure of interest rates. The purpose of this paper is to describe the evolution of interest rates as diffusion process .(chen . R, etal. ,2003)[4] to describe the evolution of interest rates as a diffusion   process. (Chen. R, et al. ,2003) [4] Provides estimation the parameters in one , two and three factor using the approximation maximum likelihood estimation in a state-space model. And did (Gibson. R, et al. ,2012) [9] offer a general view of the term structure of interest rates. (Tilley. P. , 2002)[16] present the CIR model and the results include the extended CIR model and easier construction for it , an effective method of price general interest rate

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derivatives within this model, and price of bonds by the Laplace transform of functional of the initial process, which is squared Bessel. And (Georges . P. ,2003) [8] offers two models are CIR and vasicek of the interest rates that has been introduction into a macro-economic stochastic simulating model (SSM). The aim in this paper will be using the SSM with alternative term structures of interest rates . (M. R. Grasselli, et al. ,2005) [13] they reformulated the CIR model into the chaotic representation ,starting with the squared Gaussian representation of the CIR model, finding a simple expression for the fundamental random variable X_∞, also they derive an explicit form for the nth term of the Wiener chaos expansion of the CIR model, for n = 0, 1, 2,.., at last conclude a new expression for the price of a zero coupon bond which reveals a connection between Gaussian measures and ricatti differential equations. (Chatterjee . S . ,2005) [3] the aim of this paper is improve the term structure of interest rate in the sterling and euro treasury bond markets over the interval 1999-2003. (Alfonsi. A. ,2005) [1] the aim in this thesis is simulation the CIR process by using many methods include the implicit and explicit , as well as the strong and weak convergence of this methods. He also study the numerically their behavior. At last comparing them with the schemes proposed. (Assee . J . ,2009) [2] present the one- factor CIR model for model interest rates. In this thesis he refers to that by using finite difference techniques boundary behavior serves as a boundary condition and guarantees a uniqueness of solutions if the boundary is attainable, as well as the boundary condition is not needed to guarantee uniqueness if the boundary is non-attainable, The finite difference solution is verified by use of positive numerical approximations. (Zhu. L. ,2014) [17] present the CIR process with Hawkes jumps .It can be consider as a generation of the CIR process and the Hawkes process with exponential function. In this paper, we focus on a stochastic system modelled by a Cox Ingersoll Ross (CIR) process as described in [5]. Estimation the constant parameters of this process by using two methods which are Maximum Likelihood Estimation (MLE) method and Bayesian Estimation of the CIR Process by using R programing .

This paper is structured as follows. In section 2 present the Cox Ingersoll Ross (CIR) process. In section 3 present the methods of estimation the CIR process and Section 4 cantinas the simulation study. In section 5 present the conclusion.

2. Cox Ingersoll Ross (CIR) Process

The short rate r_t are governed by the stochastic differential equation (2.1) [11] :

$$dy_t = (\theta_1 - \theta_2 y_t)\, dt + \theta_3 \sqrt{y_t}\, dW_t \qquad ...(2.1)$$

where $\theta_1, \theta_2, \theta_3$ are positive parameters. The parameters have the following explanations : θ_1 is the speed of mean reversion, θ_2 is the long term average value of the process, θ_3 is the volatility, W_t is Brownian motion and y_t is a short rate. The CIR process remains non-negative, show that whenever $2\theta_1\theta_2 > \theta_3$, the interest rate is strictly larger than zero. Furthermore, there is empirical evidence that whenever interest rates are high, the volatility is likely to be high as well, which justifies the volatility term in the CIR process [5].

The distribution function of y(t) is the non-central chi-square , with $2q + 2$ degrees of freedom and parameter of non-centrality 2u proportional to the current spot rate.

