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Pairwise Lower Separation Axioms in Čech Fuzzy Soft Bi-Closure Spaces

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**ABSTRACT**

The idea of Čech fuzzy soft bi-closure space (Čfs bicsp) \((M, \ell_1, \ell_2, D)\) is a new one, and its basic features are defined and studied in [12]. In this paper, separation axioms, namely pairwise, \(T_i\) (\(i = 0,1,2\)), pairwise semi-(respectively, pairwise pseudo and pairwise Uryshon) \(T_2\)-Čfs bicsp's are introduced and studied in both Čech fuzzy soft bi-closure space and their induced fuzzy soft bitopological spaces. It is shown that hereditary property is satisfied for \(T_i\), \(i = 0,1\) with respect to Čech fuzzy soft bi-closure space but for other mentioned types of separations axioms, hereditary property satisfies for closed subspaces of Čech fuzzy soft bi-closure space.

**1. Introduction**

In 1965, Zadeh [23] proposed the concept of a fuzzy set for the first time. Fuzzy sets were established to provide a mathematical representation for situations in which there are no definite membership requirements for ill-defined groups or collections of items. In 1999, Moldtsov [17] published a paper on the creation of soft set theory, a modern technique of showing complexity and confusion. The theory of soft sets has a wide range of potential applications, some of which Molodsov presented in [17]. In 2001 Maji et al. [11] combined fuzzy sets and soft sets to establish the concept of fuzzy soft sets. Later in 2011, Tanay and Kandemir [20] proposed a notion of topological structure based on fuzzy soft sets.

In 1966, Čech [2] presented the notion of Čech closure spaces \((X, C)\), where \(C: P(X) \to P(X)\) is a mapping fulfilling \(C(\emptyset) = \emptyset, V \subseteq C(V)\) and \(C(V \cup F) = C(V) \cup C(F)\), the mapping termed Čech closure operator on \(X\), which is analogous to a topological closure operator with the distinction that it does not have to be idempotent. That is, Čech satisfies just three of the four Kuratowski...

Majeed [8] recently established the definition of Čech fuzzy soft closure spaces, which were motivated by Chang’s concepts of fuzzy soft set and fuzzy soft topology [3]. Majeed and Maibed also studied the architecture of Čech fuzzy soft closure spaces, including separation axioms and connectedness [12, 13, 9, 10]. As a generalization to Čech fuzzy soft closure space [8], we recently presented and researched the concept of Čech fuzzy soft bi-closure spaces (Čfs bicsp’s) [15] and give more study of Čfs bicsp’s in [16].

This work aims to introduce and study lower separation axioms in Čfs bicsp’s. Section 3, devoted to defining pairwise \(T_0\) and \(T_1\) in Čfs bicsp’s and their induced fs-bits’s. We discuss the relation between \(PT_0\) and \(PT_1\), and study the hereditary property on Čfs bicsp’s. Also, we give the relation between Čfs bicsp \((M, \ell_1, \ell_2, D)\) and its induced fs-bits \((M, T_{\ell_1}, T_{\ell_2}, D)\) when induced fs-bits is \(PT_i, i = 0,1\). In this section 4, pairwise \(T_2\)-Čfs bicsp and other types, namely, pairwise semi- (respectively, pairwise pseudo and pairwise Uryshon) \(T_2\)-Čfs bicsp’s are introduced, the properties of each type are discussed as in Section 3. In addition, the relationships between separation axioms introduced in the current section and the previous section are obtained.

2. Preliminaries

Throughout this work, \(M\) is the universal set, \(I=\{0,1\}\), \(I 0=(0,1)\), \(D\) is the set of parameters for \(M\), and \(\mathcal{A}\) ba a nonempty subset of \(D\). A fuzzy set \(\beta\) is a mapping from \(M\) into \(I\) [23]. \(I^M\) refer to the family of all fuzzy sets of \(M\).

**Definition 2.1** [22] A fuzzy soft set \((fss)\) \(\beta_\mathcal{A}\) on the universe set \(M\) is a mapping from \(D\) to \(I^M\), i.e., \(\beta_\mathcal{A}:D \to I^M\), where \(\beta_\mathcal{A}(d) \neq \emptyset\) if \(d \in \mathcal{A} \subseteq D\) and \(\beta_\mathcal{A}(d) = \emptyset\) if \(d \notin \mathcal{A}\), where \(\emptyset\) is the empty fuzzy set on \(M\). The family of all fss’s over \(M\) denoted by \(FS(M, D)\).

**Definition 2.2** [22] Let \(\beta_\mathcal{A}, \mu_B \in FS(M, D)\). Then,

1. \(\beta_\mathcal{A} \subseteq \mu_B\) iff \(\beta_\mathcal{A}(d) \subseteq \mu_B(d),\) for all \(d \in D\).
2. \(\beta_\mathcal{A} = \mu_B\) iff \(\beta_\mathcal{A} \subseteq \mu_B\) and \(\mu_B \subseteq \beta_\mathcal{A}\).
3. \(\rho_{\mathcal{A} \cup B} = \beta_\mathcal{A} \cup \mu_B\) iff \(\rho_{\mathcal{A} \cup B}(d) = \beta_\mathcal{A}(d) \lor \mu_B(d),\) for all \(d \in D\).
4. \(\rho_{\mathcal{A} \cap B} = \beta_\mathcal{A} \cap \mu_B\) iff \(\rho_{\mathcal{A} \cap B}(d) = \beta_\mathcal{A}(d) \land \mu_B(d),\) for all \(d \in D\).
5. The complement of \(\beta_\mathcal{A}\) is denoted by \(\beta_\mathcal{A}^c\) where \(\beta_\mathcal{A}^c(d) = \overline{1} - \beta_\mathcal{A}(d),\) for all \(d \in D\), where \(\overline{1}(x) = 1 \forall x \in M\).
6. \(\beta_s\) is called null fss, denoted \(\overline{0}_D\), if \(\beta_s(d) = \emptyset,\) for all \(d \in D\).
7. \(\beta_D\) is called universal fss, denoted \(\overline{1}_D\), if \(\beta_D(d) = \overline{1},\) for all \(d \in D\).

**Definition 2.3** [1] A fss \(\beta_\mathcal{A} \in FS(M, D)\) is called fs-point (fs-point), denoted by \(x^s_\mathcal{A}\), if there exists \(x \in M\) and \(s \in D\) such that \(\beta_\mathcal{A}(s)(x) = t\) \((0 < t \leq 1)\) and \(\emptyset\) otherwise for all \(y \in M - \{x\}\).
The fs-point $x_t^\epsilon$ is said to belong to the fss $\beta_{\mathcal{A}}$, denoted by $x_t^\epsilon \in \beta_{\mathcal{A}}$, if for the element $s \in \mathcal{M}, t \leq \beta_{\mathcal{A}}(s) (x)$.

**Definition 2.4** [20] A triple $(\mathcal{M}, \mathcal{T}, \mathcal{D})$ is called a fuzzy soft topological space where $\mathcal{T}$ is a family of $\mathcal{T}'$s over $\mathcal{M}$ which satisfying the following properties.
1. $\overline{D}, 1_D \in \mathcal{T}$,
2. $\beta_{\mathcal{A}}, \mu_B \in \mathcal{T} \Rightarrow \beta_{\mathcal{A}} \cap \mu_B \in \mathcal{T}$,
3. $(\beta_{\mathcal{A}})_i \in \mathcal{T}$ for all $i \Rightarrow \bigcup_{i \in I} (\beta_{\mathcal{A}})_i \in \mathcal{T}$.

