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strongly N-extending Strongly N-extending Modules

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ABSTRACT

Relative extending modules and relative (quasi-)continuous modules were introduced and studied by Oshiro as a generalizations of extending modules and (quasi-) continuous respectively. On other hand, Oshiro, Rizvi and Permouth introduced N-extending and N-(quasi-) continuous modules depending $\mathcal{A} = \mathcal{A}(\mathrm{N}, \mathrm{M}) = \{\mathrm{A} \subseteq \mathrm{M} | \exists \mathrm{X} \subseteq \mathrm{N}, \exists \mathrm{f}\mathrm{Hom}(\mathrm{X}, \mathrm{M}) \text{such that } \mathrm{f}(\mathrm{X}) \text{is essential in A}\}$ where N and M are modules. $\mathcal{A}(\mathrm{N}, \mathrm{M})$ is closed under submodules, essential extension and isomorphic image. A module M is N-extending if for each submodule $\mathrm{A} \in \mathcal{A}(\mathrm{N}, \mathrm{M})$, there is a direct summand B of M such that A is essential in B. Moreover, a module M is strongly extending if every submodule is essential in a stable (equivalently, fully invariant) direct summand of M.

In this paper, we introduce and study classes of modules which are proper stronger than that of N-extending modules and N-(quasi-)continuous modules. Many characterizations and properties of these classes are given.

1. Introduction

Let R be a ring with identity and M be a left unitary R-module. A nonzero submodule N of M is essential if every non-zero submodule of M has non-zero intersection with N. Also, a submodule N of M is closed in M, if it has no proper essential extensions in M. Also, Let N be a submodule of M, a relative complement of N in M is any submodule N' of M which is maximal with respect to the property that $N \cap N' = 0$. For details of these concepts see [3]. Recall that a module M is extending if, every submodule of M is essential in a direct summand of M.

The notion of the extending modules and their generalizations studied extensively by many authors. Oshiro in [7], introduced relative extending modules a generalization of the concept of extending modules. Following [7], let $\mathcal{A}$ be subfamily of the family of all submodules of an R-module M, a module M is $\mathcal{A}$ -extending if every submodule which with belong to$\mathcal{A}$ is essential in a direct summand of M. On the other hand, K. Oshiro, S. Riziv and S.Permouth in [6],

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introduced N-extending modules, depending on the family $\mathcal{A}(N,M) = \{ A \subseteq M \mid \exists X \subseteq N , \exists f \in Hom\,(X,M)\,, f\,(X)$ is essential in $A\}$ where N, M are modules. They called a module M is N-extending (or M has$\mathcal{A}(N,M)\mathrm{C}_1$) if for each $A \in \mathcal{A}\,(N,M)$, there exists A' which is a direct summand of M such that A is essential in A'.

The author in [2] introduced and studied a class of modules which is stronger than extending modules. A module M is strongly extending if, every submodule in M is essential in a stable (equivalently, fully invariant) direct summand of M.

In paper, we introduce and study relative strongly extending modules which are stronger concepts of relative extending modules.

2. Strongly N-extending modules.

Recall that a submodule N of a module M is fully invariant if $f(N)\subseteq N$ for each $f \in \mathrm{End}_R(M)$ and a module M is called Duo if every submodule of M is fully invariant [8]. A submodule N of a module M is called stable if, $f(N)\,)\subseteq N$ for each homomorphism $f: N\to M$. A module M is called fully stable if each submodule of M is stable [1].

Definition (1.1): Let N be a module. A module M is said to be strongly N-extending if for each submodule A of M with $A \in \mathcal{A}\,(N,M)$, is essential in a stable direct summand of M.

Remarks and Examples (1.2):

(1) Every strongly extending is strongly N-extending for each module N. But the converse is not true. For example, the Z-module $M = Z \oplus Z$ is Z_3-extending which is not strongly extending.

(2) Every strongly N-extending is N-extending. But the converse is not true in general. For example, the Z-module $M = Z_2 \oplus Z$ is Z_2- extending [6], which is not strongly Z_2-extending. Since $Z_2 \oplus 0$ is closed submodule of M and belong to $\mathcal{A}(Z_2,M)$. Let $f: Z_2 \oplus (0) \longrightarrow Z_2 \oplus Z$ by $f(\overline{x},0)= (0,x)$ for each $(\overline{x},0) \in Z_2 \oplus (0)$. f is Z-homomorphism while $f(Z_2 \oplus (0)) \nsubseteq Z_2 \oplus (0)$ so $Z_2 \oplus 0$ is not stable submodule of M.