The probability density of the interest rate at time s, conditional on its value at the present time t as [5]:

$$f(r(s),s \; ; r(t),t) = ce^{-u-v}\left(\frac{u}{v}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2}) \quad \dots (2.2)$$

where

$$c = \frac{2k}{\sigma^2(1-e^{-k(t-s)})} \qquad , \dots(2.3)$$

$$u = cr_t \, e^{-k(t-s)} \qquad , \dots(2.4)$$

$$v = cr_s \qquad , \dots(2.5)$$

$$q = \frac{2k\theta}{\sigma^2} - 1 \qquad , \dots(2.6)$$

$I_q(x)$ is the modified Bessel function of the first kind of order q.

$$I_q(x) = \left(\frac{x}{2}\right)^q \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{k! \, \Gamma(q+k+1)}$$

The characteristics of the distribution of the future interest rates are those expected , The mean goes to θ and the variance to zero , if θ_1 approaches infinity .The conditional mean goes to the current interest rate and the variance to $\theta_3^2 \, r(t) *(s - t)$ if θ_1 approaches zero .

If the interest rate does display mean reversion ($\theta_1, \theta_2 > 0$), then as s becomes large its distribution will approaching a gamma distribution. The steady state density function is [5] :

$$f[r(\infty), \infty; r(t), t] = \frac{\omega^v}{\Gamma(v)} \, r^{v-1}e^{-\omega r} \quad , \text{ where } \omega = \frac{2k}{\sigma^2} \; , v = 2k\frac{\theta}{\sigma^2}$$

The steady state mean is θ and variance is $\frac{\sigma^2\theta}{2k}$.

The expected , variance and auto-covariance conditional of CIR are [6] :

$$E[r(t)|r(s)] = \frac{\theta_1}{\theta_2} + (r_0 - \frac{\theta_1}{\theta_2})e^{-\theta_2 t}$$

$$\text{Var}[r(t)|r(s)] = \theta_3^{\,2}\left(\frac{e^{-\theta_2 t}-e^{-2\theta_2 t}}{\theta_2}\right)r_0 + \frac{\theta_1\theta_3^{\,2}}{2\theta_2^{\,2}}(1 - e^{-2\theta_2 t})$$

$$\text{Auto-covariance} = \theta_3^{\,2}\frac{e^{-\theta_2 t}-e^{-\theta_2(t+s)}}{\theta_2}r_0 + \theta_3^{\,2}\frac{\theta_1}{2\theta_2}(e^{-\theta_2(t-s)} - e^{\theta_2(t+s)})$$

3. parameters Estimation methods for the CIR process

3. 1 Maximum Likelihood Estimate (MLE) for CIR Process

Suppose a discrete time version on the time grid $t_1, t_2 ,...., t_n$ with time step $\Delta t = t_i - t_{i-1}$. If X_{ti} is given, the conditional density function g of X_{ti+1} is [15] :

$$f(r(s),s \; ; r(t),t)= Ce^{-u-v}\left(\frac{u}{v}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2})$$

As we mention the equation (2.3),(2.4),(2.5),(2.6)

The likelihood function for the interest rate r_t with n observations is :

$$L(\theta_1,\theta_2,\theta_3)=\prod_{i=1}^{n-1} Ce^{-u-v}\left(\frac{u}{v}\right)^{\frac{q}{2}} I_q(2(uv)^{1/2}) \quad \dots(3.1)$$

The log-likelihood function is as following :

$$\text{Ln } L(\theta_1,\theta_2,\theta_3) = (n-1) \ln (C)+\sum_{i=1}^{n-1}(-u_{t_i} - v_{t_{i+1}}) + \frac{1}{2} q \ln\left(\frac{v_{t_{i+1}}}{u_{t_i}}\right) + \ln[I_q(2\sqrt{u_{t_i} v_{t_{i+1}}})]) \quad \dots(3.2)$$

where $u_{t_i} = Cx_{t_i}e^{-\alpha\Delta t}$ and $v = Cx_{t_{i+1}}$.