$\mathcal{T}$ is called a topology of $\mathcal{T}'$s on $\mathcal{M}$. Each member of $\mathcal{T}$ is called an $\mathcal{T}$-open $\mathcal{T}'$. $\mu_B$ is called a $\mathcal{T}$-closed $\mathcal{T}'$ in $(\mathcal{M}, \mathcal{T}, \mathcal{D})$ if $\mu_B^c \in \mathcal{T}$.

**Definition 2.5** [18] A quadruple $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{D})$ is said to be a fuzzy soft bi-topological space where $\mathcal{T}_1, \mathcal{T}_2$ are arbitrary fuzzy soft topologies on $\mathcal{M}$.

The following recall the concept of Čfs bicsp and its fundamental properties. For $i, j = 1, 2$ where $i \neq j$. Other than that we will mention the value of $i$ and $j$.

**Definition 2.6** [15] A Čfs bicsp is a quadruple $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ where $\mathcal{M}$ is a nonempty set, and $\ell_1, \ell_2: FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ are two fuzzy soft closure operators on $\mathcal{M}$ which are correct according to the following axioms:

1. $\ell_i(0_D) = 0_D$,
2. $\beta_{\mathcal{A}} \subseteq \ell_1(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$,
3. $\ell_i(\beta_{\mathcal{A}} \cup \mu_B) = \ell_i(\beta_{\mathcal{A}}) \cup \ell_i(\mu_B)$ for all $\beta_{\mathcal{A}}, \mu_B \in FS(\mathcal{M}, \mathcal{D})$.

**Definition 2.7** [15] A fss $\beta_{\mathcal{A}}$ of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be $\ell_1$-closed (respectively, $\ell_1$-open, respectively) fss if $\ell_1(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ (respectively, $\ell_1(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}^c$). And, it is called a closed fss if $\ell_1(\ell_j(\beta_{\mathcal{A}})) = \beta_{\mathcal{A}}$. For $i, j = 1, 2$ where $i \neq j$. The complement of a closed fss is called an open fss.

**Proposition 2.8** [15] Let $\beta_{\mathcal{A}}$ be a fss of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then, $\beta_{\mathcal{A}}$ is a closed fss in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ if and only if $\beta_{\mathcal{A}}$ is $\ell_1$-closed fss.

**Definition 2.9** [15] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $\mathcal{H} \subseteq \mathcal{M}$. The quadruple $(\mathcal{H}, \ell_1, \ell_2, \mathcal{D})$ is called a Čech fuzzy soft bi-closure subspace (Čfs bi-csubsp) of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$, where $\ell_{3\mathcal{H}}: FS(\mathcal{H}, \mathcal{D}) \to FS(\mathcal{H}, \mathcal{D})$ defind by $\ell_{3\mathcal{H}}(\beta_{\mathcal{A}}) = \overline{\mathcal{H}} \cap \ell_i(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{H}, \mathcal{D})$. The Čfs bi-csubsp $(\mathcal{H}, \ell_1, \ell_2, \mathcal{D})$ is said to be a closed (respectively, open) subspace if $\overline{\mathcal{H}}$ is a closed (resp. open) fss over $\mathcal{M}$.

**Proposition 2.10** [15] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $(\mathcal{H}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Let $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. Then, $\beta_{\mathcal{A}}$ is a closed fss over $\mathcal{H}$ if and only if $\ell_{3\mathcal{H}}(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$.

**Definition 2.11** [16] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp, the induced fuzzy soft bitopological space (induced fs-bits, for short) of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$, denoted by $(\mathcal{M}, T_1^\ell, T_2^\ell, \mathcal{D})$ where $T_1^\ell = \{ \beta_{\mathcal{A}}: \ell_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}} \}$.

**Definition 2.12** [16] Let $(\mathcal{M}, T_1^\ell, T_2^\ell, \mathcal{D})$ be the induced fs-bits of the Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ and $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D}), \beta_{\mathcal{A}}$ is called an $T_1^\ell$-open fss in $(\mathcal{M}, T_1^\ell, T_2^\ell, \mathcal{D})$, if $\beta_{\mathcal{A}} \in T_1^\ell$. The complement of an
open fss $\beta_a$ is a $T_{\ell_1}$-closed fss, and $\beta_a$ is called an open fss in $(M, T_{\ell_1}, T_{\ell_2}, D)$, if $\beta_a$ is an $T_{\ell_i}$-open, for $i = 1, 2$.

**Definition 2.13** [16] Let $(M, T_{\ell_1}, T_{\ell_2}, D)$ be the induced fs-bits of $(M, \ell_1, \ell_2, D)$ and let $\beta_a \in FS(M, D)$. The fuzzy soft closure of $\beta_a$ for $\ell_i$ and $\ell_j$, denoted by $T_{\ell_i}T_{\ell_j}\text{-}cl(\beta_a)$, is the intersection of all closed fuzzy soft supersets of $\beta_a$. i.e., $T_{\ell_i}T_{\ell_j}\text{-}cl(\beta_a) = \cap \{ \rho_c : \beta_a \subseteq \rho_c \text{ and } \ell_i(\rho_c) = \rho_c \}$.

**Theorem 2.14** [16] Let $(M, \ell_1, \ell_2, D)$ be Čfs bicsp and $(M, T_{\ell_1}, T_{\ell_2}, D)$ be the induced fs-bits of $(M, \ell_1, \ell_2, D)$. Then for any $\beta_a \in FS(M, D)$.

$$T_{\ell_i}T_{\ell_j}\text{-}int(\beta_a) \subseteq \ell_i\left(\text{int}(\beta_a)\right) \subseteq \beta_a \subseteq \ell_i\left(\ell_j(\beta_a)\right) \subseteq T_{\ell_i}T_{\ell_j}\text{-}cl(\beta_a).$$

**Proposition 2.15** [16] Let $(M, \ell_1, \ell_2, D)$ be Čfs bicsp and $(M, T_{\ell_1}, T_{\ell_2}, D)$ be the induced fs-bits of $(M, \ell_1, \ell_2, D)$. Then for any $\beta_a \in FS(M, D)$.

$$T_{\ell_i}\text{-}int(\beta_a) \subseteq \ell_i(\beta_a) \subseteq \ell_i(\ell_i(\beta_a)) \subseteq T_{\ell_i}\text{-}cl(\beta_a).$$

**Definition 2.16** [16] Let $(M, \ell_1, \ell_2, D)$ and $(M, \ell_1^*, \ell_2^*, D)$ be two Čfs bicsp’s on $M$. Then $\ell_i$ is said to finer than $\ell_i^*$, or equivalently $\ell_i^*$ is coarser than $\ell_i$, if for each $\beta_a \in FS(M, D)$, $\ell_i(\beta_a) \subseteq \ell_i^*(\beta_a)$, for $i = 0, 1$.

3. **Pairwise $T_i$-Čech fuzzy soft bi-closure spaces, $i = 0, 1$**

**Definition 3.1** A Čfs bicsp $(M, \ell_1, \ell_2, D)$ is said to be pairwise $T_0$-Čfs bicsp (PT$_0$-Čfs bicsp), if for every pair of distinct fs-points $x_k^i$ and $y_k^i$, either $x_k^i \notin \ell_i(y_k^i)$ or $y_k^i \notin \ell_i(x_k^i)$, for $i = 1$ or $2$.

Now we give some examples to illustrate Definition 3.1.