(3) Every uniform module is strongly N-extending for each R-module N. In particular, Z as Z-module is strongly Z_2-extending.

(4) But the converse of (3) every uniform module is strongly N-extending is not true. For example, Z_6 as Z-module is strongly Z_2-extending which is not uniform.

(5) Since the set of all submodules of a modules M is coincide with $\mathcal{A}(M,M)$[6]. The following statements are equivalent:

(i) $M$is strongly extending;

(ii) $M$is strongly M- extending;

(iii) M is strongly N-extending for every module N.

The proof is direct from the fact " the set of all submodules of a modules M is coincide with $\mathcal{A}(\mathrm{M},\mathrm{M})$[6]".

It is proved in [2] that a module M is stronglyextending if and only if every closed submodule of M is a stable direct summand of M.This result leads us to give the following characterization of strongly N-extending modules.

Proposition (1.3): A module M is strongly N-extending if and only if every closed submodule of M which with belongs to $\mathcal{A}(N, M)$ is a stable direct summand.

Proof: ($\Longrightarrow$)Suppose that M is strongly N-extending module. Let K be a closed submodule of M with $K \in \mathcal{A}(N, M)$. Since M is strongly N-extending, then there exists a stable direct summand B of M such that K is essential in B. But K is closed submodule of M, hence $K = B$ (i.e.) K is a stable direct summand of$M$.

($\Longleftarrow$)Let A be a submodule of M with $A \in \mathcal{A}(N, M)$. Thus, by Zorn's lemma, there exists a closed submodule H of M such that A is essential in H. Since $\mathcal{A}(N, M)$ is closed under essential extension, so $H \in \mathcal{A}(N, M)$. By hypothesis, H is a stable direct summand of M. Therefore, M is strongly N-extending.□

The next results give us characterizations of strongly N-extending modules.

Proposition (1.4): A module M is strongly N-extending if and only if for each submodule A of M with $A \in \mathcal{A}(N, M)$, there is a direct decomposition $M = M_1 \oplus M_2$ such that A $\subseteq M_1$ where M_1 is a stable submodule of M and $A \oplus M_2$ is essential of M.

Proof:($\Longrightarrow$)Suppose that M is strongly N-extending R-module. Let A be a submodule of M with $A \in \mathcal{A}(N, M)$.Thus A is essential in a stable direct summand (say)K of M (i.e.)$M = K \oplus K_1$, where K_1 is a submodule of M. Also, since A is essential in K and K_1 is essential in K_1, thus $A \oplus K_1$ is essential in $K \oplus K_1 = M$. Hence $A \oplus K_1$ is essential submodule of M.

($\Longleftarrow$)Let A be a submodule of M with $A \in \mathcal{A}(N, M)$. By hypothesis, there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$where M_1 is a stable submodule of M and $A \oplus M_2$ is essential in M. We claim that $A$is essential in M_1. Let K be a nonzero submodule of M_1, hence K is a submodule of , so $(A \oplus M_2) \cap K \neq (0)$ (since $A \oplus M_2$ is essential in M). Let $k = a + m_2 (\neq 0)$, where $k \in K, a \in A$ and $m_2 \in M_2$, thus $m_2 = k - a$ which implies $m_2 \in M_1 \cap M_2 = (0)$, therefore $0 \neq k = a \in K \cap A$, then $K \cap A \neq (0)$, hence A is essential in M_1. Thus M is strongly$N$-extending.□

Following [2], every fully invariant direct summand is stable. By using this fact, we have directly the following characterization of strongly N-extending modules.

Proposition (1.5): A module M is strongly N-extending if and only if every submodule of M with belong $\mathcal{A}(N, M)$ is essential in a fully invariant direct summand of M.□

Remark (1.6): From above Proposition, it can be restated that all results with stable direct summand being replaced by fully invariant direct summand.

In the following results, we discuss when a submodule of strongly N-extending module is strongly N-extending.

Proposition (1.7): A closed submodule of strongly N-extending is strongly N-extending.