The log-likelihood function must be maximization by taking partial derivatives of equation (2.1) with respect to $\theta_1,\theta_2,\theta_3$ make them equal to zero yield three equations:

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_1}|_{\widehat{\theta_1}} = 0 \quad \dots(3.3)$$

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_2}|_{\widehat{\theta_2}} = 0 \quad \dots(3.4)$$

$$\frac{\partial \ln L(r_t;\theta_1,\theta_2,\theta_3)}{\partial\theta_3}|_{\widehat{\theta_3}} = 0 \quad \dots(3.5)$$

By Solving these equations (3.3), (3.4) ,(3.5) will yield the maximum likelihood estimates:

$$\hat{\gamma} = (\widehat{\theta_1},\widehat{\theta_2},\widehat{\theta_3}) = \arg\max_\gamma \ln L(\gamma)$$

3.2 Bayesian estimation

We assume the unknown parameters are random variables. Then, these parameters have prior information. The difficulty in this type of estimation lies in collecting information about un known parameters and accurately determining the prior probability distribution for them due to the difficulty of obtaining the previous information or the inaccuracy of this information. This information is put in the form of an prior probability distribution known as the " prior probability density function" . It is combined with the maximum Likelihood to get what is called the " posterior probability density function".

Bayesian Estimation provides parameter estimates with perfect statistical property, parsimonious descriptions of observed data and predictions for missing data and forecasts of future data, a computational framework for model estimation, selection and validation[14].

Posterior distribution is indicates to the knowledge about θ with all known information.
Prior distribution is indicates to the knowledge about the θ without any information about the data.
Likelihood is the probability of the observed data depending on the parameter θ [10]

Mathematically, can be expressed as :

$$P(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

$$P(\theta|y) \propto P(y|\theta)P(\theta)$$

Where,

$p(y, \theta)$: the joint probability density function of the random variables y, θ.

$P(\theta)$: the prior probability density function.

$p(y|\theta)$: Maximum Likelihood Estimation.

$p(\theta|y)$: posterior probability density function

The Bayesian Estimation of CIR process as the following :-

The CIR model which is based on the following stochastic differential equation is [7] :

$$dy_t = (\theta_1 - \theta_2 y_t)\, dt + \theta_3\sqrt{y_t}\, dW_t$$

where $\theta_1, \theta_2, \theta_3$ are random variable , W_t is Brownian motion. We approximate the CIR process by use Euler-Maruyama scheme. As :

$y(t_{i+1}) = y(t_i) + (\theta_1 - \theta_2 y(t_i))\Delta t + \theta_3\sqrt{|y(t_i)|}\,\sqrt{\Delta t}\, Z_i$ …(3.2)

Where $Z_i \sim N(0, \Delta t)$,

Let

$Y = (y_1, \dots, y_T)$ refer to the group of all observation data , $Y^* = (y_1^*, \dots, y_{T-1}^*)$ refer to the group of all augmented data , And

$y_t^* = \{y_{t,1}^*, \dots, y_{t,M}^*\}$, Where T is observations, and M augmented data between each pair of outgo observations.

e refer the parameter vector $(\theta_1, \theta_2, \theta_3{}^2)$ as μ=$(\psi, \theta_3{}^2)$ where ψ= $(,\theta_1, \theta_2)$ due the drift term $(\theta_1 - \theta_2 y(t))$ is linear in k , θ and the term $\theta_3\sqrt{y(t)}$ is the output of the parameter σ and a function of y(t) . Firstly we assume that the priori

probability density of the parameters satisfy $P(\theta_1, \theta_2, \theta_3) \propto \frac{1}{\sigma}$, to determine the fully conditional posterior distributions of y_t^*, ψ and σ^2.

$f(y_t^*|Y, \mu) = f(y_{t,1}^*, \dots, y_{t,M}^*|Y, \mu) = \prod_{j=1}^M f(y_{t,j}^* | y_{t,j-1}^*, \theta_2)$

We suppose $y_{t,j}^*$ is a Markov process for $j = 1, \dots M$, for each $t > 0$, $\Delta = \frac{\Delta t}{M+1}$, Where $y_{t,0}^* = y_t$ And

$y_{t,j+1}^*|y_{t,j}^*, \mu \sim N(y_{t,j}^* + (\theta_1 - \theta_2 y_{t,j}^*)\Delta, \theta_3^2 \Delta y_{t,j}^*)$ the fully conditional probability density of Y^* .

