[Al-Qadisiyah Journal of Pure Science](https://qjps.researchcommons.org/home)

Volume 26 | Number 4 Article 32

8-15-2021

On Φ-Prime Submodules

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Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (Special issue num.4) (2021) pp. 288–292

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Article History

Received on: 19/6/2021

Revised on: 30 /7/2021

Accepted on: 5/8/2021

Keywords:

Prime ideal

Prime submodule

ϕ-prime submodule

DOI: https://doi.org/10.29350/ jops.2021.26. 4.1374

ABSTRACT

In this paper, we study some properties of ϕ-prime submodules and we give another charactrization for it. For given submodules N and K of a module M with $K \subseteq N$, we prove that N is ϕ-prime submodule if and only if $\frac{N}{K}$ is ϕ_K-prime submodule. Finally, we show that any finite sum of ϕ-prime submodules is ϕ-prime.

1. Introduction

Throughout this paper R is a commutative ring with identity and M is a unitary R-module. Prime ideals play an essential role in ring theory. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodule (see [2], [3] and [4]). These have led to more information on the structure of the R-module M. A proper submodule P of M is called a prime submodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that $r \in (P: M)$ or $x \in P$. It is easy to see that P is a prime submodule of M if and only if $(P: M)$ is a prime ideal of R. A submodule $P \neq M$ is said to be weak prime submodule of M if $r \in R$ and $x \in M$, $0 \neq rx \in P$ gives that $r \in (P: M)$ or $x \in P$. So any prime submodule is weak prime. Let $S(M)$ be the set of all submodules of M and $\phi: S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is said to be a ϕ-prime submodule of M if $r \in R$ and $x \in M$, $rx \in P - \phi(P)$ implies that $r \in (P: M)$ or $x \in P$. Since $P - \phi(P) = P - (P \cap \phi(P))$, so without loss of generality, throughout this paper we will consider $\phi(P) \subseteq P$.

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2. Main Results

We begin this section with the following definition.

Definition (2.1) ([5, Definition 8.18]): Let M be an R-module, we say that a non-zero element $m \in M$ is zero divisor on M if there is a non-zero element $r \in R$ such that $rm = 0$.

For a submodule N of a module M, we can consider $\frac{M}{N}$ as $\frac{R}{(N:M)}$-module by defining the module multiplication as $(r + (N:M))(m + N) = rm + N$. If a submodule N of M is prime, then the $\frac{R}{(N:M)}$-module $\frac{M}{N}$ is torsion-free [1, Lemma 1], where torsion-free module means the annihilator of every non zero element is zero. But this is not true in the case of ϕ-prime submodules as we see in the following proposition which is a generalization of [6, Proposition 2.9].

Proposition (2.2): Suppose N is ϕ-prime submodule of M. Then we have the following.

1. If $m + N$ is zero divisor on the $\frac{R}{(N:M)}$-module $\frac{M}{N}$, then $(N:M)m \subseteq \phi(N)$.
2. If $\frac{L}{N}$ is a submodule of $\frac{M}{N}$ contains zero divisors on $\frac{M}{N}$, then $(N:M)L \subseteq \phi(N)$.

Proof:

1. As m is a zero divisor on $\frac{M}{N}$, there is $r + (N:M) \in R/(N:M)$ such that $(r + (N:M))(m + N) = rm + N = N$, that is $rm \in N$. Since N is ϕ-prime submodule of M, $rm \in \phi(N)$. Let $s \in (N:M)$. Then $sx \in N$, for each $x \in M$. Thus $sm \in N$ and $(s + r)m \in N$. Since $r \notin (N:M)$, $s + r \notin (N:M)$. As N is ϕ-prime submodule of M, if $(s + r)m \notin \phi(N)$, then $s + r \in (N:M)$ or $m \in N$ which contradict the hypothesis, hence $(s + r)m \in \phi(N)$ and so $sm \in \phi(N)$, for each $s \in (N:M)$. Thus $(N:M)m \subseteq \phi(N)$.

2. Let $r \in (N:M)$ and $m \in L$. Since $m + N$ is zero divisor on $\frac{M}{N}$, by part (1), $(N:M)m \subseteq \phi(N)$ and hence $(N:M)L \subseteq \phi(N)$.

Let N be a submodule of M. We define $\phi_N: S(\frac{M}{N}) \to S(\frac{M}{N}) \cup \{\emptyset\}$ by $\phi_N(\frac{L}{N}) = \emptyset$ if $\phi(L) = \emptyset$, otherwise $\phi_N(\frac{L}{N}) = \frac{(\phi(L)+N)}{N}$ for $N \subseteq L$. It is clear for submodules N and K of M that $(N: M) = (\frac{N}{K}:\frac{M}{K})$. In the following theorem, we show that every ϕ_N-prime submodule of $\frac{M}{N}$ is of the form $\frac{P}{N}$, where P is ϕ-prime submodule of M.

Theorem (2.3): For ϕ-prime submodule N of M, there is a one to one corresponding between ϕ_N-prime submodules of $\frac{M}{N}$ and ϕ-prime submodules of M that contains N.

