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ABSTRACT

This paper studies the concept of a quasi-inner product space and its completeness to get and prove some properties of quasi-Hilbert spaces. The best examples of this notion are spaces L_p , where $0<p<\infty$.

1. Introduction

The most important in functional analysis is the concept of normed space and its completeness which is related to other concepts such as a metric space, inner product space, and a quasi-normed space. The set of all measurable functions $L_p[a,b]$, $0<p<\infty$, $[a, b] \subset \mathbb{R}$ is a good example on these concepts ([3,4,8]).

In the normed spaces, mathematicians have used Gâteaux derivatives to introduce a notion of quasi- inner product space ([5,7]). This paper has used a quasi-normed space to give a quasi-inner product space and quasi-Hilbert space. It is studied the relationship between a notion of quasi-Hilbert space and others, with study some properties of it.

Section one of the paper includes definitions of a quasi-normed space, a quasi- Banach space and others with some valuable results. The second section presents a quasi- inner product space and quasi-Hilbert space with some essential properties.

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2. Quasi-Banach Spaces

Definition (2.1). Quasi-metric space (U, d_q), where U is a nonempty set X with a quasi-metric d_q which differs from a metric function by the inequality :

$$d_q(u,v) \le C\left(d_q(u,w) + d_q(w,v)\right) \text{ for all } u,v,w \in U, \text{ where } 1 \le C < \infty.$$

A function d_q be a metric if C = 1, thus it is generalization of a metric . Every a metric function is quasi-metric, but not the converse in generality [6].

Definition (2.2). A quasi-norm $_q\|\cdot\|$ on a vector space U over the field of real numbers $\mathbb{R}$ is a function $_q\|\cdot\|$: $U \longrightarrow [0,+\infty)$ with the properties:

(1) $_q\| v\| \ge 0, \forall v \in U,\ _q\| v\| = 0 \leftrightarrow v = 0$.

(2) $_q\| \alpha v\| = |\alpha|\,_q\| v\|,\ \forall v \in U, \forall \alpha \in \mathbb{R}$.

(3) $_q\| v+w\| \le C\left(_q\| v\| + {}_q\| w\|\right)\ \forall\ v, w, \in U$, where a constant $C \ge 1$.

If $C = 1$, then the quasi-norm is a norm function. A quasi-normed space is $\left(U, {}_q\|\cdot\|\right)$ or simply U. Since every quasi-normed space U is a quasi- metric space by $d_q(v,w) = {}_q\| v-w\|$, the concept of completeness is given. A quasi-Banach space is a complete quasi-normed space ([3,6]).

Remark (2.3). It is clear, every quasi-normed space is a quasi-metric space, conversely may be not true, indeed,

Take (U, d_q), where $d_q(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|u_k - v_k|}{1+|u_k - v_k|}$ is a quasi-metric space, but it is not a quasi-normed space (see[8]).

Remark (2.4). It is obvious, any inner product function satisfies Definition (2.2) and generates a quasi - norm which is $_q\| v\| = (< v,v >)^{1/2}\ \forall v \in U$.

Theorem (2.5) [1]. A quasi-normed space U is an inner product space iff equality:

$$_q\| v+w\|^2 + {}_q\| v-w\|^2 = 2\,_q\| v\|^2 + 2\,_q\| w\|^2 \qquad \forall u,v \in U, \qquad (2.1)$$

is satisfied by the quasi-norm of U.

Definition (2.6) [4]. The vector space $L_p[a,b]$, 0<p<∞, [a, b] ⊂ $\mathbb{R}$ is a set of all measurable functions f on [a, b] for which $\int_a^b |f(x)|^p\, dx < \infty$.

Theorem (2.7) [3,4]. A space $L_p\ [a,b]$, with the function $_q\| f \| = \left(\int_a^b | f(x)|^p \ dx \right)^{1/p}$ is a quasi-Banach space when $0<p<\infty$, a Banach space when $1<p<\infty$, and $L_2\,[a,b]$ is only a Hilbert space.