The fully conditional posterior distribution of ψ is :

$p(\psi|Y, Y^*, \theta_3^2) = p(\theta_1, \theta_2|Y, Y^*, \theta_3)$

$\propto p(Y^*, Y|\mu)p(\mu)$

$\propto p(Y^*, Y|\mu)$

$\propto \prod_{t=1}^{T-1} \prod_{j=0}^M f(, y_{t,j+1}^*| y_{t,j}^*, \theta_2)$

Where $y_{t,M+1}^* = y_{t+1}$

$\propto \exp\{\sum_{t=1}^{T-1} \sum_{j=0}^M \frac{-\{y_{t,j+1}^* - [y_{t,j}^* + (\theta_1 - \theta_2 y_{t,j}^*)\Delta]\}^2}{2\theta_3^2 \Delta y_{t,j}^*}\}$

$\propto \exp\{-\sum_{t=1}^{T-1} \sum_{j=0}^M \frac{[(\theta_1 - \theta_2 y_{t,j}^*)\Delta + y_{t,j}^*]^2 - 2y_{t,j+1}^*(\theta_1 - \theta_2 y_{t,j}^*)\Delta)\}}{2\theta_3^2 \Delta y_{t,j}^*}\}$

$\propto \exp\{-\sum_{t=1}^{T-1} \sum_{j=0}^M \frac{(\theta_1 - \theta_2 y_{t,j}^*)^2 \Delta^2 + 2\Delta (y_{t,j}^* - y_{t,j+1}^*)(\theta_1 - \theta_2 y_{t,j}^*)}{2\theta_3^2 \Delta y_{t,j}^*}\}$

Since

$(\theta_1 - \theta_2 y_{t,j}^*)^2 \Delta^2 + 2\Delta (y_{t,j}^* - y_{t,j+1}^*)(\theta_1 - \theta_2 y_{t,j}^*)$

$= \Delta^2 k^2 + \Delta^2 y_{t,j}^{*2} \theta_2^2 - 2\Delta^2 y_{t,j}^* \theta_1 \theta_2 + 2\Delta(y_{t,j}^* - y_{t,j+1}^*)\theta_1 - 2\Delta(y_{t,j}^* - y_{t,j+1}^*)y_{t,j}^* \theta_2$,

We have

$P(\psi|Y, Y^*, \theta_3^2)$

$\propto \exp\{-\frac{\Delta Ak^2 + \Delta B\theta^2 - 2\Delta(T-1)(M+1)k\theta - 2Ck - 2D\theta}{2\theta_3^2}\}$

Where

$$A = \sum_{t=1}^{T-1} \sum_{j=0}^M \frac{1}{y_{t,j}^*}$$

$$B = \sum_{t=1}^{T-1} \sum_{j=0}^{M} y_{t,j}^{*}$$

$$C = -\sum_{t=1}^{T-1} \sum_{j=0}^{M} \frac{y_{t,j}^{*} - y_{t,j+1}^{*}}{y_{t,j}^{*}}$$

$$D = \sum_{t=1}^{T-1} \sum_{j=0}^{M} (y_{t,j}^{*} - y_{t,j+1}^{*})$$

$$n = (T-1)(M-1)$$

The fully conditional posterior distribution of ψ is

$$\psi | Y, Y^{*}, \theta_3{}^2) \sim N(\mu , \Lambda^{-1})$$

Where

$$\Lambda = \begin{pmatrix} \frac{\Delta}{\theta_3{}^2} A & -\frac{\Delta}{\theta_3{}^2} (T-1)(M+1) \\ -\frac{\Delta}{\theta_3{}^2} (T-1)(M+1) & \frac{\Delta}{\theta_3{}^2} B \end{pmatrix},$$

The fully conditional posterior distribution for $\theta_3{}^2$ is

$$P(\theta_3{}^2 | Y, Y^{*}, \psi)$$

$$\propto p(Y, Y^{*} | \theta_1, \theta_2, \theta_3{}^2) p(\theta_1, \theta_2, \theta_3{}^2)$$

$$\propto p(Y, Y^{*} | \theta_1, \theta_2, \theta_3{}^2) p(\theta_1, \theta_2, \theta_3{}^2) (\frac{d\theta_3{}^2}{d\theta_3})^{-1}$$