Proof: Let H be a closed submodule of strongly N-extending module M. Let K be a closed submodule of H with $K \in \mathcal{A}(N,H)$. Since H is closed submodule of M, then K is closed of M. Also, $K \in \mathcal{A}(N,M)$. Now, since M is strongly N-extending, thus K is a stable direct summand of M. But $K \subseteq H$, then K is direct summand of H. Also, we claim that K is a stable submodule of H. Let $f: K \longrightarrow H$ be any homomorphism and consider the sequence $\xrightarrow{f} H \xrightarrow{i} M$, where i is the inclusion mapping. Then $(i \circ f): K \to M$, and since K is stable of M, then $(i \circ f)(K) \subseteq K$. So $f(K) \subseteq K$.Then K is a stable direct summand of H. Therefore H is strongly N-extending.□

Corollary (1.8): A direct summand of strongly N-extending module is strongly N-extending.□

Corollary (1.9): A direct summand A of strongly N-extending module with $A \in \mathcal{A}(N,M)$ is strongly extending.□

Proposition (1.10): Every submodule H of strongly N-extending module M and with the property that the intersection of H with any stable direct summand of M is stable direct summand of H, is strongly N-extending.

Proof: Let A be a submodule of H with $A \in \mathcal{A}(N,H)$. Since M is strongly N-extending and A is a submodule of M, then there is a stable direct summand K of M such that A is essential in K of M. But $A \subseteq K \cap H \subseteq K$, thus A is essential in $K \cap H$ and by hypothesis, $K \cap H$ stable direct summand of H. Hence H is strongly N- extending.□

It is well known that a direct sum of (strongly) extending modules need not be (strongly) extending ([2])[9].

Here, we see that, a direct sum of strongly N-extending need not be strongly N-extending. In fact, Z and $Z_2$are Z_2-extending as Z-modules but we can concluded that From remarks and examples (1.2) (2), $Z_2 \oplus Z$ is not strongly Z_2-extending as Z-module.

In the next result, we obtain when a direct sum of strongly N- extending module is strongly N-extending module.

Theorem(1.11): Let $M = M_1 \oplus M_2$where M_1 and M_2 are strongly N-extending.Then M is strongly N-extending if and only if every closed submodule K belong to $\mathcal{A}(N,M)$ with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is stable direct summand.

Proof: The necessity condition is valid by Proposition (1.8). Conversely, let L be a closed submodule of M with $L \in \mathcal{A}(N,M)$. By Zorn's lemma, there exists a closed submodule H in L such that $L \cap M_2$ is essential in H. Since L is a closed submodule of M, so H is a closed submodule of M. Clearly since $L \cap M_2$ is essential in H *and* M_1 *is essential in* M_1 *so* $(L \cap M_2) \cap M_1$ is essential in $H \cap M_1$ Since $(L \cap M_2) \cap M_1$ is essential in $H \cap M_1$ *and*$(L \cap M2) \cap M_1 = 0$so $H \cap M_1 = 0$. Then by hypothesis, H is a stable direct summand of M. Let $M = H \oplus H'$ for some submodule H' of M. But $L \in \mathcal{A}(N,M)$ and H is a submodule of L, then $H \in \mathcal{A}(N,M)$ (since $\mathcal{A}(N,M)$ is closed under submodules). Now, $L = L \cap M = L \cap (H \oplus H') = H \oplus (L \cap H')$. So$(L \cap H')$ is closed in M with $L \cap H' \in \mathcal{A}(N,M)$ (Since$(L \cap H')$ is a submodule of L and $L \in \mathcal{A}(N,M)$. Also, $(L \cap H') \cap M_2 = 0$. By hypothesis, $(L \cap H')$is a stable direct summand of M, and hence of H' (since $(L \cap H') \subseteq H'$). Thus, $H' = (L \cap H') \oplus K$, where K is a submodule of H'. Now, $M = H \oplus H' = H \oplus (L \cap H') \oplus K) = (H \oplus (L \cap H') \oplus K) = L \oplus K$. It follows that L is a direct summand of M. Since H and $L \cap H'$ are stable submodule of M and $L = H \oplus (L \cap H')$, then L is stable of M [1]. So L is a stable direct summand of M. Theretofore, M is strongly N-extending.□

Following [6], if M be an R-module and $0 \to N' \to N \to N'' \to 0$ be an exact sequence of modules, then $\mathcal{A}(N', M) \cup \mathcal{A}(N", M) \subseteq \mathcal{A}(N, M)$.

