Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 4 Article 21

8-15-2021

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Recommended Citation

Abdulmajeed, Ahmed Y. and Khudair, Ayad R. (2021) "Solving Riccati Type q–Difference Equations via Difference Transform Method," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 4, Article 21.
DOI: 10.29350/qjps.2021.26.4.1318
Available at: https://qjps.researchcommons.org/home/vol26/iss4/21

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Article History

Received on: 2/6/2021

Revised on: 15/7/2021

Accepted on:25/7/2021

Keywords:

Time scales calculus; Differential transform method; Riccati type q −difference equations

DOI:*https://doi.org/10.29350/ jops. 2021.26. 4.1318*

ABSTRACT

In this paper, we deal with a time scale that its delta derivative of graininess function is a nonzero positive constant. Based on the Taylor formula for this time scale, we investigate the difference transform method (DTM). This method has been applied successfully to solve Riccati type q −difference equations in quantum calculus. To demonstrate the ability and efficacy of this method, some examples have been provided.

1. Introduction

One of the simplest and more important type of nonlinear differential equations are Riccati differential equations [32]. Due to their close connection to the Bessel function, these equations have often appeared in many physical problems, likes; static Schrödinger equation [12], Newton's laws of motion [29], 3D-Gross-Pitaevskii equation [26], cosmology problem [33]. Also, it relates to many mathematical subjects, including; projective differential geometry [2], calculus of variations [41], optimal control [30], and dynamic programming [6]. Several techniques have been used to solve constant coefficients Riccati differential equations, such as; operation matrix method [31], variational iteration method [17], polynomial least squares method [9], homotopy perturbation method [1], Legendre wavelet method [3], and Adomian's decomposition method[15].

Riccati difference equations are not different from Riccati differential equations, as they have many applications in various fields. Where it arises in the filtering problem [27], the optimal control problem [5] and it has been studied by numerous scholars [4, 42, 38, 28]. In fact, the first study appeared on the difference Riccati equations was in 1905 by H. Tietze [39].

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The q-difference equation (q-DEs)is a type of difference equation that is based on q-calculus. Indeed, the old references refer to the beginning of q-calculus was in the late 20th century to make links between mathematics and physics [16]. It has a variety of uses in mathematics, engineering and science, including basic hypergeometric functions [10, 11, 36, 35, 34], orthogonal polynomials [25], combinatorics [19], and quantum theory [21]. In recent years, several scholars have attempted to solve many types of q-DEs by using semi-analytic methods, including; the q-differential transformation method (q-DTM)[13, 14], variational iteration method [40], succes- sive approximation method, and homotopy analysis method [37].

In this paper, we deal on time scale $\mathbb{T}\mathbb{T}$ that its delta derivative of graininess function is a nonzero positive constant, that is $\mu^{\Delta} = \zeta\zeta > 0$. Based on Taylor formula for this time scale, we introduce some fundamental theorems related to DTM in order to solve the following Riccati type q −difference equations on the time scale $\mathbb{T}\mathbb{T} = \overline{q^{\mathbb{N}}} = \{0\} \cup \{q^t | t \in \mathbb{N}, 0 < q < 1\}$:

$$\Psi^{\Delta}(t) = g_1(t)\Psi(t) + g_2(t)\Psi(t)^2 + g_3(t), \quad \Psi(t) = A. \tag{1}$$

where $g_c(t), r = 1,2,3$ are analytic function on $\mathbb{T}\mathbb{T}$.