**Example 3.2** Let $(M, \ell_1, \ell_2, D)$ be the discrete Čfs bicsp (i.e., $\ell_i(\beta_a) = \beta_a$ for all $\beta_a \in FS(M, D)$), then $(M, \ell_1, \ell_2, D)$ is a PT$_0$-Čfs bicsp.

**Example 3.3** Let $(M, \ell_1, \ell_2, D)$ be the trivial Čfs bicsp (i.e., for all $\beta_a \in FS(M, D)$, $\ell_i(\beta_a) = \varnothing_D$ if $\beta_a = 0_D$ and $\ell_i(\beta_a) = \bar{1}_D$ otherwise). Then $(M, \ell_1, \ell_2, D)$ is not PT$_0$-Čfs bicsp because for any distinct fs-points $x_k^i$ and $y_k^i$, we have $x_k^i \notin \ell_i(y_k^i)$ and $y_k^i \notin \ell_i(x_k^i)$.

**Example 3.4** Let $M = \{x, y\}$, $D = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(M, D)$ such that $\Gamma = \{s_1, x_{t_1} \lor y_{k_1}\}, (s_2, x_{t_2} \lor y_{k_2}); \ t_1, t_2, k_1, k_2 \in \text{l}_0\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(M, D) \rightarrow FS(M, D)$ as follows:
Then \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is a Čfs bicsp. To show \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is \(PT_0\)-Čfs bicsp, we have three cases for distinct fs-points in \(\mathcal{M}\).

**Case (1):** If \(x \neq y\) and \(s_1 = s_2\), then we have \(x_{t_1}^{s_1}\) and \(y_{k_1}^{s_1}\) are distinct fs-points. It is clear that \(y_{k_1}^{s_1} \notin \ell_1(x_{t_1}^{s_1})\) because \(k_1 > (x_{t_1}^{s_1})(y) = (x_{t_1}(y)) = 0\). Similarly, \(x_{t_2}^{s_2}\) and \(y_{k_2}^{s_2}\) are distinct fs-points and \(y_{k_2}^{s_2} \notin \ell_1(x_{t_2}^{s_2})\).

**Case (2):** If \(x = y\) and \(s_1 \neq s_2\), then \(x_{t_1}^{s_1}\) and \(x_{t_2}^{s_2}\) are distinct fs-points. It is clear that \(x_{t_1}^{s_1} \notin \ell_1(x_{t_2}^{s_2})\) because \(t_1 > \ell_1(x_{t_2}^{s_2})(s_1)(x) = 0\). Similarly, \(y_{k_1}^{s_1}\) and \(y_{k_2}^{s_2}\) are distinct fs-points and \(y_{k_2}^{s_2} \notin \ell_1(x_{t_2}^{s_2})\).

**Case (3):** If \(x \neq y\) and \(s_1 \neq s_2\), then we have \(x_{t_1}^{s_1}\) and \(y_{k_2}^{s_2}\) are distinct fs-points such that \(y_{k_2}^{s_2} \notin \ell_1(x_{t_2}^{s_2})\). Similarly, \(x_{t_2}^{s_2}\) and \(y_{k_2}^{s_1}\) are distinct fs-points and \(y_{k_2}^{s_1} \notin \ell_1(x_{t_2}^{s_2})\). Hence, \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is \(PT_0\)-Čfs bicsp.

**Theorem 3.5** Let \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) be a \(PT_0\)-Čfs bicsp, then for any two distinct fs-points \(x_i^s\) and \(y_k^{s'}\), \(\ell_i(x_i^s) \neq \ell_i(y_k^{s'})\), for \(i = 1 \text{ or } 2\).

**Proof:** Let \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) be a \(PT_0\)-Čfs bicsp, and let \(x_i^s\) and \(y_k^{s'}\) be any two distinct fs-points. Suppose that \(\ell_i(x_i^s) = \ell_i(y_k^{s'})\) for \(i = 1 \text{ and } 2\). Since \(x_i^s \notin \ell_i(x_i^s)\) and \(y_k^{s'} \notin \ell_i(y_k^{s'})\) for \(i = 1 \text{ and } 2\). Then from the hypothesis, \(x_i^s \notin \ell_i(y_k^{s'})\) and \(y_k^{s'} \notin \ell_i(x_i^s)\) for \(i = 1 \text{ and } 2\). This implies \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is not \(PT_0\)-Čfs bicsp, which is a contradiction. Hence, \(\ell_i(x_i^s) \neq \ell_i(y_k^{s'})\) for \(i = 1 \text{ or } 2\).

The converse of the above theorem is not true, as the following example show.

**Example 3.6** Let \(\mathcal{M} = \{x, y\}\), \(\mathcal{D} = \{s_1, s_2\}\) and let \(\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})\) such that \(\Gamma = \{(s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2}); t_1, t_2, k_1, k_2 \in l_0\}\). Define fuzzy soft closure operators \(\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \rightarrow FS(\mathcal{M}, \mathcal{D})\) as follows:
\[ \ell_1(\beta_A) = \begin{cases} \emptyset_D & \text{if } \beta_A = \emptyset_D, \\ \{(s_1, x_{0.2} \lor y_{0.1})\} & \text{if } \beta_A \subseteq x_{0.2}^{s_1}, \\ \{(s_1, x_{0.1} \lor y_{0.2})\} & \text{if } \beta_A \subseteq y_{0.2}^{s_1}, \\ \beta_A & \text{if } \beta_A \in \{x_{t_1}^{s_1}; \ 0.2 < t_1 \leq 1\} \\ \beta_A & \text{if } \beta_A \in \{y_{k_1}^{s_1}; \ 0.2 < k_1 \leq 1\}, \\ x_1^{s_2} & \text{if } \beta_A \in \{x_{t_2}^{s_2}; \ t_2 \in I_0\}, \\ y_1^{s_2} & \text{if } \beta_A \in \{y_{k_2}^{s_2}; \ k_2 \in I_0\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & \text{if } \beta_A \in \Gamma. \end{cases} \]

And

\[ \ell_2(\beta_A) = \begin{cases} \emptyset_D & \text{if } \beta_A = \emptyset_D, \\ \{(s_1, x_{0.7} \lor y_{0.5})\} & \text{if } \beta_A \subseteq x_{0.7}^{s_1}, \\ \{(s_1, x_{0.5} \lor y_{0.7})\} & \text{if } \beta_A \subseteq y_{0.7}^{s_1}, \\ \beta_A & \text{if } \beta_A \in \{x_{t_1}^{s_1}; \ 0.7 < t_1 \leq 1\}, \\ \beta_A & \text{if } \beta_A \in \{y_{k_1}^{s_1}; \ 0.7 < k_1 \leq 1\}, \\ x_1^{s_2} & \text{if } \beta_A \in \{x_{t_2}^{s_2}; \ t_2 \in I_0\}, \\ y_1^{s_2} & \text{if } \beta_A \in \{y_{k_2}^{s_2}; \ k_2 \in I_0\}, \\ \{\ell_2(x_{t_1}^{s_1}) \cup \ell_2(y_{k_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & \text{if } \beta_A \in \Gamma. \end{cases} \]

Then \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is a \(\mathcal{C}\)fs bicsp. It is clear that for any two distinct fs-points \(x_i^s\) and \(y_k^{s'}\), where \(s, s' \in \{s_1, s_2\}\) and \(t, k \in I_0\) we have \(\ell_i(x_i^s) \neq \ell_i(y_k^{s'})\) for \(i = 1 \ or \ 2\). However, \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is not \(PT_0\)-Csfs bicsp, since there exist two distinct fs-point \(x_{0.1}^{s_1}\) and \(y_{0.1}^{s_1}\) such that \(x_{0.1}^{s_1} \in \ell_1(y_{0.1}^{s_1})\) and \(x_{0.1}^{s_1} \notin \ell_1(y_{0.1}^{s_1})\) for \(i = 1, 2\).

