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Invariants Of Uniform Conjugacy On Uniform Dynamical System

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ABSTRACT

In this paper, We present some important dynamical concepts of uniform space such as the uniform minimal systems and uniform shadowing. We explain some definitions and theorems such as uniform expansive, weak uniform expansive, uniform generator. we study the relations among types of compact uniform space and uniform shadowing as well as uniform conjugacy. Moreover we show that if $T: X \rightarrow X$ and $S: Y \rightarrow Y$ are two uniform homeomorphisms on compact uniform spaces X and Y , if $\varphi: X \rightarrow Y$ is a uniform conjugacy from T to S , then $CR_U(S) = \varphi(CR_U(T))$.

1. Introduction

Anosov [1] introduced the notion of shadowing property (or pseudo-trajectory tracing property) turned out to be one of the very important and useful dynamical properties of continuous maps on uniform space since its inception. The dynamical systems are one of the famous domains of researches in Mathematics [2].

Various generalizations of shadowing property have been obtained and studied in detail the concepts of uniform shadowing, uniform expansive, uniform weak expansive, uniform generator, uniform transitive, uniform mixing, uniform chain transitive, uniform conjugacy, and uniform pseudo-trajectory tracing property. The trajectory was defended by Pilyugin [6] and the expansive of uniform dynamical systems introduced by I. J. Khadim and A. A. Ali [5].

For a given real number, $U \in \mathcal{U}$ a sequence of points $\{x_i, a < i < b\}$ of uniform space (X, \mathcal{U}) is called all pseudo-trajectory of a continuous map $T: X \rightarrow X$ if $(T(x_i), x_{i+1}) \in U$ for each $i \in (a, b - 1)$. Given $V \in \mathcal{U}$, a U -pseudo-trajectory $\{x_i, a < i < b\}$ is called V -traced by $x \in X$, if $(T^i(x), x_i) \in V, \forall i \in (a, b)$. Here the symbols a and b are taken as $-\infty \leq a < b \leq \infty$, if T is bijective and as $a, b \in [0, \infty]$, [3].

If T is not bijective, we say that T has the shadowing property or (pseudo-trajectory tracing property) for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that every U -pseudo-trajectory of T can be traced by some point of x . The notion of U -pseudo-trajectory is quite a natural notion since on account of rounded errors, a computer will calculate a pseudo-trajectory rather than a trajectory. Moreover, V -tracing shows that a pseudo-trajectory is uniformly approximated by a genuine trajectory if X is compact then the shadowing property several problems including properties of maps possessing shadowing property and its relation with other dynamical properties have been studied in detail.

We prove that T is a homeomorphism and (X, \mathcal{U}) be a compact uniform space, then T is uniform weak expansive iff (if and only if) T has a uniform generator. Although we show that if $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms on compact uniform spaces X and Y and $\varphi: X \rightarrow Y$ is a uniform conjugacy from T to S , then $\Omega_U(S) = \varphi(\Omega_V(T))$.

In [7] they present invariant dynamical properties under G -conjugacy. Moreover, we introduce the notions of G -minimal systems and limit G -shadowing property and we show that these properties are invariant under topological G -conjugacy.

Definition(1.1): [5]. Let X be a set. A uniform construction on X is a non-empty collection \mathcal{U} include of subsets of the Cartesian square $X \times X$ fulfilling:

- (1) if $U \in \mathcal{U}$, then $\Delta_X \subset U$, such that $\Delta_X = \{(x, x): x \in X\}$
- (2) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$;
- (3) if $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
- (4) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
- (5) if $U \in \mathcal{U}$, then $\exists V \in \mathcal{U}$ where $V \circ V \subset U$;

The elements of \mathcal{U} are called the entourage of the uniform construction and the set X is called a uniform space. The uniformity \mathcal{U} is named separating (and X is called separated) if $\bigcap \{U: U \in \mathcal{U}\} = \Delta$.

We note that (3), (4), and (5) hint that, for any entourage, U found a symmetric entourage V such that $V \circ V \subset U$. Let X be a set and $U \subset X \times X$. If $x \in X$, we define the subset $U_{[x]} \subset X$ by $U_{[x]} = \{y \in X: (x, y) \in U\}$.