2. Preliminaries

This section has provided a brief overview of time scale preliminary information and their relationship to q-calculus

Definition 2.1 *[18] A time scale is a non-empty arbitrary closed subset of real numbers denoted by* $\mathbb{T}\mathbb{T}$. *Time scale examples,* $[0,1]$, *the natural numbers set* $\mathbb{N}$, *the real numbers set* $\mathbb{R}$, $[0,1] \cup [2,3]$ *and the cantor set whereas the set of rational numbers* $\mathbb{Q}$, *complex numbers* $\mathbb{C}$, *and* $[0,1)$, $(0,1]$, $(0,1)$, $(0,1] \cup \{2\,b - a\}$ *are not time scales.*

Definition 2.2 *[8] Let* $\mathbb{T}\mathbb{T}$ *be any time scale and* $r \in \mathbb{T}\mathbb{T}$. *Operator of a forward jump* $\sigma: \mathbb{T}\mathbb{T} \to \mathbb{T}\mathbb{T}$ *is given as:*

$$\sigma(r) = \inf\{s \in \mathbb{T}\mathbb{T}: s > r\} \tag{2}$$

while, the Operator of a backward jump $\rho: \mathbb{T}\mathbb{T} \to \mathbb{T}\mathbb{T}$ at all $r \in \mathbb{T}\mathbb{T}$ is given as:

$$\rho(r) = \sup\{s \in \mathbb{T}\mathbb{T}: s < r\} \tag{3}$$

Definition 2.3 *[7]*

For $s \in \mathbb{T}\mathbb{T}$, the function $\mu: \mathbb{T}\mathbb{T} \to [0, \infty)$ defined by

$$\mu(s) = \sigma(s) - s \tag{4}$$

is called graininess function.

assume that

$$\mathbb{T}\mathbb{T} = \overline{q^{\mathbb{N}}} = \{q^t | t \in \mathbb{N}, 0 < q < 1\} \cup \{0\}$$

Let the q −shift factorial is given by

$$(t;q)_0 = 1 \quad and \quad (t;q)_m = \prod_{jj=0}^{m-1} (1 - aq^{jj}), \quad m = 1, \dots, m$$

such that t is real number.

Definition 2.4 *[7] For $t \in \overline{q^{\mathbb{N}}}$, the delta q −derivative of a function $g(t)$ on $\mathbb{TT} = \overline{q^{\mathbb{N}}}$ is given by*

$$g^{\Delta}(t) = \begin{cases} \frac{gg(qt)-gg(t)}{(q-1)t}, & iiff \quad t \in \overline{q^{\mathbb{N}}} \\ \lim\limits_{n\to\infty} \frac{gg(q^n)-gg(0)}{q^n}, & iiff \quad t = 0 \end{cases} \tag{5}$$

Definition 2.5 *[22] Let $G: \overline{q^{\mathbb{N}}} \to \mathbb{R}$ a pre-antiderivative of the function $g: \overline{q^{\mathbb{N}}} \to \mathbb{R}$.such that $G^{\Delta\Delta}(t) = g(t)$. The indefinite integral of the function g is define by*

$$\int g(t)\Delta t = G(t) + c, \tag{6}$$

where c is a constant. Moreover, the definite integral is defined by

$$\int_b^{dd} g(t)\Delta t = G(d) - g(b), \forall b, d \in \overline{q^{\mathbb{N}}} \tag{7}$$

Definition 2.6 *[8] The monomials $h_n: \mathbb{TT} \times \mathbb{TT} \to \mathbb{R}$, $n \in \mathbb{N}_0$ on a time scale $\mathbb{TT}$ are defined by*

$$h_0(r,s) = 1$$
$$h_{n+1}(r,s) = \int_s^r h_n(r,s)dr, \quad n \in \mathbb{N}, r, s \in \mathbb{TT}. \tag{8}$$

Hence, the Δ −derivative of $h_n(r,s)$ with respect to r given by

$$h_n^{\Delta}(r,s) = h_{n-1}(r,s), n \geq 1 \tag{9}$$

Example 2.1 *[8]*

1. When $\mathbb{TT} = \overline{q^{\mathbb{N}}}$, we get

$$h_n(t,s) = \prod_{\omega=0}^{n-1} \frac{t - sq^{\omega}}{\sum_{n=0}^{\omega} q^n}, \forall n \in \mathbb{N} \tag{10}$$