Next, we show that \(PT_0\) is a hereditary property on \(\mathcal{C}\)fs bi-csubsp.

**Theorem 3.7** Every \(\mathcal{C}\)fs bi-csubsp of a \(PT_0\)-Csfs bicsp, is a \(PT_0\)-Csfs bi-csubsp.

**Proof** Let \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) be a \(PT_0\)-Csfs bicsp and \((\mathcal{H}, \ell_{1\mathcal{H}}, \ell_{2\mathcal{H}}, \mathcal{D})\) be a \(\mathcal{C}\)fs bi-csubsp of \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\). Let \(x_i^s\) and \(y_k^{s'}\) be any two distinct fs-points in \((\mathcal{H}, \ell_{1\mathcal{H}}, \ell_{2\mathcal{H}}, \mathcal{D})\) such that \(x_i^s \in \ell_{1\mathcal{H}}(y_k^{s'})\) and \(y_k^{s'} \in \ell_{1\mathcal{H}}(x_i^s)\) for \(i = 1 \ or \ 2\). Then \(x_i^s \in \ell_{1\mathcal{H}}(y_k^{s'})\) implies \(x_i^s \in \ell_{2\mathcal{H}}(y_k^{s'})\) or \(y_k^{s'} \in \ell_{2\mathcal{H}}(x_i^s)\). Hence \((\mathcal{H}, \ell_{1\mathcal{H}}, \ell_{2\mathcal{H}}, \mathcal{D})\) is \(PT_0\)-Csfs bi-csubsp.

**Definition 3.8** The induced fs-bits \((\mathcal{M}, \ell_{T_1}, \ell_{T_2}, \mathcal{D})\) of a \(\mathcal{C}\)fs bicsp \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is said to be pairwise \(T_0\)-fsbs (\(PT_0\)-fsbs, for short), if for every two distinct fs-points \(x_i^s\) and \(y_k^{s'}\), either \(x_i^s \in \ell_{T_1}(y_k^{s'})\) or \(y_k^{s'} \in \ell_{T_2}(x_i^s)\), for \(i, j = 1 \ or \ 2\).

To give the relationship between the induced fs-bits \((\mathcal{M}, \ell_{T_1}, \ell_{T_2}, \mathcal{D})\) which is \(PT_0\)-fs-bits and \(\mathcal{C}\)fs bicsp \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) we need first to introduce the following lemma.

**Lemma 3.9** Let \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) be a \(\mathcal{C}\)fs bicsp. Then for any \(\beta_A \in FS(\mathcal{M}, \mathcal{D})\), \(\beta_A \subseteq \ell_1(\beta_A) \subseteq \ell_i(\ell_j(\beta_A))\).
Proof. Obvious. ■

Theorem 3.10 If \((M, T_{\ell_1}, T_{\ell_2}, D)\) is a \(PT_0\)-fs-bits, then \((M, \ell_1, \ell_2, D)\) is also \(PT_0\)-\(\tilde{C}\)fs bicsp.

Proof: Let \(x^s_i\) and \(y^s_j^{'}\) be any two distinct fs-points in \((M, \ell_1, \ell_2, D)\). Since \((M, T_{\ell_1}, T_{\ell_2}, D)\) is a \(PT_0\)-fs-bits, then \(x^s_i \notin T_{\ell_1} T_{\ell_2} \cdot cl(y^s_j^{'})\) or \(y^s_j^{'} \notin T_{\ell_1} T_{\ell_2} \cdot cl(x^s_i)\), for \(i, j = 1, 2\). By Theorem 2.14 and Lamma 3.9, we get, \(x^s_i \notin \ell_i(y^s_j^{'})\) or \(y^s_j^{'} \notin \ell_j(x^s_i)\). This implies \((M, \ell_1, \ell_2, D)\) is \(PT_0\)-\(\tilde{C}\)fs bicsp. ■

The converse Theorem 3.10 is not true, as we shown in the following example.

Example 3.11 Let \(M = \{x, y\}, D = \{s_1, s_2\}\) and. Define fuzzy soft closure operators \(\ell_1, \ell_2: FS(M, E) \rightarrow FS(M, E)\) as follows:

\[
\ell_1(\beta_{\alpha}) = \begin{cases} 
\emptyset & \text{if } \beta_{\alpha} = \emptyset, \\
\{(s_1, x_1), (s_2, y_1)\} & \text{if } \beta_{\alpha} \subseteq x^s_1, \\
\{(s_1, y_1), (s_2, x_1)\} & \text{if } \beta_{\alpha} \subseteq y^s_1, \\
\{(s_1, x_1), (s_2, x_1)\} & \text{if } \beta_{\alpha} \subseteq x^s_2, \\
\{(s_1, y_1), (s_2, y_1)\} & \text{if } \beta_{\alpha} \subseteq y^s_2, \\
\{(s_1, x_1, (s_2, x_1 \vee y_1))\} & \text{if } \beta_{\alpha} \subseteq \{(s_1, x_1, (s_2, x_1)); \beta_{\alpha} \notin \{x^s_1, i = 1, 2, t \in I_0\}, \\
\{(s_1, x_1 \vee y_1), (s_2, y_1)\} & \text{if } \beta_{\alpha} \subseteq \{(s_1, x_1, (s_2, y_1)); \beta_{\alpha} \notin \{x^s_1, y^s_2; t, k \in I_0\}, \\
\{(s_1, x_1 \vee y_1), (s_2, x_1)\} & \text{if } \beta_{\alpha} \subseteq \{(s_1, y_1, (s_2, x_1)); \beta_{\alpha} \notin \{x^s_2, y^s_1; t, k \in I_0\}, \\
\{(s_1, y_1), (s_2, x_1 \vee y_1)\} & \text{if } \beta_{\alpha} \subseteq \{(s_1, y_1, (s_2, y_1)); \beta_{\alpha} \notin \{y^s_1, i = 1, 2, t \in I_0\}, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

And \(\ell_2(\beta_{\alpha}) = \beta_{\alpha}\) for all \(\beta_{\alpha} \in FS(M, E)\). Then, \((M, \ell_1, \ell_2, D)\) is \(PT_0\)-\(\tilde{C}\)fs bicsp. But \((M, T_{\ell_1}, T_{\ell_2}, D)\) is not \(PT_0\)-fs-bits, because for any two distinct fs-points \(x^s_i\) and \(y^s_j^{'}\), where \(s, s' \in \{s_1, s_2\}\) and \(t, k \in I_0\) we have \(x^s_i \notin T_{\ell_1} T_{\ell_2} \cdot cl(y^s_j^{'})\) and \(y^s_j^{'} \notin T_{\ell_1} T_{\ell_2} \cdot cl(x^s_i)\). Thus \((M, \ell_1, \ell_2, D)\) is \(PT_0\)-\(\tilde{C}\)fs bicsp.

Now, we introduce the notion of pairwise \(T_1\)-\(\tilde{C}\)fs bicsp’s.

Definition 3.12 A \(\tilde{C}\)fs bicsp \((M, \ell_1, \ell_2, D)\) is said to be pairwise \(T_1\)-\(\tilde{C}\)fs bicsp \((PT_1\)-\(\tilde{C}\)fs bicsp), if for every two distinct fs-points \(x^s_i\) and \(y^s_j^{'}\), we have \(x^s_i \notin \ell_i(y^s_j^{'}\)) and \(y^s_j^{'} \notin \ell_j(x^s_i)\).