Definition (1.2): Let $T: X \rightarrow X$ be a continuous map on a uniform space (X, \mathcal{U}) . A sequence $\{x_n: 0 \leq n < \infty\} \subset X$ is called a true trajectory of T and it is denoted by $\text{tra}_T(x_0, T) = \{x_0, T(x_0), \dots\}$ if $x_{j+1} = T(x_j)$, for all $j \in \mathbb{N}$.

Definition (1.3): Let $T: X \rightarrow X$ be a continuous map on a uniform space (X, \mathcal{U}) . A sequence $\{x_n \in X: n \in \mathbb{Z}\}$ is named a U -pseudo-trajectory for T , if $(T(x_n), x_{n+1}) \in U, \forall n \in \mathbb{Z}$.

Definition (1.4): A U -pseudo-trajectory $\{x_n\}$ for T is called V -shadowed by $x \in X$ if $(T^n(x), x_n) \in V, \forall n \in \mathbb{Z}$.

Definition (1.5): Let $T: X \rightarrow X$ be a continuous map on a uniform space (X, \mathcal{U}) . A map T has the uniform shadowing property (USP) if $\forall V \in \mathcal{U}, \exists U \in \mathcal{U}$ where for any U -pseudo-trajectory for T there is $x \in X$ such that $\forall n \in \mathbb{Z}$ we have $(T^n(x), x_n) \in V$.

Let (X, \mathcal{U}) be a compact uniform space with compact. Then the family $\{\theta_U: U \in \mathcal{U}\}$ is equicontinuous.

Theorem (1.6): Let $T: X \rightarrow X$ be a homeomorphism on a compact uniform space (X, \mathcal{U}) . Then T has USP iff T^{-1} has USP

Proof: Assume that T has USP then for each $V \in \mathcal{U}, \exists U \in \mathcal{U}$ such that any U -pseudo-trajectory for T is V -shadowed with some point in X . Choose $\mathcal{C} \in U$ so that for any $x, y \in X$ then we have $(x, y) \in \mathcal{C}$ implies that $(T(x), T(y)) \in \mathcal{C}$, now let $\{x_n\}$ be a U -pseudo-trajectory for T^{-1} and put $y_n = x_{-n}$. So $\forall n \in \mathbb{Z}$ then $(T^{-1}(x_{-n}), x_{-n+1}) \in U$.

Hence $\forall n \in \mathbb{Z}$, we have $(x_{-n}, T(x_{-n+1})) \in \mathcal{C}$. So $(y_n, T(y_{n-1})) \in U$. Thus $\exists x \in X$ so that $(T^n(x), y_n) \in V$ for each $n \in \mathbb{Z}$. So $(T^{-n}(x), x_n) \in V$ that is $\{x_n\}$ is V -shadowed.

Conversely, similarly, we can show that if T^{-1} has USP then T has USP.

Definition(1.7): A Uniform homeomorphism $T: X \rightarrow X$ is called uniform expansive on condition that $\exists U \in \mathcal{U}$ such that $\forall x, y \in X$ with $tra_T(x) \neq tra_T(y)$, there exists $n \in \mathbb{Z}$ such that $((T^n(u), T^n(v))) \notin U$ for all $u \in tra_T(x), v \in tra_T(y)$, such that $u, v \in X$. The u is entourage called uniform expansive entourage for T .

Definition(1.8): A uniform homeomorphism $T: X \rightarrow X$ on a uniform space (X, U) is called uniform weak expansive on condition that $\exists V \in \mathcal{U}$ such that $\forall x, y \in X$ with $tra_T(x) \neq tra_T(y)$ if $u \in tra_T(x)$ and $v \in tra_T(y)$, then there exists $n = n(u, v) \in \mathbb{Z}$ such that $(T^n(u), T^n(v)) \notin U$. The entourage U is named a uniform weak expansive entourage for T .

The following definition is a generalization of the definition of generator in[6]

Definition(1.9): A uniform generator for T is finite open cover α of X is named if for each bisequence $\{A_n, -\infty < n < \infty\} \subset \alpha$, the set $(T^{-n} \times T^{-n})(\overline{A_n})$ contains at most one trajectory.

Theorem (1.10): A uniform homeomorphism $T: X \rightarrow X$ on a uniform space (X, U) . Then T is uniform weak expansive iff T has uniform generator.