Proposition (1.12): Let $0 \to N' \to N \to N" \to 0$ be an exact sequence. Then, a module M is strongly N-extending if and only if $M$is strongly N'-extending and M is strongly $N"$-extending.

Proof:($\Longrightarrow$)It is clear by using the fact $\mathcal{A}(N', M) \cup \mathcal{A}(N", M) \subseteq \mathcal{A}(N, M)$.

($\Longleftarrow$) Suppose that M is strongly N'-extending and strongly $N"$-extending. Since $0 \to N' \to N \to N" \to 0$ is exact sequence. So one can assume that $N' \subseteq N$ and $N" = N / N'$. Let $K \in \mathcal{A}(N, M)$, then there exists L be a submodule of N and there exists $g \in Hom\,(L, M)$ such that $g\,(L)$is essential in K. Let K' be a submodule of K which is closure of the submodule $g\,(L \cap N')$. Since M is strongly N'-extending, then K' is a stable direct summand of M. Let $M = K' \oplus K"$, for some submodule $K"$ of M and $K = K' \oplus (K \cap K")$. Let $\pi: K \to K \cap K"$ denote the canonical projection. Define $\theta: (L + N') / N' \to Mby\theta\,(x + N') = \pi g\,(x)$, for all $x \in L$. Note that given $x \in L$, if $x \in N'$then $g\,(x) \in g\,(L \cap N') \subseteq K'$and hence $\theta\,(x + N') = \pi g\,(x) = 0$. Thus, θ is well-defined and clearly $\theta \in Hom\,((L + N') / N', M)$. Now let $0 \neq y \in K \cap K"$. Then $0 \neq yr = g\,(u)$ for some $r \in R, u \in L$.Since $g\,(u) \in K \cap K"$ it follows that $0 \neq yr = g\,(u) = \pi g\,(u) = \theta\,(u + N')$. Thus $\theta\,((L + N') / N')$ is essential in the closed submodule $K \cap K"$ of the strongly $N"$-extending module M. Hence $K \cap K"$ is a stable direct summand of M. Since K' and $K \cap K"$ are direct summand of M and $K = K' \oplus (K \cap K")$, then K is direct summand of M. In other, direction, since K' and $K \cap K"$ are stable of M and $K = K' \oplus (K \cap K")$, then $K$is a stable of M .So K is stable direct summand of M. Therefore, M is strongly N-extending.□

Relative (quasi-)continuous modules have been considered by several authors. In [4], S. Dogrouz considers (quasi-)continuous modules relative to a class of modules. On other hand, Lopez, Oshiro and Rizivi in [6] introduced (quasi-)continuous modules relative a given R-module N. A module M is called N-continuous if satisfies N-extending and $\mathcal{A}(N, M)C_2$: Every submodule of M which is belong in $\mathcal{A}(N, M)$, which is isomorphic to a direct summand of $M$is a direct summand of M. A module M is called N-quasi-continuous if satisfies N-extending and $\mathcal{A}(N, M)C_3$: If two direct summand A and X of M which A belong in $\mathcal{A}(N, M)$ have zero intersection, then their sum is a direct summand of M.

Here, we introduce relative strongly (quasi-)continuous modules.

Firstly, we can consider the following conditions for modules M and N:

$\mathcal{A}(N, M)SC_1$: Every submodule of M which is belong in $\mathcal{A}\,(N, M)$ is essential in a stable direct summand of M.

$\mathcal{A}(N, M)SC_2$: Every submodule of M which is belong in $\mathcal{A}(N, M)$, which is isomorphic to a direct summand of $M$is a stable direct summand of M.

$\mathcal{A}(N,M)$SC$_3$: If two direct summand A and X of M which A belong in $\mathcal{A}(N,M)$ have zero intersection, then their sum is a stable direct summand of M.

Definition (1.13): A module M is said to be strongly N-continuous if M satisfies, $\mathcal{A}(N,M)$SC$_1$ and $\mathcal{A}(N,M)$SC$_2$.

Definition (1.14): A module M is said to be strongly N-qausi- continuous if M is satisfies, $\mathcal{A}(N,M)$SC$_1$ and $\mathcal{A}(N,M)$SC$_3$.