$$\propto \prod_{t=1}^{T-1} \prod_{j=0}^{M} \frac{1}{\sqrt{2\pi\theta_3}\sqrt{\Delta}\sqrt{y_{t,j}^{*}}} \exp\left\{\frac{-\{y_{t,j+1}^{*} - [y_{t,j}^{*} + (\theta_1 - \theta_2 y_{t,j}^{*})\Delta]\}^2}{2\theta_3{}^2 \Delta y_{t,j}^{*}}\right\} \cdot \frac{1}{\theta_3} \cdot \frac{1}{\theta_3}$$

$$\propto \frac{1}{(\theta_3)^{n/2}} \prod_{t=1}^{T-1} \prod_{j=0}^{M} \exp\left\{\frac{-\{y_{t,j+1}^{*} - [y_{t,j}^{*} + (\theta_1 - \theta_2 y_{t,j}^{*})\Delta]\}^2}{2\theta_3{}^2 \Delta y_{t,j}^{*}}\right\} \cdot \frac{1}{\theta_3} \cdot \frac{1}{\theta_3}$$

$$\propto (\theta_3{}^2) - \frac{(T-1)(M+1)}{2} - 1 \exp\{-\frac{\sum_{t=1}^{T-1} \sum_{j=0}^{M} \frac{\{y_{t,j+1}^{*} - [y_{t,j}^{*} + (\theta_1 - \theta_2 y_{t,j}^{*})\Delta]\}^2}{2 y_{t,j}^{*}}}{\theta_3{}^2}\}$$

Therefore,

$$\sigma | Y, Y^{*}, \psi \sim \text{Inverse} - \text{Gamma} (E, F),$$

Where E = (T-1)(M+1)/2 , And

$$F = \sum_{t=1}^{T-1} \sum_{j=0}^{M} \left\{ \frac{\{y_{t,j+1}^* - [y_{t,j}^* + (\theta_1 - \theta_2 y_{t,j}^*)\Delta]\}^2}{2\, y_{t,j}^*} \right\}$$

4. Simulation study

In this section, we will use R languages to simulation the CIR process driven by Brownian motion. In the main model the initial values of the parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$ also that the initial value of the interest rate y =10. In simulation side ,we will simulate realizations $\theta_1, \theta_2, \theta_3$ and from their posterior distribution by using 10000 iterations of the Metropolis Hastings algorithm. shows the CIR process in a simulation state . Figure 4.1 represents the CIR process driven by Brownian motion using discrete-time. As it is clear from that the random process is clear in this figure and that the process is always positive.
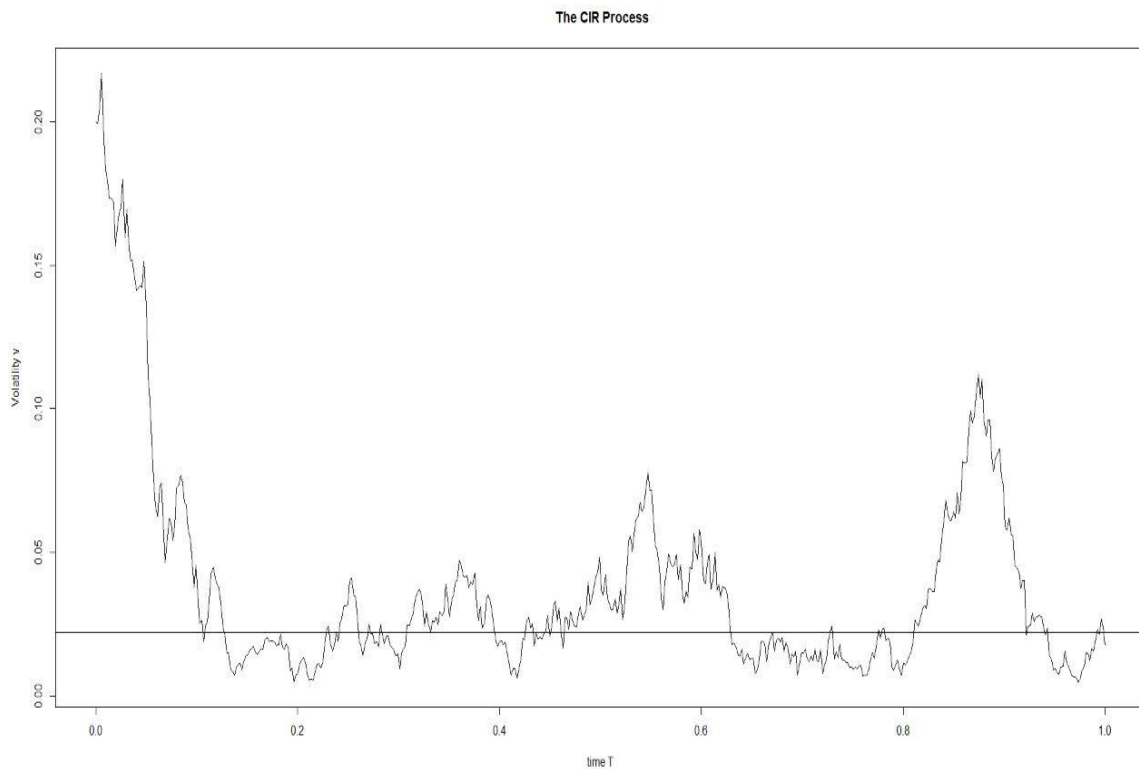