2. When $\mathbb{TT} = \mathbb{R}$, we get

$$h_n(t,s) = \frac{(t-s)^n}{n!}, \quad \forall n \in \mathbb{N} \tag{11}$$

3. When $\mathbb{TT} = \mathbb{Z}$, we get

$h_n(t,s) =$ � $t - s$ � $\forall n \in \mathbb{N}$ (12)

Theorem 2.1 *[8] For all $t, s \in \mathbb{TT}$ and $jj \in \mathbb{N}_0$, we have*

$$0 \leq h_{jj}(t,s) \leq \frac{(t-s)^{jj}}{jj!}, \quad \forall r \geq s \tag{13}$$

Let $m \in \mathbb{N}$ and $g: \mathbb{T} \to \mathbb{R}$ is $m -$ tiimes differentiable function on $\mathbb{T}^{k^m}$, $t \in \mathbb{T}$.

Let $s \in \mathbb{T}^{k^{m-1}}$, then

$$g(t) = \sum_{jj=0}^{m-1} h_{jj}(t,s) g^{\Delta^{jj}}(s) + R_m(t) \quad (14)$$

is called Taylors formula and the remainder term $R_m(t)$ is defined by

$$R_m(t) = \int_s^t g^{\Delta^m}(r) h_{m-1}(t,\sigma(r))\Delta r$$

and it tends to zero as $m \to \infty$.

Proposition 2.1 *[8] Let $g: \mathbb{TT} \to \mathbb{R}$ is an analytic function at s and at all $t \in (s - \varepsilon\varepsilon, +\infty) \cap \mathbb{TT}$ holds that $g(t) = \sum_{jj=0}^{\infty} a_{jj} h_{jj}(t,s)$. Then $g(t)$ is infinitely jj −times differentiable at s and $g^{\Delta\Delta^{jj}}(s) = a_{jj}$*

Theorem 2.2 *[24] For any $t, s \in \mathbb{TT}$ with $\mu^{\Delta\Delta} = \zeta\zeta > 0$ a constant. Then the product of monomials h_c and $h_{\kappa\kappa}$ as follows*

$$h_c(t,s) h_{\kappa\kappa}(t,s) = \sum_{u=\kappa\kappa}^{c+\kappa\kappa} F(u,\kappa,r) h^{\sigma\sigma^{\kappa\kappa}}_{c+\kappa\kappa-u}(s,s) h_u(t,s), \quad (15)$$

such that

$$F(u,\kappa,r) = \begin{cases} \sum_{\omega=1}^{\kappa\kappa} \sum_{l=1}^{u-\kappa\kappa} (-1)^{c-1} \varphi_l(u) u^{l(\omega+1)-\frac{r(r+1)}{2}}, & ffor \quad u > r \\ 1, & ffor \quad u = \kappa \end{cases} \quad (16)$$

$$\varphi_l(u,r) = \prod_{\substack{s \neq l \\ 1 \le s \le r}} \frac{1}{(u^{-u} - u^{-s})}, \varphi_1(u) = 1 \quad and \quad u = 1 + \zeta\zeta \quad (17)$$

Remark 2.1 $F(u,\kappa,r)$ *in theorem(2.2) can be computed in another way according to the following formula [24]*

$$F(u,\kappa,r) = \sum_{jj_1=0}^{u-\kappa\kappa} \sum_{jj_2=jj_1}^{u-\kappa\kappa} \sum_{jj_3=jj_2}^{u-\kappa\kappa} \cdots \sum_{jj_{\kappa\kappa-1}=jj_{\kappa\kappa-2}}^{u-\kappa\kappa} \sum_{jj_{\kappa\kappa}=jj_{\kappa\kappa-1}}^{u-\kappa\kappa} u^{\sum_{u=1}^{\kappa\kappa} jj_u} \quad (18)$$