Next, we give an example of a \(PT_1\)-\(\tilde{C}\)fs bicsp.

Example 3.13 Let \(M = \{x, y\}, D = \{s_1, s_2\}\) and let \(\Gamma \subseteq FS(M, D)\) such that \(\Gamma = \{(s_1, x_1), \vee y_{k_1}^{s_1}, (s_2, x_2 \vee y_{k_2}^{s_2}); t_1, t_2, k_1, k_2 \in I_0\}\). Define fuzzy soft closure operators \(\ell_1, \ell_2: FS(M, D) \rightarrow FS(M, D)\) as follows:
\[
\ell_1(\beta_A) = \begin{cases} 
\emptyset_D & \text{if } \beta_A = \emptyset_D, \\
x_{t_1}^{s_1} & \text{if } \beta_A \in \{x_{t_1}^{s_1}; t_1 \in I_0\}, \\
x_{t_2}^{s_2} & \text{if } \beta_A \in \{x_{t_2}^{s_2}; t_2 \in I_0\}, \\
y_{k_1}^{s_1} & \text{if } \beta_A \in \{y_{k_1}^{s_1}; k_1 \in I_0\}, \\
y_{k_2}^{s_2} & \text{if } \beta_A \in \{y_{k_2}^{s_2}; 0 \leq k_2 \leq 1\}, \\
\{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & \text{if } \beta_A \in I. 
\end{cases}
\]

And

\[
\ell_2(\beta_A) = \begin{cases} 
\emptyset_D & \text{if } \beta_A = \emptyset_D, \\
x_{t_1}^{s_1} & \text{if } \beta_A \in \{x_{t_1}^{s_1}; t_1 \in I_0\}, \\
x_{t_2}^{s_2} & \text{if } \beta_A \in \{x_{t_2}^{s_2}; 0 < t_2 < 0.9\}, \\
y_{k_1}^{s_1} & \text{if } \beta_A \in \{y_{k_1}^{s_1}; 0.9 \leq t_2 \leq 1\}, \\
y_{k_2}^{s_2} & \text{if } \beta_A \in \{y_{k_2}^{s_2}; k_2 \in I_0\}, \\
\{\ell_2(x_{t_1}^{s_1}) \cup \ell_2(y_{k_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & \text{if } \beta_A \in I. 
\end{cases}
\]

Then \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is \(PT_1\)-\(\mathcal{C}f_{s}\) bicsp. Since for any two distinct \(fs\)-points \(x_{t_1}^{s_1}\) and \(y_{k_2}^{s_2}\) we have \(x_{t_1}^{s_1} \notin \ell_1(y_{k_2}^{s_2})\) and \(y_{k_2}^{s_2} \notin \ell_2(x_{t_1}^{s_1})\) and \(x_{t_1}^{s_1} \notin \ell_1(y_{k_2}^{s_2})\) and \(y_{k_2}^{s_2} \notin \ell_1(x_{t_1}^{s_1})\).

In the next, we give the relation between \(PT_1\) and \(PT_0\)-\(\mathcal{C}f_{s}\) bicsp’s.

**Proposition 3.14** Every \(PT_1\)-\(\mathcal{C}f_{s}\) bicsp is \(PT_0\)-\(\mathcal{C}f_{s}\) bicsp.

**Proof:** Follows directly from the definition of \(PT_1\)-\(\mathcal{C}f_{s}\) bicsp.  ■

The converse of Proposition 3.14 is not true, as seen in the following example.

**Example 3.15** Let \(\mathcal{M} = \{x, y\}, \mathcal{D} = \{s_1, s_2\}\) and let \(I \subseteq FS(\mathcal{M}, \mathcal{D})\) such that \(I = \{(s_1, x_{t_1}), (s_2, x_{t_2} \lor y_{k_2})\}; t_1, t_2, k_1, k_2 \in I_0\). Define fuzzy soft closure operators \(\ell_1, \ell_2: FS(\mathcal{M}, \mathcal{D}) \rightarrow FS(\mathcal{M}, \mathcal{D})\) as follows:

\[
\ell_1(\beta_A) = \begin{cases} 
\emptyset_D & \text{if } \beta_A = \emptyset_D, \\
x_{t_1}^{s_1} & \text{if } \beta_A \in \{x_{t_1}^{s_1}; 0 < t_1 < 0.7\}, \\
x_{t_2}^{s_2} & \text{if } \beta_A \in \{x_{t_2}^{s_2}; t_2 \in I_0\}, \\
y_{k_1}^{s_1} & \text{if } \beta_A \in \{y_{k_1}^{s_1}; k_1 \in I_0\}, \\
y_{k_2}^{s_2} & \text{if } \beta_A \in \{y_{k_2}^{s_2}; k_2 \in I_0\}, \\
\{\ell_1(x_{t_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_1}^{s_1})\} & \text{if } \beta_A \in I. 
\end{cases}
\]

And
Then \((M, \ell_1, \ell_2, D)\) is \(PT_0\)-Čfs bicsp. However, it is not \(PT_1\)-Čfs bicsp because there exist \(x_{0.5}^{s_1}\) and \(y_{0.4}^{s_2}\) are two distinct fs-points such that \(x_{0.5}^{s_1} \notin \ell_1(y_{0.2})\).

### Theorem 3.16
If every fs-point in a Čfs bicsp \((M, \ell_1, \ell_2, D)\) is a closed fs ss, then \((M, \ell_1, \ell_2, D)\) is a \(PT_1\)-Čfs bicsp.

**Proof.** Let \(x_i^s\) and \(y_k^{s'}\) be any two distinct fs-points in \((M, \ell_1, \ell_2, D)\). From hypothesis, we have \(\ell_i(x_i^s) = x_i^s\) and \(\ell_i(y_k^{s'}) = y_k^{s'}\) for \(i = 1, 2\). This implies \(x_i^s \notin \ell_i(y_k^{s'})\) and \(y_k^{s'} \notin \ell_j(x_i^s)\). Thus, \((M, \ell_1, \ell_2, D)\) is a \(PT_1\)-Čfs bicsp.  

The converse of the above theorem is not true in general as we have seen in the following example.

### Example 3.17
In Example 3.13, \((M, \ell_1, \ell_2, D)\) is a \(PT_1\)-Čfs bicsp, yet there exists fs-point \(y_{n_2}^{s_2}\) such that \(\ell_1(y_{n_2}^{s_2}) = y_{n_2}^{s_2}\) and \(x_{n_3}^{s_2}\) such that \(\ell_2(x_{n_3}^{s_2}) = x_{n_3}^{s_2}\).

### Theorem 3.18
Every Čfs bi-csubsp of \(PT_1\)-Čfs bicsp is \(PT_1\)-Čfs bi-csubsp.

**Proof:** Similar to the proof of Theorem 3.7.

In the next definition the concept of \(PT_1\) in the induced fs-bits is given.

### Definition 3.19
The induced fs-bits \((M, T_{\ell_1}, T_{\ell_2}, D)\) of a Čfs bicsp \((M, \ell_1, \ell_2, D)\) is said to be pairwise \(T_1\)-fs-bits (\(PT_1\)-fs-bits, for short) if for every two distinct fs-points \(x_i^s\) and \(y_k^{s'}\), we have \(x_i^s \notin T_{\ell_1}\ell_{ij}^{-1}cl(y_k^{s'})\) and \(y_k^{s'} \notin T_{\ell_1}\ell_{ij}^{-1}cl(x_i^s)\), for \(i, j = 1 or 2, i \neq j\).