Proof:((\Rightarrow))

Assume that T is uniform weak expansive. If $(x, y) \in (T^{-n} \times T^{-n})(\overline{A_n})$ then for each $n \in \mathbb{Z}$. We have $(T^n(x), T^n(y)) \in U$. Let $\{A_n\}_{n=-\infty}^{\infty}$ be a bisequence such that $\{A_n\}_{n=-\infty}^{\infty} \subset \alpha$, that is $A_n \in \alpha$, such that $n \in \mathbb{Z}$, A_n is a uniform of $\mathcal{U}, \forall n \in \mathbb{Z}$. Such that $(x, y) \in (T^{-n} \times T^{-n})(\overline{A_n}), \forall n \in \mathbb{Z}$. Then $(x, y) \in (T^{-n} \times T^{-n})(\overline{A_n}), \forall n \in \mathbb{Z}, (T^n(x), T^n(y)) \in \overline{A_n}$,

$n \in \mathbb{Z}$. By hypothesis each $A_n \in \mathcal{U}$. Let $z_n \in A_n, \forall n \in \mathbb{Z}$ then $(T^n \times T^n)(x, y) \in \overline{A_n}$, then $(T^n(x) \times T^n(y)) \in \overline{A_n} \Rightarrow (T^n(x) \times T^n(y))$ uniformity $z_n \in A_n$.

If $(x_n, y_n) \in A_n \Rightarrow (x, y) \in T^{-n} \times T^{-n}(\overline{A_n}) \Rightarrow (x, y) \in (T^n \times T^n)(\overline{A_n})$,
 $\forall n \in \mathbb{Z} \Rightarrow \exists (x_n, y_n) \in \overline{A_n}$, such that $X = (T^{-n} \times T^{-n})(x_n), Y = (T^{-n} \times T^{-n})(y_n)$,
 $\forall n \in \mathbb{Z}$. If $(x_n, y_n) \in \overline{A_n}, \forall n \in \mathbb{Z} \Rightarrow \exists (x_n^k, y_n^k) \in \overline{A_n}$, such that $x_n^k \rightarrow x_n, y_n^k \rightarrow y_n$,
as $k \rightarrow \infty, (T^{-n} \times T^{-n})(x_n^k) \rightarrow (T^{-n} \times T^{-n})(x_n) = x \forall n$, as $k \rightarrow \infty, (T^{-n} \times T^{-n})(y_n^k) \rightarrow$
 $(T^{-n} \times T^{-n})(y_n) = y \forall n$, as $k \rightarrow \infty$, then $(x_n, y_n) = (T_x^n, T_y^n) \in \mathcal{U}$.

((\Leftarrow)) Assume that α is a uniform generator for T and \mathcal{U} is the uniform space. If T is not uniform weak expansive then $\exists x, y \in X$, so that $\text{tra}(x) \neq \text{tra}(y), \forall n \in \mathbb{Z}$. we conclude that $(T^n(x), T^n(y)) \in U$.

Thus for each n there exists $A_n \in \mathcal{U}$ so that $(T^n(x), T^n(y)) \in A_n$.

Hence $(x, y) \in T^{-n} \times T^{-n}(\overline{A_n})$.

Which is a contradiction ■

2. Uniform conjugacy.

Definition (2.1): Let T and S be two uniform homeomorphisms in uniform space X and Y . Then T is uniform conjugate to S if there is equivariant uniform homeomorphism $\varphi: X \rightarrow Y$ satisfy $\varphi \circ T = S \circ \varphi$. The uniform homeomorphism φ is named uniform conjugacy.

Definition (2.2): For given points $x, y \in X$ we inscribe $x \rightsquigarrow_U^v y$ if there exist finite V -pseudo-trajectory $x = x_0, x_1, \dots, x_n = y$ and $y = y_0, y_1, \dots, y_m = x$, for T . If for every $V \in \mathcal{U}, x \rightsquigarrow_U^v y$, then x is called linked to y (we write $x \sim_U y$).

Definition (2.3): Let $T: X \rightarrow X$ be a continuous map on a uniform space (X, \mathcal{U}) . Then a point $p \in X$ is called U -chain recurrent point of T if $p \sim_U p$.

Definition (2.4): A homeomorphism $T: X \rightarrow X$ is called uniform transitive condition that for every non-empty open subset U and V of X , then $\exists n > 0, n \in \mathbb{Z}$ such that $((T^n \times T^n)(U)) \cap V \neq \emptyset$.