3. The qq −differential transform method

In 1986, Zhou proposed the DTM and applied it to analyze the electric circuit problems [43]. Inspired Zhou's idea and based on q-Taylor's formula, the q-DTM has been introduced [20]. In 2011, ElShahed has been extended the q-DTM to two dimensional for solving partial q-DEs [13]. In the same year, the damped q-DEs with strongly nonlinear has been used successfully by using the q-DTM [23]. This section devoted to derive some important formula related to DTM. Now, let $\psi\psi(t)$ be is $N - tiimes$ q-differentiable on $\overline{q^{\mathbb{N}}}$, then by using theorem (2.1) with t_0 one can approximate the function $\psi\psi(t)$ as follows:

$$\Psi(t) = \sum_{u=0}^{\infty} \Psi_q[u] h_u(t,0) \quad (19)$$

where

$$\Psi_q[u] = \Psi^{\Delta^u}(0), \forall u = 0,1,2, \ldots \quad (20)$$

The Eq.((20)) is called the DTM, while Eq. ((19)) is called inverse of DTM.

Suppose that the functions $\Phi(t)$, $\Psi(t)$, and $\Xi(t)$ are approximate as $\Phi(t) = \sum_{\kappa\kappa=0}^{\infty} \Phi_q[\kappa] h_{\kappa\kappa}(t,0)$, $\Psi(t) = \sum_{\kappa\kappa=0}^{\infty} \Psi_q[\kappa] h_{\kappa\kappa}(t,0)$, and $\Xi(t) = \sum_{\kappa\kappa=0}^{\infty} \Xi_q[\kappa] h_{\kappa\kappa}(t,0)$ respectively, then the essential mathematical operations achieved by DTM are presented in the next theorems.

Theorem 3.1 *For any real constants a, and b , if* $\Xi(t) = a\,\Phi(t) \mp b\,T(t)$, *then* $\Xi_q[\kappa] = a\,\Phi_q[\kappa] \mp b\,T_q[\kappa], \forall \kappa = 0,1,2,\ldots$

Lemma 3.1 *If* $0 \in \mathbb{T}$ *and* $\mu^{\Delta} = \zeta$ *is nonzero constant, then multiplying any two monomials,* $h_c(t,0)$ *and* $h_{\kappa}(t,0)$ *, is given as follows:*

$$h_c(t,0)h_{\kappa}(t,0) = F(r+\kappa,\kappa,r)h_{c+\kappa}(t,0), \quad \kappa, r \neq 0, \quad \forall t \in \mathbb{T} \tag{21}$$

Proof. Since $\sigma^m(0) = 0$ for all $m = 0,1,2,\ldots$, we have

$$h_c^{\sigma^m}(0,0) = \begin{cases} 1, & r = 0 \\ 0, & o.w., \end{cases} \quad \forall m = 0,1,2,\ldots \tag{22}$$

Using theorem(2.2), one can get

$$h_c(t,0)h_{\kappa}(t,0) = \sum_{u=\kappa}^{c+m} F(u,\kappa,r)h^{\sigma^{\kappa}}_{c+\kappa-u}(0,0)h_u(t,0), \tag{23}$$

The result can be get it by substitute Eq.((22)) in Eq.((23)).

Theorem 3.2 *If* $T(t) = \Xi^{\Delta}(t)$, *then* $T_q[\kappa] = \Xi_q[\kappa+1], \forall \kappa = 0,1,2,\ldots$

Theorem 3.3 *If* $T(t) = \Xi(t)\Phi(t)$, *then*

$$\Psi_q[0] = \Xi_q[0]\,\Phi_q[0]$$

$$\Psi_q[1] = \Xi_q[1]\,\Phi_q[0] + \Xi_q[0]\,\Phi_q[1]$$

$$\Psi_q[r] = \Xi_q[r]\,\Phi_q[0] + \Xi_q[0]\,\Phi_q[r] + \sum_{\kappa=1}^{c-1} \Xi_q[r-\kappa]\,\Phi_q[\kappa]F(r,r-\kappa,\kappa), \quad r = 2,3,4,\cdots$$