Examples (1.15):

(1) Every strongly N-(quasi-)contenuous is N-(quasi-)continuous and the converse is not true in general. For example, $M = Z_2 \oplus Z$ is Z_2-(quasi)-continuous as Z-module [6] which is not strongly Z_2-quasi-continuous Z-module.

(2) Every strongly N-continuous module is strongly N-quasi-continuous and the converse is not true in general. For example, Z as Z-module is strongly Z-quasi-continuous but it is not strongly Z-continuous, since Z as Z-module does not satisfy $\mathcal{A}(N,M)$SC$_2$ condition. In fact, $3Z \in \mathcal{A}(Z,Z)$ and $3Z \cong Z$ and is direct summand of Z. But $3Z$ is not stable direct summand of Z.

Recall that a module M is SS-module if every direct summand of M is stable [2] which is equivalent to weakly Duo module [8], that is, every direct summand is fully invariant. We introduce the following useful concept.

Definition (1.16): Let M and N are modules. M is called SS-N-module if, every direct summand D of M with $D \in \mathcal{A}(N,M)$ is a stable.

It is clear that strongly N-extending modules, uniform modules, SS-modules, Duo modules and fully stable modules are examples of SS-N-module.

Remarks and Examples (1.17):

(1) Every SS-module is SS-N-extending for each module N. But the converse is not true in general. For example, the Z-module $M = Z \oplus Z$ is SS-N-module for each semisimple module N. But M is not SS-module, in fact, $Z \oplus (0)$ is direct summand of M which is not stable submodule of $M = Z \oplus Z$.

(2) A direct sum of SS-N-module need not be SS-N-module. For example, consider Z and Z_2 as Z-modules. Since Z and Z_2 are uniform Z-modules, then they are SS-Z_2-module. But $M = Z_2 \oplus Z$ as Z-module is not SS-Z_2-module. In fact the only proper direct summand of M are $(0) \oplus (0)$, $Z_2 \oplus (0)$, $(0) \oplus Z$ and $Z \oplus Z_2$. One can easily check that $\mathcal{A}(Z_2,M) = \{(0), Z_2 \oplus (0)\}$. But $Z_2 \oplus (0)$ is not stable submodule of M.

(3)A direct summand of SS-N-module is SS-N-module.

Proof: Let M be SS-N-module and H be a direct summand of M. Let K be a direct summand of H with $K \in \mathcal{A}(N,H)$. Since H is a direct summand of M, then K is a direct summand of M. Also $K \in \mathcal{A}(N,M)$. Now since M is SS-N-module, thus K is a stable submodule of M. But, $K \subseteq H$, then K is direct summand of H. We claim that K is stable of H. Let $f: K \longrightarrow H$ be any homomorphism. Thus $(i \circ f): K \to M$ where $i: H \to M$ is inclusion mapping and so $(i \circ f)(K) \subseteq K. (i.e.) f(K) \subseteq K$. Thus K is stable submodule of H. Hence H is SS-N-module.□

In the following lemmas, we give a characterization of modules satisfy the conditions $\mathcal{A}(N,M)SC_1$, $\mathcal{A}(N,M)SC_2$ and $\mathcal{A}(N,M)SC_3$.

Lemma (1.18): A module M has $\mathcal{A}(N,M)SC_1$ condition if and only if M has $\mathcal{A}(N,M)C_1$ condition and M has SS-N-extending.

Proof: (⟹) Let M has $\mathcal{A}(N,M)$ SC_1condition.Then, clearly M satisfy $\mathcal{A}(N,M)C_1$ condition. Let D be a direct summand of M such that $D \in \mathcal{A}(N,M)$. So, D is closed which belong to $\mathcal{A}(N,M)$, thus by Proposition (1.3), D is stable.

(⟸)Let H be a submodule of M with $H \in \mathcal{A}(N,M)$. Then by $\mathcal{A}(N,M)C_1$ property of M, H is essential in a direct summand D of M. But $D \in \mathcal{A}(N,M)$, since $\mathcal{A}(N,M)$is closed under essential extension. So by SS- N-module $D$is a stable. Therefore, M has $\mathcal{A}(N,M)SC_1$.□

Lemma (1.19): A module M has $\mathcal{A}(N,M)SC_2$ condition if and only if M has $\mathcal{A}(N,M)C2$ condition and M is SS-N-module.

