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# Application Of Strip Domain And Parabolic Region On Univalent Holomorphic Functions

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# Application of strip domain and parabolic region on univalent holomorphic functions

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**ABSTRACT**

In this paper, by using univalent functions connected with the strip domain, parabolic starlike and parabolic uniformly convex functions are introduced. Some relations between these classes are proved.

## 1. Introduction

Let $\alpha$ and $\beta$ be real numbers with $\alpha < 1$ and $\beta > 1$.

The function $S_{\alpha,\beta}(z)$ defined by

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\Pi} i \, log\left(\frac{1 - e^{i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}}{1 - e^{-i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}}\right), \tag{1.1}$$

where $z$ is the unit disk $\Delta = \{z \in \mathbb{C}, |z| < 1\}$, is analytic and univalent in $\Delta$ with $S_{\alpha,\beta}(0) = 1$. (For more details see [2]). In addition the function $S_{\alpha,\beta}(z)$ maps $\Delta$ onto the strip domain $\omega$ such that $\alpha < Re(\omega) < \beta$.

The function $S_{\alpha,\beta}(z)$ can be written as follow:

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$$S_{\alpha,\beta}(z)=\frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{\Pi}i\,log\left(\frac{ie^{-i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}+(-i)e^{i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}e^{-i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}}{1-e^{-i\frac{\Pi(1-\alpha)}{(\beta-\alpha)}z}}\right). \tag{1.2}$$

Also, it is easy to see that

$$\begin{aligned}S_{\alpha,\beta}(z)&=1+\sum_{n=1}^{\infty}B_n\,z^n\\&=1+B_1z+\sum_{n=2}^{\infty}B_n\,z^n,\end{aligned} \tag{1.3}$$

where

$$B_n=\frac{2(\beta-\alpha)}{n\Pi}sin\left(\frac{n\Pi(1-\alpha)}{\beta-\alpha}\right),\qquad (n=1,2,\dots). \tag{1.4}$$

We consider

$$H(z)=S_{\alpha,\beta}(z)-1. \tag{1.5}$$

We define the operator $G_{B_1,\eta}(H(z))=G_{B_1,\eta}(z)$ as follow:

$$G_{B_1,\eta}(z)=(i-B_1\eta)z+\eta\int_0^z\frac{H(t)}{t}dt. \tag{1.6}$$

**Definition (1.1):** A function $f(z)$ is said to be parabolic starlike function in $\Delta$ denoted by PS

$$\left|\frac{zf'}{f}-1\right|<Re\left(\frac{zf'}{f}\right),\qquad z\in\Delta. \tag{1.7}$$

**Definition (1.2):** A function $f(z)$ is said to be uniformly parabolic convex function in $\Delta$ denoted by UPC if,

$$\left|\frac{zf''}{f'}-1\right|<Re\left(1+\frac{zf'''}{f'}\right). \tag{1.8}$$

For other subclasses of univalent functions, one may refer to [1,3,4] and [5].

## 2. Main Results

In this section we give some relations between PS an UPC.

**Theorem (2.1):** $G_{B_1,\frac{1}{B_1}}(z)$ is in UPC if and only if $H(z)\in PS$.

**Proof:** By (1.6) after a simple calculation we have

$$G_{B_1,\eta}(z)=z+\sum_{n=2}^{+\infty}A_nz^n, \tag{2.1}$$

where,

$$A_n=\frac{B_n}{B_1n}. \tag{2.2}$$

Since $G_{B_1,\frac{1}{B_1}}(z) \in UPC$, then by (1.8) we have:

$$\left|\frac{z(G_{B_1,\frac{1}{B_1}}(z))''}{(G_{B_1,\frac{1}{B_1}}(z))'}\right| < Re\left\{1+\frac{z(G_{B_1,\frac{1}{B_1}}(z))''}{(G_{B_1,\frac{1}{B_1}}(z))'}\right\},$$

or equivalently by putting (1.6) in the above inequality we have

$$\left|\frac{z(\frac{H(z)}{z})'}{\frac{H(z)}{z}}\right| < Re\left(1+\frac{z(\frac{H(z)}{z})'}{\frac{H(z)}{z}}\right),$$

or equivalently

$$\left|\frac{z(H(z))'}{H(z)}-1\right| < Re\left(\frac{z(H(z))'}{H(z)}\right),$$

then by definition 1.1, $H(z) \in PS$. □

**Definition (2.2):** A function $H(z)$ defined by (1.5) is said parabolic of order $\gamma$ type $\theta$ in the unit disk $\Delta$ denoted by $P(\gamma,\theta)$ if

$$\left|\frac{z(G_{B_1,\eta}(z))''}{(G_{B_1,\eta}(z))'}+1-(\gamma+\theta)\right| < (\theta-\gamma)+Re\left[1+\frac{z(G_{B_1,\eta}(z))''}{(G_{B_1,\eta}(z))'}\right], \qquad (2.3)$$

where $G_{B_1,\eta}(z)$ is defined by (1.6).

**Theorem (2.3):** $H(z) \in P(\gamma,\theta)$ if and only if for every $z \in \Delta$, the values of

$$\frac{z(G_{B_1,\eta}(z))''}{(G_{B_1,\eta}(z))'}+1$$

lie in the interior of the parabolic region.