Proof. Let $\Psi(t) = \Xi(t)\Phi(t)$ so one can have

$$\sum_{\kappa=0}^{\infty} \Psi_q[\kappa]h_{\kappa}(t,0) = \left(\sum_{\kappa=0}^{\infty} \Xi_q[\kappa]h_{\kappa}(t,0)\right)\left(\sum_{\kappa=0}^{\infty} \Phi_q[\kappa]h_{\kappa}(t,0)\right)$$

$$= \sum_{\nu=0}^{\infty}\sum_{\kappa=0}^{\infty} \Xi_q[\nu]\Phi_q[\kappa]h_{\kappa}(t,0)h_{\nu}(t,0)$$

By using lemma(3.1), one can have

$$\sum_{\kappa=0}^{\infty} \Psi_q[\kappa]h_{\kappa}(t,0) = \sum_{\kappa=0}^{\infty} \Xi_q[\kappa]\Phi_q[0]h_{\kappa}(t,0) + \sum_{\kappa=1}^{\infty} \Xi_q[0]\Phi_q[\kappa]h_{\kappa}(t,0)$$
$$+\sum_{c=1}^{\infty}\sum_{\kappa=1}^{\infty} \Xi_q[r]\Phi_q[\kappa]F(\kappa+r,r,\kappa)h_{\kappa+c}(t,0) \tag{24}$$

Now, change the index in the third sum of Eq.((24)), we have

$$\sum_{\kappa=0}^{\infty} \Psi_q[\kappa]h_{\kappa}(t,0) = \sum_{\kappa=0}^{\infty} \Xi_q[\kappa]\Phi_q[0]h_{\kappa}(t,0) + \sum_{\kappa=1}^{\infty} \Xi_q[0]\Phi_q[\kappa]h_{\kappa}(t,0)$$
$$+\sum_{c=2}^{\infty}\sum_{\kappa=1}^{c-1} \Xi_q[r-\kappa]\Phi_q[\kappa]F(r,r-\kappa,\kappa)h_c(t,0) \tag{25}$$

Finally, the coefficients of $h_c(t,0)$ are compared, and the result is obtained directly.

Theorem 3.4 *If* $f(t)$ *is analytic function on time scale* $\mathbb{T} = \overline{q^{\mathbb{N}}}$ *and* $T(t) = f(t)\Phi(t)$, *then*

$$\Psi_q[0] = f(0)\,\Phi_q[0]$$

$$\Psi_q[1] = ff^{\Delta}(0)\ \Phi_q[0] + ff(0)\ \Phi_q[1]$$

$$\Psi_q[r] = ff^{\Delta^r}(0)\ \Phi_q[0] + ff(0)\ \Phi_q[r] + \sum_{\kappa\kappa=1}^{c-1} ff^{\Delta^{r-\kappa\kappa}}(0)\ \Phi_q[\kappa]F(r, r-\kappa, \kappa), \quad r = 2,3,4,\cdots$$

Proof. Since $ff(t)$ is analytic function on time scale $\mathbb{TT} = \overline{q^{\mathbb{N}}}$, one can get $ff(t) = \sum_{\kappa\kappa=0}^{\infty} ff^{\Delta^{\kappa\kappa}}(0) h_{\kappa\kappa}(t, 0)$. Therefore, the result can be obtained directly using theorem(3.3).

Theorem 3.5 *If $ff(t)$ is analytic function on time scale $\mathbb{TT} = \overline{q^{\mathbb{N}}}$ and $T(t) = ff(t)\phi\phi^2(t)$, then*

$$\Psi_q[0] = ff(0)\ \Phi_q^2[0]$$

$$\Psi_q[1] = ff^{\Delta}(0)\ \Phi_q^2[0] + 2ff(0)\ \Phi_q[1]\Phi_q[0]$$

$$\Psi_q[r] = ff^{\Delta^r}(0)\Phi_q^2[0] + ff(0)\Phi_q[r]\Phi_q[0] + ff(0)\Phi_q[0]\Phi_q[r] + ff(0)\sum_{\kappa\kappa=1}^{c-1} \Phi_q[r-\kappa]\Phi_q[\kappa]F(r, r-\kappa, \kappa)$$