3. Some Properties of Quasi-Iinner Product Spaces.

Definition(3.1) [1]. Aquasi-normed space U is called a quasi-inner product space, if :

$$_q\| v+w \|^4 - {}_q\| v-w \|^4 = 8\,(\ {}_q\| v \|^2\ \tau(v,w) + {}_q\| w \|^2\ \tau(w,v)\ ),\ \forall\ v,w \in U, \qquad (3.1)$$

is satisfied, where $\tau(v,w)$ and $\tau(w,v)$ are Gateaux derivatives, A Gâteaux derivative of $_q\| v \|$, $\tau(v,w)$ at $v \in U$ in the direction $w \in U$ is defined as:

$$\tau(v,w) = \frac{_q\| v \|}{2}\left(\lim_{h\to +0} \frac{_q\| v+hw \| - {}_q\| v \|}{h} + \lim_{h\to -0} \frac{_q\| v+hw \| - {}_q\| v \|}{h} \right), \text{where } h \in \mathbb{R}.$$

Similarly, $\tau(w,v)$ at $w \in U$ in the direction v is defined. If U is a quasi-Banach space then it is called a quasi-Hilbert space.

Proposition (3.2). If U is a quasi-normed space then $\forall\ v,w \in U$,

(1) $\tau(v,w) \geq 0$, $\tau(0,w) = \tau(v,0) = 0$, and $\tau(v,v) = {}_q\| v \|^2$.

(2) If the right limit = the left limit then: $\tau(v,w) = {}_q\| v \|\ \lim_{h\to 0} h^{-1}\left({}_q\| v+hw \| - {}_q\| v \| \right)$.

Proof:

The proof is obvious from Definition (3.1)**.**

Proposition (3.3). If U is an inner-product space then $\tau(v,w) = < v,w >$ and $\tau(w,v) = < w,v >$

$\forall\ v,w \in U$.

Proof:

From Definition (3.1), $\forall\ v,w \in U$.

$$\tau(v,w) = \frac{_q\| v \|}{2}\left(\left(\lim_{h\to +0} \frac{_q\| v+hw \|^2 - {}_q\| v \|^2}{h\ {}_q\| v+hw \| + {}_q\| v \|} \right) + \left(\lim_{h\to -0} \frac{_q\| v+hw \|^2 - {}_q\| v \|^2}{h\ {}_q\| v+hw \| + {}_q\| v \|} \right) \right).$$

Using Remark (2.4), we get $\tau(v,w) = < v,w >$. Similarly, $\tau(w,v) = < w,v >$.

Remark (3.4). Using the proof of Proposition (3.3) with the binomial theorem [2] in Definition (3.1) to get the functional $\tau\,(f,g)$ of the following:

(1) $\forall\ f,g \in L_1[a,b]$, we have $\tau(f,g) = {}_q\| f \| \int_a^b (\text{sng } f(x))g(x)\ dx$. where,

$$\text{sng } f(x) = \begin{Bmatrix} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \end{Bmatrix}.$$

(2) $\forall\ f,g \in L_4[a,b]$, we have $\tau(f,g) = {}_q\| f \|^{-2} \int_a^b |f(x)|^3 (\text{sng } f(x))g(x)\ dx$.

Similarly, $\tau(g,f)$ is defined for (1) and (2).

Example (3.5). If $f, g \in L_1[0,1]$, such that $f(x) = x\ \forall\ x \in [0, \frac{1}{2}]$ and $g(x) = x\ \forall\ x \in [\frac{1}{2}, 1]$.

From Remark (3.4), $\tau(f,g) = \tau(w,v) = \frac{3}{64}$ and the right hand of equation(3.1) = $\frac{15}{32}$, but the left hand = $\frac{3}{64}$, thus $L_1[0,1]$ is not a quasi-inner product space. Also, it is not an inner product space, since 1 = the right hand of equation (2.1) $\neq$ the left hand = $\frac{5}{16}$.

Theorem (3.6). A quasi-inner product space U is an inner-product space if and only if the equality (2.1) is satisfied.