The next theorem gives the relationship between the induced fs-bits \((M, T_{\ell_1}, T_{\ell_2}, D)\) which is \(PT_1\)-fs-bits and Čfs bicsp \((M, \ell_1, \ell_2, D)\).

### Theorem 3.20
If \((M, T_{\ell_1}, T_{\ell_2}, D)\) is a \(PT_1\)-fs-bits, then \((M, \ell_1, \ell_2, D)\) is also \(PT_1\)-Čfs bicsp.

**Proof:** Similar to the proof of Theorem 3.19.

### Example 3.21
Let \(M = \{x, y\}, D = \{s_1, s_2\}\) and let \(\Gamma \subseteq FS(M, D)\) such that \(\Gamma = \{(s_1, x_t^s \vee y_{k_2}), (s_2, x_t^s \vee y_{k_2}), t_1, t_2, k_1, k_2 \in I_0\}\). Define fuzzy soft closure operators \(\ell_1, \ell_2: FS(M, D) \rightarrow FS(M, D)\) as follows:

\[
\ell_2(\beta_A) = \begin{cases} 
\bar{o}_D & \text{if } \beta_A = \bar{o}_D, \\
_{x_1}^{s_1} & \text{if } \beta_A \in \{x_1^{s_1}, t_1 \in I_0\}, \\
_{x_2}^{s_2} & \text{if } \beta_A \in \{x_2^{s_2}, t_2 \in I_0\}, \\
y_0^{s_2} & \text{if } \beta_A \in \{y_0^{s_2}, 0 < k_2 < 0.4\}, \\
y_1^{s_2} & \text{if } \beta_A \in \{y_1^{s_2}, 0.4 \leq k_2 \leq 1\}, \\
\{\ell_2(x_1^{s_1}), \ell_2(x_2^{s_2}) \cup \ell_2(y_0^{s_2})\} & \text{if } \beta_A \in \Gamma, \\
1_D & \text{otherwise.}
\end{cases}
\]
And $\ell_2(\beta, a) = \beta, a$ for all $\beta, a \in FS(M, D)$. Then $(M, \ell_1, \ell_2, D)$ is $PT_1$-Čfs bicsp. But $(M, T_{\bar{\xi}}, T_{\bar{\eta}}, D)$ is not $PT_1$-fs-bits, because for any two distinct fs-points $x_{i}^{s}$ and $y_{i}^{s'}$, where $s, s' \in \{s_1, s_2\}$ and $t, k \in I_0$ we have $x_{i}^{s} \in T_{\bar{\xi}}T_{\bar{\eta}}-cl(y_{k}^{s'}) = \overline{1}_D$ and $y_{k}^{s'} \in T_{\bar{\xi}}T_{\bar{\eta}}-cl(x_{i}^{s}) = \overline{1}_D$ for $i, j = 1$ or 2.

**Proposition 3.22** If $(M, \ell_1, \ell_2, D)$ is $PT_1$-Čfs bicsp and $\ell_1^{*}, \ell_2^{*}$ are Č-fsco’s on $U$ such that $\ell_1^{*}$ and $\ell_2^{*}$ is finer than $\ell_1$ and $\ell_2$ respectively, then $(M, \ell_1^{*}, \ell_2^{*}, D)$ is $PT_1$-Čfs bicsp for $i = 1, 2$.

**Proof:** We prove the proposition when $i = 1$ and the proof is similar for $i = 2$. Let $x_{i}^{s}$ and $y_{k}^{s'}$ be any two distinct fs-points in $(M, \ell_1^{*}, \ell_2^{*}, D)$. From hypothesis $(M, \ell_1, \ell_2, D)$ is $PT_1$-Čfs bicsp, this yield $x_{i}^{s} \notin \ell_i^{*}(y_{k}^{s'})$ and $y_{k}^{s'} \notin \ell_i^{*}(x_{i}^{s})$, for $i, j = 1, 2, i \neq j$. Since $\ell_1^{*}$ and $\ell_2^{*}$ is finer than $\ell_1$ and $\ell_2$ respectively, that means $\ell_1^{*}(\beta, a) \subseteq \ell_1(\beta, a)$ and $\ell_2^{*}(\beta, a) \subseteq \ell_2(\beta, a)$ for all $\beta, a \in FS(M, D)$. This implies, $x_{i}^{s} \notin \ell_i^{*}(y_{k}^{s'})$ and $y_{k}^{s'} \notin \ell_i^{*}(x_{i}^{s})$. Hence, $(M, \ell_1^{*}, \ell_2^{*}, D)$ is $PT_1$-Čfs bicsp. ■

### 4. Pairwise $T_2$-Čech fuzzy soft bi-closure spaces.

**Definition 4.1** A Čfs bicsp $(M, \ell_1, \ell_2, D)$ is said to be pairwise $T_2$-Čfs bicsp ( $PT_2$-Čfs bicsp), if for every two distinct fs-points $x_{i}^{s}$ and $y_{k}^{s'}$, there exist disjoint $\ell_1$-open $fs$ $\beta, a$ and $\ell_2$-open $fs$ $E_B$ such that $x_{i}^{s} \in \beta, a$ and $y_{k}^{s'} \in E_B$.

Next, we give an example of $PT_2$-Čfs bicsp.

**Example 4.2** Let $M = \{x, y, z\}$, $D = \{s\}$. Define fuzzy soft closure operators $\ell_1, \ell_2: FS(M, D) \rightarrow FS(M, D)$ as follows:

$\ell_1(\beta, a) = \begin{cases} \overline{0}_D & \text{if } \beta, a = \overline{0}_D, \\ \{(s, x_{t+0.2}, y_{t+0.2})\} & \text{if } \beta, a \in \{x_{i}^{s}; 0 < t < 0.8\}, \\ \{(s, x_t, y_{t})\} & \text{if } \beta, a \in \{x_{i}^{s}; 0.8 \leq t \leq 1\}, \\ \{(s, y_{k+0.2})\} & \text{if } \beta, a \in \{y_{k}^{s}; 0 < k < 0.8\}, \\ \{(s, y_k)\} & \text{if } \beta, a \in \{y_{k}^{s}; 0.8 \leq k \leq 1\}, \\ \{(s, z_{t+0.2})\} & \text{if } \beta, a \in \{z_{r}^{s}; 0 < r < 0.8\}, \\ \{(s, z_r)\} & \text{if } \beta, a \in \{z_{r}^{s}; 0.8 \leq r \leq 1\}, \\ \ell_1(x_{i}^{s}) \cup \ell_1(y_{k}^{s}) & \text{if } \beta, a \in \{(s, x_t \lor y_k); t, k \in I_0\}, \\ \ell_1(z_{r}^{s}) \cup \ell_1(z_{r}^{s}) & \text{if } \beta, a \in \{(s, z_{t} \lor z_{r}); t, r \in I_0\}, \\ \ell_1(z_{r}^{s}) \cup \ell_1(x_{i}^{s}) \cup \ell_1(y_{k}^{s}) & \text{if } \beta, a \in \{(s, x_t \lor y_k \lor z_{r}); t, k, r \in I_0\}. 
\end{cases}$

And $\ell_2(\beta, a) = \beta, a$ for all $\beta, a \in FS(M, D)$. Then $(M, \ell_1, \ell_2, D)$ is Čfs bicsp which is $PT_2$-Čfs bicsp. To explain that we have three cases for distinct fs-points as follows:
Case (1): \(x^s_i, y^s_k\) are distinct fs-points, there exist \(\ell_1\)-open \(fss\ \beta_A = \{(s, x_1)\}\) and there exist \(\ell_2\)-open \(fss\ \mathcal{E}_B = \{(s, y_1)\}\) such that \(x^s_i \in \beta_A\) and \(y^s_k \in \mathcal{E}_B\), and \(\beta_A \cup \mathcal{E}_B = \emptyset_D\). And there exist \(\ell_2\)-open \(fss\ \beta_A = \{(s, x_1)\}\) and there exist \(\ell_1\)-open \(fss\ \mathcal{E}_B = \{(s, y_1)\}\) such that \(x^s_i \in \beta_A\) and \(y^s_k \in \mathcal{E}_B\), and \(\beta_A \cup \mathcal{E}_B = \emptyset_D\).