Definition (2.5): A uniform homeomorphism $T: X \rightarrow X$ is called uniform mixing if for every non-empty open subset U and V of X , then $\exists N \in \mathbb{Z}$ such that $\forall n \geq N$, we have $((T^n \times T^n)(U)) \cap V \neq \emptyset$.

Definition (2.6): A uniform homeomorphism $T: X \rightarrow X$ is called uniform minimal if $\forall x \in X$ then $\overline{\text{tra}_T(x)} = X$ where $\text{tra}_T(x) = \{T^n(x) : n \in \mathbb{Z}\}$ is the trajectory of x .

Theorem (2.7): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are conjugate then T is uniform transitive iff S is uniform transitive.

Proof: Assume that T and S are uniform conjugate by φ that is $\varphi \circ T = S \circ \varphi$. Let T be uniform transitive and, $U, V \subset Y$ such that U, V are open sets. Then $(\varphi^{-1} \times \varphi^{-1})(U)$ and

$(\varphi^{-1} \times \varphi^{-1})(V) \subset X$ such that $(\varphi^{-1} \times \varphi^{-1})(U), (\varphi^{-1} \times \varphi^{-1})(V)$ are open sets. Then $\exists n > 0$ where $(T^n \times T^n)((\varphi^{-1} \times \varphi^{-1})(U)) \cap ((\varphi^{-1} \times \varphi^{-1})(V)) \neq \emptyset$. Consequently $(\varphi^{-1} \times \varphi^{-1})(S^n \times S^n)(U) \cap ((\varphi^{-1} \times \varphi^{-1})(V)) \neq \emptyset$ and therefore $(\varphi^{-1} \times \varphi^{-1})(S^n \times S^n)(U) \cap (V) \neq \emptyset$. ■

Theorem (2.8): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are uniform conjugates then T is uniform mixing iff S is uniform mixing.

Proof: Let T be uniform mixing and $U, V \subset Y$ such that U, V are open sets. Then $(\varphi^{-1} \times \varphi^{-1})(U)$ and $(\varphi^{-1} \times \varphi^{-1})(V) \subset X$ such that $(\varphi^{-1} \times \varphi^{-1})(U), (\varphi^{-1} \times \varphi^{-1})(V)$ are open sets. Then $\exists N \in \mathbb{Z}$ such that $\forall n \geq N$ so that $((T^n \times T^n)(\varphi^{-1} \times \varphi^{-1})(U)) \cap (\varphi^{-1} \times \varphi^{-1})(V) \neq \emptyset$. Therefore $\forall n \geq N$ then $(\varphi^{-1} \times \varphi^{-1})((S^n \times S^n)(U)) \cap (\varphi^{-1} \times \varphi^{-1})(V) \neq \emptyset$ and $((S^n \times S^n)(U) \cap V) \neq \emptyset$. ■

Theorem (2.9): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are uniform conjugates then T is uniform weak expansive iff S is uniform weak expansive.

Proof: Let V be a weak expansive entourage of T . Then $\exists U \in \mathcal{U}$ where $(x, y) \in U$ implies to $(\varphi^{-1}(x), \varphi^{-1}(y)) \in V, \forall y \in Y$. Let $\text{tra}_T(x) \neq \text{tra}_T(y)$, for each $n \in \mathbb{Z}$, $(S^n(x), S^n(y)) \in U$. Therefore $\forall n \in \mathbb{Z}$, $(\varphi^{-1} S^n(x), \varphi^{-1} S^n(y)) \in V$.

Hence $(T^n \varphi^{-1}(x), T^n \varphi^{-1}(y)) \in V, \forall n \in \mathbb{Z}$. Then $\text{tra}_T(\varphi^{-1}(x)) = \text{tra}_T(\varphi^{-1}(y))$. Therefore $\varphi^{-1}(\text{tra}_T(x)) = \varphi^{-1}(\text{tra}_T(y))$.

Which is a contradiction. ■

Theorem (2.10): Let (X, \mathcal{U}) be uniform space, let $T: X \rightarrow X, S: Y \rightarrow Y$ be two continuous maps and $\varphi: X \rightarrow Y$ be a uniform homeomorphisms. If T and S are uniform conjugates then T has the USP iff S has the USP.