Proof:

If U *is* a quasi-inner product space such that the equality (2.1) is satisfied, then by Theorem (2.5), it is an inner product space.

If U is an inner-product space, using Remark (2.4) we get :

$${}_q\| v+w \|^2 = \langle v+w, v+w \rangle = {}_q\| v \|^2 + 2\langle v, w \rangle + {}_q\| w \|^2 \Rightarrow \left({}_q\| v+w \|^2\right)^2 = \left({}_q\| v \|^2 + {}_q\| w \|^2\right)^2 + 4\langle v,w \rangle \left({}_q\| v \|^2 + {}_q\| w \|^2\right) + 4(\langle v,w \rangle)^2,\ \forall\ v,w \in U.$$

Also, ${}_q\| v-w \|^2 = {}_q\| v \|^2 - 2\langle v, w \rangle + {}_q\| w \|^2 \Rightarrow$

$${}_q\| v-w \|^4 = \left({}_q\| v \|^2 + {}_q\| w \|^2\right)^2 - 4\langle v,w \rangle\left({}_q\| v \|^2 + {}_q\| w \|^2\right) + 4(\langle v,w \rangle)^2.$$

This implies that, ${}_q\| v+w \|^4 - {}_q\| v-w \|^4 = 8({}_q\| v \|^2 + {}_q\| w \|^2)\langle v,w \rangle$.

Thus, from Proposition (3.3), an equation (3.1) is satisfied, and the proof is finished.

Example (3.7). Since the right hand of an equation (3 .1) = the left hand =

$8(\int_a^b |f(x)|^3 (\text{sng } f(x))\, g(x)\, dx + \int_a^b |g(x)|^3 (\text{sng } g(x)) f(x)\ dx)$ with $L_4[a,b]$, then it is a quasi-inner product spaces, but an equation(2.1) fails, indeed, If we take $L_4\,[-1,1]$ such that $f(x) = -x \;\forall\, x \in [-1,0]$ and $g(x) = x \;\forall\, x \in [0,1]$, then it is not an inner product space, since 1 = the right hand of equation(2.1) ≠ the left hand ≅ 1. 265.

Proposition (3.8). A quasi-inner product space U is an inner-product space if and only if the following equivalence holds:

$$ {}_q\| v + w \| = {}_q\| v - w \| \leftrightarrow \tau(v, w) = 0, \;\forall\; v , w \in U \qquad (3.2)$$

Proof:

The proof of this proposition proceeds in a same way into version in a normed space [5].

Remark (3.9). (1) Since a space $L_p\,[a, b]$is a quasi-Banach space, then it is a quasi-Hilbert space if it is a quasi-inner product space for example a space $L_4\,[a,b]$ is a quasi-Hilbert space, while $L_1\,[a,b]$ is not .

(2) A space $L_2\,[a,b]$ is one only which be a quasi-Hilbert space and Hilbert space together , as shown in the following:

Theorem (3.10). $L_2\,[a,b]$, [a, b] ⊂ ℝ , is a quasi-Hilbert space .

Proof:

Since a space $L_2\,[a,b]$ is Hilbert space from Theorem(2.6), then $\forall$ f , g ∈ $L_2\,[a,b]$, $< f, g > = \tau\,(f, g) = \int_a^b |f(x)|\;\; (\text{sng}\, f(x)\,)\, g(x)\, dx$, and $< g, f > = \tau\,(g , f) = \int_a^b |g(x)|\;\; (\text{sng}\, g(x)\,) f(x)\;\, dx$, according to Proposition (3.3) and application the binomial theorem in Definition (3.1). Thus, the right hand of equation (3.1) is $8(\int_a^b |f(x)\,|^3 (\text{sng}\, f(x)\,)\, g(x)\, dx \int_a^b |g(x)\,|^3 (\text{sng}\, g(x)\,) f(x)\;\, dx\,)$ which is the same value of its left hand . Therefore, $L_2\,[a,b]$ is a quasi-Hilbert space.

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