Proof: (⟹) Let M has $\mathcal{A}(N,M)SC_2$ condition. Then, clearly, M has $\mathcal{A}(N,M)C_2$ condition. Let H is a direct summand of M with $H \in \mathcal{A}(N,M)$.Since $H \cong H$ and hence by $\mathcal{A}(N,M)$ SC_2 property, H is a stable submodule of M. So M is SS-N-module.

(⟸) Let H be a submodule of M with $H \in \mathcal{A}(N,M)$ such that $H \cong D$ where D is a direct summand of M. Thus, by $\mathcal{A}(N,M)C_2$ property H is a direct summand of M . Also, by SS-N-module property of M we have H is a stable of M. Therefore, M has $\mathcal{A}(N,M)SC_2$ condition.□

Lemma (1.20): A module M has $\mathcal{A}(N,M)SC_3$condition if and only if M has $\mathcal{A}(N,M)C_3$ condition and M is SS-N-module.

<u>Proof:</u>(⟹) Clearly every module M satisfies $\mathcal{A}(N,M)SC_3$ has $\mathcal{A}(N,M)$ C_3. Let H be a direct summand of M with $H \in \mathcal{A}(N,M)$. Let $D = (0)$. So D and H are direct summands of M such that $H \cap D = (0)$. Thus, by $\mathcal{A}(N,M)$ SC_3 property $H \oplus D = (0) \oplus H = H$ is stable submodule of M.Then M is SS-N-module.

(⟸) Let $H \in \mathcal{A}(N,M)$ such that H and K are direct summands of $M$and $H \cap K = (0)$ .Thus, by $\mathcal{A}(N,M)C_3$ property for $M, H \oplus D$ is a direct summand of M. Also, $H \oplus D \in \mathcal{A}(N,M)$(since $H\oplus D$ is submodule of H and $H \in \mathcal{A}(N,M)$and since $\mathcal{A}(N,M)$ is closed under submodule). Then by SS-N-module $H \oplus D$ stable of M. Hence M satisfies $\mathcal{A}(N,M)SC_3$ property.□

By using above three lemmas, we have the following characterizations of strongly N-(quasi-)continuous modules. The proofs of these propositions are direct.

Proposition (1.21): A module M is strongly N-continuous if and only if M is N-continuous and M is SS-N-module.□

Proposition (1.22): A module M is strongly N-quasi-continuous if and only if M is N-quasi-continuous and M is SS-N-module.□

Proposition (1.23): A module M is strongly N-continuous if and only if M is satisfies the condition $\mathcal{A}(N,M)SC_1$and $\mathcal{A}(N,M)C_2$.□

Proposition (1.24): A module M is strongly N-quasi-continuous if and only if M is satisfies the condition $\mathcal{A}(N,M)SC_1$ and $\mathcal{A}(N,M)C_3$.□

Proposition (1.25): A module M is strongly N-continuous if and only M satisfies the condition $\mathcal{A}(N,M)C_1$ and $\mathcal{A}(N,M)SC_2$.□

Proposition (1.26): A module M is strongly N-quasi-continuous if and only if M satisfies the condition $\mathcal{A}(N,M)C_1$ and $\mathcal{A}(N,M)$ SC_3.□

Following [6], a direct summand of N-(quasi)- continuous module is N-(quasi)-continuous. So, By Remarks and Example (1.17)(3) and Proposition(1.21) and Proposition (1.22), we assert that the strongly N-(quasi-) continuous modules is inherited by direct summands.

Proposition (1.27): A direct summand of strongly N-continuous module is strongly N-continuous.

Proposition (1.28): A direct summand of strongly N-quasi-continuous module is strongly N-quasi-continuous

By using the same argument of Proposition (1.12) we have the following results:

Proposition (1.29): Let $0 \to N' \to N \longrightarrow N" \to 0$ be an exact sequence of modules. Then a module M is strongly N-continuous if and only if M is strongly N'- continuous and strongly $N"$-continuous.□

Proposition (1.30): Let $0 \to N' \to N \longrightarrow N" \to 0$ be an exact sequence of modules. Then M is strongly N-quasi-continuous if and only if M is strongly N'-quasi-continuous and strongly $N"$-quasi- continuous.□

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