Proof: Given $(\varphi \circ \varphi) \in H$. Suppose T has the U -shadowing property, we show that $S = (\varphi \circ T \circ \varphi^{-1})$ has $H(U)$ -shadowing property. Since T has U -shadowing property, there is $V \in \mathcal{U}$ such that every V -pseudo-trajectory for T is U -traced by a point of X . S is $H(V)$ -traced by a point of Y .

Let $\{y_i: i \geq 0\}$ be a $H(V)$ -pseudo-trajectory for S . Then for each $i \geq 0$ $(S(y_i), y_{i+1}) \in H(V) \Rightarrow (T\varphi^{-1}(y_i), \varphi^{-1}(y_{i+1})) \in V$.

Then so that $(T(x_n), x_{n+1}) = (\varphi^{-1}(S(y_n)), \varphi^{-1}y_{n+1}) \in U_1$ set $\varphi^{-1}(y_i) = x_i$. Then for each $i \geq 0$, $(T(x_i), x_{i+1}) \in V$ and therefore $\{x_i: i \geq 0\}$ is U -traced by some point x of X .

Hence for each $i \geq 0$ $(T^i(x), x_i) \in U \Rightarrow (\varphi T^i(x), \varphi(x_i)) \in (\varphi \times \varphi)(U) \Rightarrow (S^i(\varphi(x)), y_i) \in H(U)$ this proves pseudo-trajectory $\{y_i: i \geq 0\}$ for S is $H(U)$ -traced by a point $\varphi(x)$ in Y . ■

Example (2.11) [7]

Consider the uniform space (X, \mathcal{U}) define as follows.

$X = [0,1)$ and \mathcal{U} be the uniformity induced by the usual uniform on X . Define $T: X \rightarrow X$ by $T(x) = x(-3x + 4)^{-1}$. We need to show that T has USP. If $U \in \mathcal{U}$ with the property that $U_\epsilon \not\subset U, \forall \epsilon > 0$.

Note that T is uniform homeomorphisms. Define the homeomorphisms $h(x): [0, \infty) \rightarrow X$ such that $h(x) = x^2(x^2 + 1)^{-1}$ and $f(x): [0, \infty) \rightarrow [0, \infty)$ such that $f(x) = \frac{1}{2}x$, then f has U_ϵ -SP, $\forall \epsilon > 0$. Since f is contraction map put $U = (h \times h)(U_\epsilon)$. Then by Theorem. 2.10., $hfh^{-1} = T$, T has U -shadowing property.

Note that $U = (h \times h)(U_\epsilon) = \{(u, v): u = x/(x^2 + 1), v = y^2/(y^2 + 1), x - \epsilon < y < x + \epsilon, x, y \geq 0\}$, does not contain $U_\epsilon, \forall \epsilon > 0$. We mention that shadowing property of map T you know on compact uniform space (X, \mathcal{U}) saved under conjugacy.

Theorem (2.12): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are uniform conjugates then T is uniform chain transitive (recurrent) iff S is uniform chain transitive (recurrent).

Proof: We prove this theorem by depending on proving of Theorem 2.10. and we note that $\forall x, y \in X$ if $x \sim_U y$ (with regard to T) then $\varphi(x) \sim_U \varphi(y)$ (with regard to S).

Theorem (2.13): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are uniform conjugates then T is uniform chain mixing iff S is uniform chain mixing.

Proof: Let T be uniform chain mixing. Given $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that we have $(x, y) \in U$ leads to $(\varphi(x), \varphi(y)) \in V, \forall x, y \in X$. If $y, \dot{y} \in Y$ then $\exists N > 0$, therefore $\forall n \geq n$ there exists a U -chain $y_0 = \varphi^{-1}(y), y_1, \dots, y_n = \varphi^{-1}(\dot{y})$ from $\varphi^{-1}(\dot{y})$ to $\varphi^{-1}(y)$. Thus $\varphi(y_0), \varphi(y_1), \dots, \varphi(y_n)$ is an V -chain from y to \dot{y} .

■

Theorem (2.14): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms. If T and S are uniform conjugates then T is uniform minimal iff S is uniform minimal.

Proof: If T is uniform minimal and $y \in Y$ such that $\overline{\text{tra}_T(\varphi^{-1}(y))} = X$. We show that $\overline{\text{tra}_S(y)} = Y$. Let $V \in \mathcal{U}_y$ be given and $\exists U \in \mathcal{U}_x$ therefore we have $(x_1, x_2) \in U$ leads to $(\varphi(x_1), \varphi(x_2)) \in V, \forall x_1, x_2 \in X$. If $z \in Y$ then $\exists n > 0$ where $(\varphi^{-1} \circ S^n(y), \varphi^{-1}(z)) = (T^n(\varphi^{-1}(y)), \varphi^{-1}(z)) \in U$. So $(S^n(y), z) \in V$, since φ is uniform homeomorphism.