$$+\sum_{\kappa\kappa=1}^{c-1} ff^{\Delta^{r-\kappa\kappa}}(0)\Phi_q[\kappa]\Phi_q[0]F(r, r-\kappa, \kappa) + \sum_{\kappa\kappa=1}^{c-1} ff^{\Delta^{r-\kappa\kappa}}(0)\Phi_q[0]\Phi_q[\kappa]F(r, r-\kappa, \kappa)$$

$$+\sum_{\kappa\kappa=1}^{c-1}\sum_{u=1}^{\kappa\kappa-1} ff^{\Delta^{r-\kappa\kappa}}(0)\Phi_q[\kappa-u]\Phi_q[u]F(\kappa, \kappa-u, u)F(r, r-\kappa, \kappa) \quad, r = 2,3,4,\cdots$$

Proof. According to theorem (3.3), we find $\Phi^2(t) = \sum_{c=0}^{\infty} \Upsilon_q[r]\ h_c(t, 0)$

Where $\Upsilon_q[r]$ define as follows:

$$\Upsilon_q[0] = \Phi_q^2[0]$$

$$\Upsilon_q[1] = 2\Phi_q[1]\Phi_q[0]$$

$$\Upsilon_q[r] = \Phi_q[r]\Phi_q[0] + \Phi_q[0]\Phi_q[r] + \sum_{\kappa\kappa=1}^{c-1}\Phi_q[r-\kappa]\Phi_q[\kappa]F(r, r-\kappa, \kappa), r = 2,3,4,\cdots$$

However, since $ff(t)$ is analytic function on time scale $\mathbb{TT} = \overline{q^{\mathbb{N}}}$, we can get $ff(t) = \sum_{\kappa\kappa=0}^{\infty} ff^{\Delta^{\kappa\kappa}}(0)h_{\kappa\kappa}(t, 0)$.

Using theorem(3.4), we have

$$\Psi_q[0] = ff(0)\ \Upsilon_q[0]$$

$$\Psi_q[1] = ff^{\Delta}(0)\ \Upsilon_q[0] + ff(0)\ \Upsilon_q[1]$$

$$\Psi_q[r] = ff^{\Delta^r}(0)\ \Upsilon_q[0] + ff(0)\ \Upsilon_q[r] + \sum_{\kappa\kappa=1}^{c-1} ff^{\Delta^{r-\kappa\kappa}}(0)\ \Upsilon_q[\kappa]F(r, r-\kappa, \kappa), \quad r = 2,3,4,\cdots$$

Now, replace the values of $\Upsilon_q[r]$ in the above equations by its equivalent values in terms $\Phi_q[r]$, we get the result directly.

4. Illustrated Examples

Example 4.1 *Consider the Riccati q-difference equation as follows:*

$$\Psi^{\Delta}(t) = 1 - \Psi(t)^2 \tag{26}$$

$$\Psi(0) = 0 \tag{27}$$

When q tends to 1, the solution exactly has the form

$$\Psi(t) = tanh(t) \tag{28}$$

Applying DTM to Eq.((26)) , we have

$$\Psi_q[1] + \Psi^2{}_q[0] - 1 = 0$$

$$\Psi_q[2] + 2\Psi_q[1]\,\Psi_q[0] = 0$$

$$\Psi_q[r+1] = -\Psi_q[r]\,\Psi_q[0] - \Psi_q[0]\,\Psi_q[r] - \sum_{\kappa\kappa=1}^{c-1} \Psi_q[r-\kappa]\,\Psi_q[\kappa]F(r, r-\kappa, \kappa),\quad r = 2,3,4,\cdots \tag{29}$$