Figure 4.1 : shows the CIR process in a simulation state when the initial value of parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$, initial value of y =10.

Table 4.1 shows estimation of parameters by using MLE for CIR process

Parameters	Initial value	Parameters estimator	Standard Error
θ_1	1	1.5757	0.955
θ_2	0.5	0.47013	0.21534
θ_3	1	1.98333	0.0444

Table 4.1 shows estimation of parameters by using MLE estimation , noting that the initial value for parameters are $\theta_1 = 1, \theta_2 = 0.5$ and $\theta_3 = 1$. Thus, the MLE estimation is so closer to the initial value. Therefore, it means that the MLE was able to estimate the parameters well, and the standard error was low especially since the θ_3 was a very good estimate because the S.D is its value 0.0444.
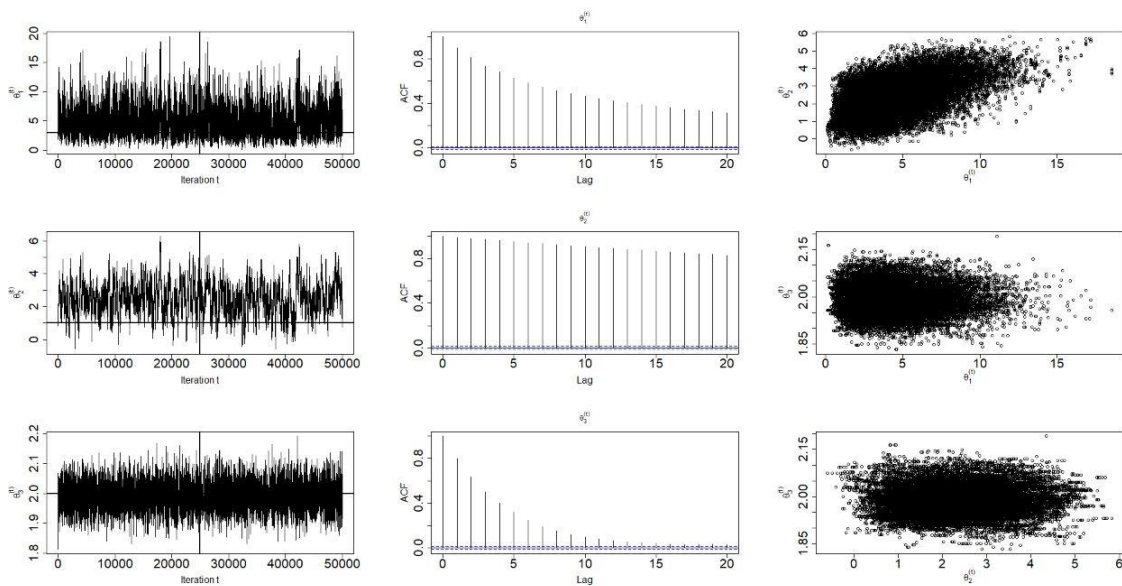

Figure 4.2: shows the results obtained from the MCMC algorithm where the first column represent the posterior distribution. The second column represent Auto-correlation for our parameters. Finally , the third column represent the spread of values between $\theta_1, \theta_3,$ and θ_3.