Proof: Assume that L is ϕ-prime submodule of M that contains N. Then by [6, Theorem 2.12 Part(1)], $\frac{L}{N}$ is ϕ_N-prime submodule of $\frac{M}{N}$. Now, suppose $\frac{L}{N}$ is ϕ_N-prime submodule of $\frac{M}{N}$. For $r \in R$ and $m \in M$, let $rm \in L - \phi(L)$. If $m \in N$, then $rm \in N - \phi(N)$ because $N \subseteq L$. As N is ϕ-prime submodule of M, $r \in (N: M) \subseteq (L: M)$ or $m \in N \subseteq L$. If $m \notin N$, then $(r + (N: M))(m + N) = rm + N \in \frac{L}{N} - \frac{(\phi(L)+N)}{N} = \frac{L}{N} - \phi(\frac{L}{N})$. Since $\frac{L}{N}$ is ϕ_N-prime submodule of $\frac{M}{N}$, $r + (N: M) \in \frac{(\frac{L}{N}:\frac{M}{N})}{(N:M)} = \frac{(L:M)}{(N:M)}$ or $m + N \in \frac{L}{N}$, that is $r \in (L: M)$ or $m \in L$ and therefore L is ϕ-prime submodule of M that contains N.

Theorem (2.4): Let N be a submodule of M. Then N is ϕ-prime submodule of M if and only if $\frac{N}{\phi(N)}$ is weakly prime submodule of the R-module $\frac{M}{\phi(N)}$.

Proof: Let N be ϕ-prime submodule of M and $r(m + \phi(N)) \in \frac{N}{\phi(N)}$ for $r \in R$ and $m \in M$ with $rm \notin \phi(N)$. Thus $rm + \phi(N) \in \frac{N}{\phi(N)}$ and so $rm \in N - \phi(N)$. As N is ϕ-prime submodule, we have $r \in (N: M)$ or $m \in M$, that is $r \in (\frac{N}{\phi(N)}:\frac{M}{\phi(N)})$ or $m + \phi(N) \in \frac{N}{\phi(N)}$. Now, let $\frac{N}{\phi(N)}$ be weakly prime submodule of $\frac{M}{\phi(N)}$. We have to show that N is ϕ-prime submodule of M. For any $r \in R$ and $m \in M$, let $rm \in N - \phi(N)$. So, $rm + \phi(N) \in \frac{N}{\phi(N)}$ and by hypothesis, either $r \in (\frac{N}{\phi(N)}:\frac{M}{\phi(N)})$ or $m + \phi(N) \in \frac{N}{\phi(N)}$. Hence $r \in (N: M)$ or $m \in N$.

Theorem (2.5): Let N and K be two ϕ-prime submodules of M such that $N + K \neq M$. Then $N + K$ is ϕ-prime submodule of M.

Proof: The proof is obtained by the isomorphism $\frac{N+K}{N} \cong \frac{K}{N \cap K}$ and Theorem 2.3.

Corollary (2.6): Any finite sum of ϕ-prime submodules is ϕ-prime provided that the sum is being proper.

Finally, for a ϕ-prime submodule K of M and an epimorphism $f: M \to N$, we define $\phi_f: S(N) \to S(N) \cup \{\emptyset\}$ by $\phi_f(f(K)) = f(\phi(K))$. Clearly $\phi_f(f(K)) \subseteq f(K)$ and if $Ker(f) \subseteq K$, then we have $(f(K): f(M)) = (K: M)$. With this notation, we can give the following result.

Proposition (2.7): Let $f: M \to N$ be an epimorphism. Then a submodule K of M is ϕ-prime submodule of M with $Ker(f) \subseteq \phi(K)$ if and only if $f(K)$ is ϕ_f-prime submodule of N.

Proof: Suppose K is ϕ-prime submodule of M with $Ker(f) \subseteq \phi(K)$ and for $r \in R$, $n \in N$, Let $rn \in f(K) - \phi_f(K) = f(K) - f(\phi(K))$. Then there exists $m \in M$ such that $rn = rf(m) = f(rm) \in f(K) - f(\phi(K))$. Hence $rm \in K + Ker(f)$ and $rm \notin \phi(K)$. Since $Ker(f) \subseteq \phi(K)$, $rm \in K - \phi(K)$. As K is ϕ-prime submodule of M, $r \in (K: M)$ or $m \in K$, that is $r \in (f(K): f(M)) = (f(K): N)$ or $f(m) \in f(K)$ and $f(K)$ becomes ϕ_f-prime submodule of N.

Now, assume that $f(K)$ is ϕ_f-prime submodule of N. To show that K is ϕ-prime submodule of M, we take $rm \in K - \phi(K)$ for $r \in R$ and $m \in M$. So $f(rm) = rf(m) \in f(K)$. If $f(rm) \in \phi_f(K) = f(\phi(K))$, then $rm \in \phi(K) + Ker(f) = \phi(K)$ which contradicts the hypothesis. Thus $rf(m) \in f(K) - \phi_f(K)$ and as $f(K)$ is ϕ_f-prime submodule, we have $r \in (f(K): N) = (K: M)$ or $f(m) \in f(K)$, that is $r \in (K: M)$ or $m \in K$.

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