Again apply DTM to the initial conditions in Eq.((27)) , one can have

$$\Psi_q[0] = 0. \tag{30}$$

Using the Maple software, one can solve the recurrence relation in Eq.((29)) with Eq.((30)) to have the value of the unknown coefficients as follows:

$$\Psi_q[1] = 1$$

$$\Psi_q[2] = 0$$

$$\Psi_q[3] = -3 + q$$

$$\Psi_q[4] = 0$$

$$\Psi_q[5] = 2(q^2 - 4q + 5)(-3 + q)^2$$

$$\Psi_q[6] = 0$$

$$\Psi_q[7] = (-3 + q)^3(q^2 - 3q + 3)(q^4 - 9q^3 + 35q^2 - 69q + 59)(q^2 - 4q + 5)$$

$$\Psi_q[8] = 0$$

$$\Psi_q[9] = 2(q^2 - 4q + 5)^2(q^2 - 3q + 3)(-3 + q)^4(2q^6 - 26q^5 + 143q^4 - 427q^3$$
$$+737q^2 - 711q + 313)(q^4 - 8q^3 + 24q^2 - 32q + 17)$$

$$\vdots$$

So, $\Psi(t) \simeq \sum_{c=0}^{9} \Psi_q[r]h_c(t,0)$ is the first ten terms of the solution of this problem. Moreover, when $q \to 1$ this solution is given by

$$\lim_{q\to 1} \Psi(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots \tag{31}$$

When $q \to 1$, the solution in Eq.((31)) agrees exactly with the Taylor series of the given solution. Also, Eq.((31)) is agreement with the result in, Example(1) [37].

Example 4.2 *Consider the Riccati q-difference equation as follows:*

$$\Psi^{\Delta}(t) = 1 + 2\Psi(t) - \Psi^2(t) \tag{32}$$

$$\Psi(0) = 0 \tag{33}$$

When q tends to 1, the solution exactly has the form

$$\Psi(t) = \sqrt{2}\tanh(\sqrt{2t} + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})) + 1 \quad (34)$$

Applying DTM to Eq.((32)) , we have

$$\Psi_q[1] = 2\Psi_q[0] - \Psi_q^2[0] + 1$$

$$\Psi_q[2] = 2\Psi_q[1] - 2\Psi_q[1]\,\Psi_q[0]$$

$$\Psi_q[r+1] = 2\Psi_q[r] - 2\Psi_q[r]\,\Psi_q[0] - \sum_{\kappa\kappa=1}^{c-1} \Psi_q[r-\kappa]\,\Psi_q[\kappa]F(r, r-\kappa, \kappa), \ \forall r = 2,3,4,\cdots \quad (35)$$

Again apply DTM to the initial conditions in Eq.((33)) , one can have

$$\Psi_q[0] = 0. \quad (36)$$

Using the Maple software, one can solve the recurrence relation in Eq.((35)) with Eq.((36)) to have the value of the unknown coefficients as follows:

$$\Psi_q[1] = 1$$

$$\Psi_q[2] = 2$$

$$\Psi_q[3] = 1 + q$$

$$\Psi_q[4] = -4q^2 + 22q - 26$$

$$\Psi_q[5] = -2(q-3)(q^3 - 9q^2 + 31q - 37)$$

$$\Psi_q[6] = -4q^7 + 56q^6 - 352q^5 + 1276q^4 - 2832q^3 + 3732q^2 - 2536q + 548$$

$$\Psi_q[7] = q^{11} - q^{10} - 239q^9 + 3230q^8 - 21721q^7 + 91686q^6 - 262608q^5 + 524197q^4$$
$$-725540q^3 + 669551q^2 - 373085q + 95377$$

$$\Psi_q[8] = 8q^{15} - 228q^{14} + 2956q^{13} - 22836q^{12} + 114738q^{11} - 375506q^{10} + 685838q^9$$
$$+144320q^8 - 5305474q^7 + 18573384q^6 - 38811708q^5 + 55564898q^4 - 55575960q^3$$
$$+37593290q^2 - 15612662q + 3034030$$

$$\Psi_q[9] = -20q^{20} + 888q^{19} - 18662q^{18} + 247048q^{17} - 2312130q^{16} + 16272346q^{15} - 89406344q^{14}$$
$$+392904980q^{13} - 1403278564q^{12} + 4115164306q^{11} - 9966897976q^{10} + 19979418384q^9$$
$$-33100759636q^8 + 45089744554q^7 - 50018425772q^6 + 44494246668q^5 - 30993249576q^4$$
$$+16286776898q^3 - 6068996878q^2 + 1427426744q - 158832042$$