Figure 4.2 we assume that, the θ_1, θ_2 distributed as log normal and θ_3 distributed as inverse gamma as the prior distributions. It is good estimating using MCMC algorithm especially θ_1, θ_3.   Also, the Autocorrelation for θ_1, θ_3 is very good comparing to θ_2. Therefore, it can be said that the Bayesian estimate of the CIR process is good. Finally, we can say that   the posterior distribution of these parameters is stable.

In addition, the MCMC estimation is better than MLE.

5. Conclusion:-

We used two method to estimate the constant parameters of the CIR process which are Maximum Likelihood estimation (MLE) method and Bayesian estimation method. From the simulation and based on the results of the MSE, we find that the Bayesian estimation works very well than MLE method.

References

1- Alfonsi.A .(2005), On the discretization schemes for the CIR (and Bessel squared) process, Monte Carlo method and application, Vol.11, Issue 4, P 355-384.

2- Assee, J.(2009), Boundary Conditions For Mean-Reverting Square Root Process, Thesis for master of Mathematics in Computational Mathematics, University of Waterloo,   https://uwaterloo.ca/computational-mathematics   /sittes/ca.computational-mathematics / files/ uploads /files/ aquanassee-copy.pdf .

3- Chatterjee, S. (January, 2005), Application of the kalman filter for estimating continuous time term structure models : the case of UK and Germany, department of economics, University of Glasgow.

4- Chen, R., Scott, L. (2003), Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests From a Kalman Filter Model, Journal of real estate finance and economics , Vol.27, P. 143-172.

5- Cox , J., Ingersoll, J. and Ross , S (March 1985), A Theory of the Term Structure of Interest Rates, Econometrica, Vol. 53, No. 2 ,pp. 385-407.

6- Dehtiar, O., Mishura, Y., and Ralchenko, K.(11 May 2020), Two method of estimation of the drift parameters of the Cox-Ingersoll-Ross process: continuous observations, Taras Shevchenko National University of Kyiv, Department of Probability, Statistics and Actuarial Mathematics,Vol. 64, Ukraine.

7- Feng, X., Xie , D. (2012), Bayesian Estimation of CIR Model, Journal of Data Science,Vol.10,No.2, pp. 271-280.

8- Georges, P. (January 2003), The vasicek and CIR models and the expectation hypothesis of the Interest rate term structure, department of finance ,Working Paper.

9- Gibson, R. , Lhabitant ,F. and talay .D, (2012), Modeling the term structure of interest rates : a review of the literature ,foundation and trends in finance, Vol.5, No.1-2, P.171.

10- Herrmann,J.,(April 2014), Bayesian Inference for a Cox-Ingersoll-Ross Model with changing Parameters and Application to Finance Data, Technical University of Berlin ,Master thesis .

11- Iacus.s.(2007), Simulation and Inference for Stochastic Differential Equations With R Examples, springer series in statistics, Italy.

12- M. A. Jafari S. Abbasian (2017),The Moments for Solution of the Cox-Ingersoll-Ross Interest Rate Model, Journal of Finance and Economics, Vol. 5, No. 1, pp.34-37.

13- M. R. Grasselli , T. R. Hurd (2005), Wiener chaos and the Cox–Ingersoll–Ross model, department of Mathematics and Statistics, McMaster University ,Canada, A 461, P. 459-479.

14-Peter D. Hoff,(2009), A first course in Bayesian statistical methods , springer Dordrecht Heidelberg London New York.

15- Sypkens, R. (September 2010), Risk properties and parameter estimation on mean reversion and Garch models, Master thesis, university of south Africa.

16- Tilley , P. (May 30-31,2002), Cox, Ingersoll And Ross Models Of Interest Rates , Universidad Anahuac in Mexico City, Mexico, Vol. 28 , No.1.

17- Zhu, L. (19 April 2014), Limit theorems for a Cox-Ingersoll-Ross process with Hawkes Jumps, Journal of Applied probability, Vol.51, Issue 3, pp. 699-712.