**Proof:** By definition 2.2, if we pot the values of

$$\frac{z(G_{B_1,\eta}(z))''}{(G_{B_1,\eta}(z))'}+1$$

equal to $\omega$, we have

$$|\omega-(\gamma+\theta)| < (\theta-\gamma)+Re(\omega), \quad or$$

$$|Re(\omega)-(\gamma+\theta)|^2+\big(Im(\omega)\big)^2 < [(\theta-\gamma)+Re(\omega)]^2, \quad or$$

$$\big(Re(\omega)\big)^2+(\gamma+\theta)^2-2(\gamma+\theta)Re(\omega)+\big(Im(\omega)\big)^2 < (\theta-\gamma)^2$$
$$+\big(Re(\omega)\big)^2+2(\theta-\gamma)Re(\omega), \quad or$$

$$[Im(\omega)]^2 < [2(\gamma-\theta)+2(\theta-\gamma)]Re(\omega) - 4\gamma\theta, \quad or$$

$$[Im(\omega)]^2 < 4\theta[Re(\omega)-\gamma],$$

and that is the interior of the parabolic region in the half-plane (right side) with vertex at $(\gamma, 0)$ and $4\theta$ is the length of the latus rectum. □

For more details about this region see [6].

**Theorem (2.4):** If $H(z)$ are $G_{B_1,\eta}(z)$ and defined by (1.5) and (1.6) respectively. Then $H(z)$ is univalently starlike of order $\nu$ if and only if $G_{B_1,\frac{1}{B_1}}(z)$ is univalently convex of order $\nu$.

**Proof:** Let $G_{B_1,\frac{1}{B_1}}(z)$ be univalently convex of order $\nu$, then

$$Re\left\{\frac{z(G_{B_1,\eta}(z))''}{(G_{B_1,\eta}(z))'}+1\right\} > \nu. \tag{2.4}$$

But by (1.6) we have

$$(G_{B_1,\frac{1}{B_1}}(z))' = B_1\left(\frac{H(z)}{z}\right), \tag{2.5}$$

and

$$(G_{B_1,\frac{1}{B_1}}(z))'' = B_1\left(\frac{H(z)}{z}\right)', \tag{2.6}$$

Thus by putting (2.5) and (2.6) in (2.4) we conclude

$$Re\left\{\frac{zH'(z)}{H(z)}\right\} > \nu.$$

So *H(z)* is univalently starlike of order *ν*. All the relations are reversible and so the proof is complete. □

**Theorem (2.5):** Let $H_k \in P(\gamma_k,\theta_k)$ with $0 \le \gamma_k < 1$, $\sum_{k=1}^{m}\gamma_k < 1$, $0 < \theta_k < \infty$, $k = 1,2,\dots,m$, $r_k > 0$ $(k = 1,2,\dots,m)$ and $\sum_{k=1}^{m} r_k = 1$. Then

$$F(z) = \prod_{k=1}^{m}(H_k)^{r_k} \tag{2.7}$$

is in $P(\gamma,\theta)$, where $\gamma = \sum_{k=1}^{m} r_k\gamma_k$ and $\theta = \sum_{k=1}^{m} r_k\theta_k$.

**Proof:** We prove this theorem when $\eta = \frac{1}{B_1}$. Since $H_k \in P(\gamma_k,\theta_k)$, $k = 1,2,\dots,m$, then by definition 2.2, we have

$$\left|\frac{z(G^k_{B_1,\frac{1}{B_1}}(z))''}{(G^k_{B_1,\frac{1}{B_1}}(z))'}+1-(\gamma_k+\theta_k)\right|<Re\left(1+\frac{z(G^k_{B_1,\frac{1}{B_1}}(z))''}{(G^k_{B_1,\frac{1}{B_1}}(z))'}\right)+(\theta_k-\gamma_k). \qquad (2.8)$$

Now we must show

$$\left|\frac{z(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))''}{(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))'}+1-(\gamma+\theta)\right|<Re\left(1+\frac{z(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))''}{(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))'}\right)+(\theta-\gamma).$$

where

$$\mathcal{F}_{B_1,\frac{1}{B_1}}(z)=\mathcal{F}_{B_1,\frac{1}{B_1}}(z)\big(F(z)\big)=\mathcal{F}_{B_1,\frac{1}{B_1}}(z)\left(\prod_{k=1}^{m}(H_k)^{r_k}\right).$$

But by a direct calculation we obtain

$$\left|\frac{z(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))''}{(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))'}+1-(\gamma+\theta)\right|=\left|\frac{zF'}{F}-(\gamma+\theta)\right|$$

$$=\left|\sum_{k=1}^{m}r_k\left(\frac{zH_k'}{H_k}-(\gamma_k+\theta_k)\right)\right|$$

$$\leq\sum_{k=1}^{m}\left[r_k\left|\frac{zH_k'}{H_k}-(\gamma_k+\theta_k)\right|\right].$$

With a simple calculation on (2.8) we obtain

$$\left|\frac{zH'}{H}-(\gamma_k+\theta_k)\right|<Re\left(\frac{zH'}{H}\right)+(\theta_k-\gamma_k),$$

and so

$$\left|\frac{z(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))''}{(\mathcal{F}_{B_1,\frac{1}{B_1}}(z))'}+1-(\gamma+\theta)\right|<\sum_{k=1}^{m}\left[r_k\left(\frac{zH_k'}{H_k}\right)+(\gamma_k+\theta_k)\right]$$

$$=Re\left(\frac{zF'}{F}\right)+(\theta-\gamma),$$

so $F\in P(\gamma,\theta)$, (when $\eta=\frac{1}{B_1}$). The proof is now complete.

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