Case (2): \(x^s_i, z^s_i\) are distinct fs-points, there exist \(\ell_1\)-open \(fss\ \beta_A = \{(s, x_1)\}\) and there exist \(\ell_2\)-open \(fss\ \mathcal{E}_B = \{(s, z_1)\}\) such that \(x^s_i \in \beta_A\) and \(z^s_i \in \mathcal{E}_B\), and \(\beta_A \cup \mathcal{E}_B = \emptyset_D\). And there exist \(\ell_2\)-open \(fss\ \beta_A = \{(s, x_1)\}\) and there exist \(\ell_1\)-open \(fss\ \mathcal{E}_B = \{(s, z_1)\}\) such that \(x^s_i \in \beta_A\) and \(z^s_i \in \mathcal{E}_B\), and \(\beta_A \cup \mathcal{E}_B = \emptyset_D\).

Case (3): \(x^s_i, z^s_i\) are distinct fs-points, there exist \(\ell_1\)-open \(fss\ \beta_A = \{(s, y_1)\}\) and there exist \(\ell_2\)-open \(fss\ \mathcal{E}_B = \{(s, z_1)\}\) such that \(y^s_k \in \beta_A\) and \(z^s_i \in \mathcal{E}_B\), and \(\beta_A \cup \mathcal{E}_B = \emptyset_D\).

Remark 4.3 If \((M, \ell_1, \ell_2, \mathcal{D})\) is \(PT_2\)-\(Csbsp\), then \((M, \ell_1, \ell_2, \mathcal{D})\) need not to be \(PT_1\)-\(Csbsp\). To see that, in Example 4.2, \((M, \ell_1, \ell_2, \mathcal{D})\) is \(PT_2\)-\(Csbsp\) but it is not \(PT_1\)-\(Csbsp\). Since there exist \(x_{3, y}^s\) and \(y_{3, y}^s\) are distinct fs-points, and \(y_{3, y}^s \in \ell_1((x_{3, y}) = \{(s, x_{0.5} \lor y_{0.5})\}).

In order to study the hereditary property in \(PT_2\)-\(Csbsp\)'s, we need first to give the following two lemmas.

Lemma 4.4 Let \((M, \ell_1, \ell_2, \mathcal{D})\) be a \(Csbsp\) and \((H, \ell_1, \ell_2, \mathcal{D})\) be a \(Cs\) bi-csubsp of \((M, \ell_1, \ell_2, \mathcal{D})\). Then for any \(\beta_A \in FS(M, \mathcal{D})\), we have \(\beta_A \cap \mathcal{H}_D = \mathcal{H}_D - (\beta_A \cap \mathcal{H}_D)\).

Proof. Let \(\beta_A \in FS(M, \mathcal{D})\). From the definition of \(\mathcal{H}_D\), it is clear that for any \(s \in \mathcal{D}\) and \(x \in H\), \(\beta_A \cap \mathcal{H}_D = \beta_A\). Now, for any \(s \in \mathcal{D}\) and \(x \in H\),

\[ [\mathcal{H}_D - (\beta_A \cap \mathcal{H}_D)](s) = (\mathcal{H}_D - \beta_A)(s) \in I_M. \]

Now,

\[ (\mathcal{H}_D - \beta_A)(s) = ((\beta_A(s) \cap \mathcal{H}_D(s)))(x) = (\mathcal{H}_D - \beta_A)(x) \cap \mathcal{H}_D. \]

Lemma 4.5 Let \((M, \ell_1, \ell_2, \mathcal{D})\) be a \(Csbsp\) and \((H, \ell_1, \ell_2, \mathcal{D})\) be a \(Cs\) bi-csubsp of \((M, \ell_1, \ell_2, \mathcal{D})\). If \(\beta_A\) is an \(\ell_1\)-open \(fss\) of \((M, \ell_1, \ell_2, \mathcal{D})\), then \(\beta_A \cap \mathcal{H}_S\) is an \(\ell_1\)-open \(fss\) in \((H, \ell_1, \ell_2, \mathcal{D})\), for \(i = 1\) or 2.

Proof. We proof the lemma when \(i = 1\) and the proof is similar for \(i = 2\). Let \(\beta_{\ell_1}\) be an \(\ell_1\)-open \(fss\) in \((M, \ell_1, \ell_2, \mathcal{D})\). Then, \(\beta_{\ell_1}\) is a \(\ell_1\)-closed \(fss\) in \((M, \ell_1, \ell_2, \mathcal{D})\). To prove \(\mathcal{H}_D \cap \beta_{\ell_1}\) is an \(\ell_1\)-open \(fss\) in \(H\). That means we must prove \(\ell_{1,1}(\mathcal{H}_D - (\mathcal{H}_D - \beta_{\ell_1})) = \mathcal{H}_D - (\mathcal{H}_D - \beta_{\ell_1})\).

Now,

\[ \ell_{1,1}(\mathcal{H}_D - (\beta_{\ell_1} \cap \mathcal{H}_D)) = \mathcal{H}_D - \ell_1(\mathcal{H}_D - (\beta_{\ell_1} \cap \mathcal{H}_D)) \]

(by definition 2.9)

\[ \subseteq \mathcal{H}_D \cap \ell_1(\beta_{\ell_1} \cap \mathcal{H}_D) \]

(by definition 2.9)

\[ = \mathcal{H}_D \cap \beta_{\ell_1} \cap \mathcal{H}_D \]

(by \(\beta_{\ell_1}\) is \(\ell_1\)-closed)

\[ = \mathcal{H}_D - (\beta_{\ell_1} \cap \mathcal{H}_D) \]

(by Lemma (4.4)).

On the other hand, \(\mathcal{H}_D - (\beta_{\ell_1} \cap \mathcal{H}_D) \subseteq \ell_{1,1}(\mathcal{H}_D - (\beta_{\ell_1} \cap \mathcal{H}_D))\). Then, \(\beta_{\ell_1} \cap \mathcal{H}_D\) is an \(\ell_1\)-open \(fss\) in \((H, \ell_1, \ell_2, \mathcal{D})\).

Now, we are ready to discuss the hereditary property in \(PT_2\)-\(Csbsp\)'s.
Theorem 4.6 Let \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) be a \(PT_2\)-Čfs bicsp and let \((H, \mathcal{L}_1^H, \mathcal{L}_2^H, D)\) be a Čfs bi-csubsp of \((M, \mathcal{L}_1, \mathcal{L}_2, D)\). Then, \((H, \mathcal{L}_1^H, \mathcal{L}_2^H, D)\) is a \(PT_2\)-Čfs bicsp of \((M, \mathcal{L}_1, \mathcal{L}_2, D)\).