$\vdots$

Therefore $\Psi(t) \simeq \sum_{c=0}^{9} \Psi_q[r]h_c(t,0)$ is the first ten terms of the solution of the given problem. When $q \to 1$ this solution is given by

$$\lim_{q\to 1} \Psi(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + \frac{197}{2835}t^9 + \cdots \quad (37)$$

When $q \to 1$, the solution in Eq.((37)) agrees exactly with the Taylor series of the given solution.

Example 4.3 *Consider the Riccati q-difference equation as follows:*

$$\Psi^{\Delta}(t) = 2\Psi^2(t) - t\Psi(t) + 1 \quad (38)$$

$$\Psi(0) = 0 \quad (39)$$

When q tends to 1, the solution exactly has the form

$$\Psi(t) = \frac{t}{1-t^2} \quad (40)$$

Applying DTM to Eq.((38)) , we have

$$\Psi_q[1] = 2\Psi_q^2[0] + 1$$

$$\Psi_q[2] = 4\Psi_q[1]\Psi_q[0] + \Psi_q[0]$$

$$\Psi_q[r+1] = 4\Psi_q[r]\Psi_q[0] + \Psi_q[r-1]F(r,1,r-1)$$

$$+2\sum_{\kappa\kappa=1}^{c-1} \Psi_q[r-\kappa]\Psi_q[\kappa]F(r,r-\kappa,\kappa), \ \ \forall r = 2,3,4,\cdots \quad (41)$$

Again apply DTM to the initial conditions in Eq.((39)) , one can have

$$\Psi_q[0] = 0. \quad (42)$$

Using the Maple software, one can solve the recurrence relation in Eq.((41)) with Eq.((42)) to have the value of the unknown coefficients as follows:

$$\Psi_q[1] = 1$$

$$\Psi_q[2] = 0$$

$$\Psi_q[3] = 9 - 3q$$

$$\Psi_q[4] = 0$$

$$\Psi_q[5] = 15(q^2 - 4q + 5)(q-3)^2$$

$$\Psi_q[6] = 0$$

$$\Psi_q[7] = -3(q^2 - 3q + 3)(q^2 - 4q + 5)(6q^4 - 54q^3 + 211q^2 - 419q + 361)(q-3)^3$$

$$-725540q^3 + 669551q^2 - 373085q + 95377$$

$$\Psi_q[8] = 0$$

$$\Psi_q[9] = 15(12q^6 - 156q^5 + 858q^4 - 2562q^3 + 4423q^2 - 4271q + 1885)(q^2 - 3q + 3)$$
$$\times (q^4 - 8q^3 + 24q^2 - 32q + 17)(q^2 - 4q + 5)^2(q - 3)^4$$
$$\vdots$$

Therefore $\Psi(t) \simeq \sum_{c=0}^{9} \Psi_q[r]h_c(t,0)$ is the first ten terms of the solution of the given problem. When $q \to 1$ this solution is given by

$$\lim_{q\to 1} \Psi(t) = t + t^3 + t^5 + t^7 + t^9 + \cdots \quad (43)$$

When $q \to 1$, the solution in Eq.((43)) agrees exactly with the Taylor series of the given solution.

5. Conclusions

In this study, we introduce the difference transform method (DTM) based on Taylor formula for any time scale with its delta derivative of graininess function is a nonzero positive constant. Riccati type q −difference equations on quantum calculus have been successfully solved and the results coincide exactly with the Taylor series of the exact solution when q − tends to 1. In fact, this method is applicable to solving any nonlinear difference equations on any time scale with $\mu^\Delta > 0$.

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