■

Definition (2.15): Let (X, \mathcal{U}) be a uniform space and $T: X \rightarrow X$ be continuous onto map. A point x in X is said to be the uniform non-wandering point of T if for every neighborhood $U_{[x]}$ based of x , there exists an integer $n > 0$ such that $((T^n)(U_{[x]}) \cap U_{[x]}) \neq \emptyset$. We shall denote the set of all uniform non-wandering points of T by $\Omega_U(T)$.

Definition (2.16): Let (X, U) and (Y, V) be uniform spaces. A map $T: X \rightarrow Y$ is called to be uniform continuous if, for every $V \in V$, there is some $U \in \mathcal{U}$ such that $(x, y) \in U$ implies that $(T(x), T(y)) \in V$. If T is one-to-one, onto and both of T and T^{-1} are uniformly

continuous. We called T a uniform homeomorphism (uniform equivalent) iff uniform continuous function is continuous.

Theorem (2.17): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms on compact uniform spaces X and Y . If $\varphi: X \rightarrow Y$ is a uniform conjugacy from T to S , then $\Omega_V(S) = \varphi(\Omega_U(T))$.

Proof: Assume that $x \in \Omega_U(T)$ and $V_{[\varphi(x)]}$ is the neighborhood based of $\varphi(x)$. The uniform continuity of φ implies to $\varphi^{-1}(V_{[\varphi(x)]})$ is the neighborhood of x . So there exist $n > 1$, therefore

$$\left(\varphi^{-1}(V_{[\varphi(x)]})\right) \cap \left(T^n\left(\varphi^{-1}(V_{[\varphi(x)]})\right)\right) \neq \emptyset,$$

$$\text{If } T^n \circ \varphi^{-1} = \varphi^{-1} \circ S^n, \text{ then } \varphi^{-1}(V_{[\varphi(x)]}) \cap \varphi^{-1}(S^n(V_{[\varphi(x)]})) \neq \emptyset.$$

$$\Rightarrow \varphi^{-1}(V_{[\varphi(x)]}) \cap S^n(V_{[\varphi(x)]}) \neq \emptyset.$$

$$\Rightarrow (V_{[\varphi(x)]}) \cap S^n(V_{[\varphi(x)]}) \neq \emptyset, \text{ then } \varphi(x) \in \Omega_V(S)$$

Hence $\varphi(x) \in \Omega_V(S)$. Thus $\varphi(\Omega_U(T)) \subseteq \Omega_V(S)$. A similar way using the uniform conjugacy $\varphi^{-1}: Y \rightarrow X$ to prove that $\varphi(\Omega_U(T)) \supseteq \Omega_V(S)$

■

Theorem (2.18): Let $T: X \rightarrow X$ and $S: Y \rightarrow Y$ be two uniform homeomorphisms on compact uniform spaces X and Y . If $\varphi: X \rightarrow Y$ is a uniform conjugacy from T to S , then $CR_V(S) = \varphi(CR_U(T))$.

Proof: Let $x \in CR_U(T)$ and $V \in \mathcal{U}$. We build a V -trajectory from $\varphi(x)$ to herself. Since φ is uniformly continuous on X , $\exists U \in \mathcal{U}$ where $(z_1, z_2) \in U$ then $(\varphi(z_1), \varphi(z_2)) \in V$, since $x \in CR_U(T)$ there exists a U -chain $x_0 = x, x_1, \dots, x_n = x$ from x to herself. $\forall i = 0, \dots, n$ put $y_i = \varphi(x_i)$.

$$\text{Then } (S(y_i), y_{i+1}) = (S \circ \varphi(x_i), \varphi(x_{i+1})) = (\varphi \circ T(x_i), \varphi(x_{i+1})) \in V.$$

Therefor $\varphi(CR_U(T)) \subset CR_V(S)$.

A similar way using the uniform conjugacy $\varphi^{-1}: Y \rightarrow X$ to prove that $\varphi(CR_U(T)) \supset CR_V(S)$.

■

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