Proof. Let \(x_{i}^s\) and \(y_{k}^{s'}\) be any two distinct fs-points in \((H, \mathcal{L}_1^H, \mathcal{L}_2^H, D)\). Then, \(x_{i}^s\) and \(y_{k}^{s'}\) are distinct fs-point in \((M, \mathcal{L}_1, \mathcal{L}_2, D)\). Since \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is a \(PT_2\)-Čfs bicsp, there exist two disjoint open fs's \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively such that \(x_{i}^s \in \beta_{\mathcal{L}_i}\) and \(y_{k}^{s'} \in \mathcal{E}_B\). Consequently, \(x_{i}^s \subseteq \beta_{\mathcal{L}_i} \cap \mathcal{H}_D, y_{k}^{s'} \subseteq \mathcal{E}_B \cap \mathcal{H}_D\) and \((\beta_{\mathcal{L}_i} \cap \mathcal{H}_D) \cap (\mathcal{E}_B \cap \mathcal{H}_D) = \emptyset_D\). By Lemma 4.5, \((\beta_{\mathcal{L}_i} \cap \mathcal{H}_D) \cap \mathcal{E}_B \cap \mathcal{H}_D\) are open fs's in \(\mathcal{L}_1^H\) and \(\mathcal{L}_2^H\) respectively. Hence \((H, \mathcal{L}_1^H, \mathcal{L}_2^H, D)\) is a \(PT_2\)-Čfs bicsp of \((M, \mathcal{L}_1, \mathcal{L}_2, D)\). ■

Definition 4.7 The induced fs-bits \((M, T_{\mathcal{L}_1}, T_{\mathcal{L}_2}, D)\) of a Čfs bicsp \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is said to be \(PT_2\)-fs-bits, if for every two distinct fs-points \(x_{i}^s\) and \(y_{k}^{s'}\), there exist an open fs's \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively such that \(x_{i}^s \in \beta_{\mathcal{L}_i}\), \(y_{k}^{s'} \in \mathcal{E}_B\) and \(\beta_{\mathcal{L}_i} \cap \mathcal{E}_B = \emptyset_D\).

Theorem 4.8 The induced fs-bits \((M, T_{\mathcal{L}_1}, T_{\mathcal{L}_2}, D)\) is \(PT_2\)-fs-bits of \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) if and only if \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is \(PT_2\)-Čfs bicsp.

Proof. Suppose \((M, T_{\mathcal{L}_1}, T_{\mathcal{L}_2}, D)\) is \(PT_2\)-fs-bits and let \(x_{i}^s\) and \(y_{k}^{s'}\) be two distinct fs-points in \(M\). Since \((M, T_{\mathcal{L}_1}, T_{\mathcal{L}_2}, D)\) is \(PT_2\)-fs-bits, there exist \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) open fs's in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively such that \(x_{i}^s \in \beta_{\mathcal{L}_i}\), \(y_{k}^{s'} \in \mathcal{E}_B\) and \(\beta_{\mathcal{L}_i} \cap \mathcal{E}_B = \emptyset_D\). Since \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) are open fs's in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively, then \(\mathcal{L}_i \cap \mathcal{L}_j = \emptyset_D\). From proposition 2.15, we have \(\mathcal{L}_i = \mathcal{L}_i \cup \mathcal{L}_j\) and \(\mathcal{L}_j = \mathcal{L}_i \cup \mathcal{L}_j\). That means there exist \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) are open fs's in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively such that \(x_{i}^s \subseteq \beta_{\mathcal{L}_i}\), \(y_{k}^{s'} \subseteq \mathcal{E}_B\) and \(\beta_{\mathcal{L}_i} \cap \mathcal{E}_B = \emptyset_D\). Hence, \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is \(PT_2\)-Čfs bicsp.

Conversely, similar to the first direction. ■

Lemma 4.9 Let \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) and \((M, \mathcal{L}_1^*, \mathcal{L}_2^*, D)\) be Čfs bicsp's, For any \(\beta_{\mathcal{L}_i} \in FS(M, D)\), if \(\ell_i(\beta_{\mathcal{L}_i}) \subseteq \ell_i^*(\beta_{\mathcal{L}_i}^*)\), then \(Int(\beta_{\mathcal{L}_i}) \subseteq Int(\beta_{\mathcal{L}_i}^*)\), for \(i = 1\) or \(2\).

Proof. We proof the lemma when \(i = 1\) and the proof is similar for \(i = 2\). Let \(\beta_{\mathcal{L}_i} \in FS(M, D)\). From hypothesis, \(\ell_1(\beta_{\mathcal{L}_i}) \subseteq \ell_1^*(\beta_{\mathcal{L}_i})\), implies \((\ell_1^*(\beta_{\mathcal{L}_i}))^c \subseteq (\ell_1(\beta_{\mathcal{L}_i}))^c\). Therefore, \(Int(\beta_{\mathcal{L}_i}) \subseteq Int(\beta_{\mathcal{L}_i}^*)\). ■

Proposition 4.10 If \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is \(PT_2\)-Čfs bicsp and \(\mathcal{L}_1^*\) and \(\mathcal{L}_2^*\) are Čfsco's on \(U\) such that \(\mathcal{L}_1^*\) is finer than \(\mathcal{L}_1\) and \(\mathcal{L}_2\) is \(\mathcal{L}_2\)-Čfs bicsp, for \(i = 1, 2\).

Proof. Let \(x_{i}^s\) and \(y_{k}^{s'}\) be any two distinct fs-points in \(M\). Since \((M, \mathcal{L}_1, \mathcal{L}_2, D)\) is \(PT_2\)-Čfs bicsp, then there exist two disjoint open fs's \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) in \(\mathcal{L}_i\) and \(\mathcal{L}_j\) respectively such that \(x_{i}^s \subseteq \beta_{\mathcal{L}_i}\) and \(y_{k}^{s'} \subseteq \mathcal{E}_B\). That is mean \(Int(\beta_{\mathcal{L}_i}) = \beta_{\mathcal{L}_i}\) and \(Int(\mathcal{E}_B) = \mathcal{E}_B\). Since \(\mathcal{L}_i^*\) is finer than \(\mathcal{L}_i\), this yields by Lemma 4.9, \(\beta_{\mathcal{L}_i} \subseteq Int(\beta_{\mathcal{L}_i}) \subseteq Int(\beta_{\mathcal{L}_i}^*)\) and \(\mathcal{E}_B \subseteq Int(\mathcal{E}_B) \subseteq Int(\mathcal{E}_B\) \subseteq \mathcal{E}_B\). Therefore, there exist two disjoint open fs's \(\beta_{\mathcal{L}_i}\) and \(\mathcal{E}_B\) in \(\mathcal{L}_i^*\) and \(\mathcal{L}_j^*\) respectively such that \(x_{i}^s \subseteq \beta_{\mathcal{L}_i}\) and \(y_{k}^{s'} \subseteq \mathcal{E}_B\). Hence, \((M, \mathcal{L}_1^*, \mathcal{L}_2^*, D)\) is \(PT_2\)-Čfs bicsp. ■
that $x_i^s \in \beta_A$ and $y_k^{s'} \in \ell_i(\beta_A)$ or there exists an $\ell_i$-open $\text{FSS } \mathcal{E}_B$ such that $y_k^{s'} \in \mathcal{E}_B$ and $x_i^{s} \notin \ell_i(\mathcal{E}_B)$, for $i = 1$ or 2.

**Example 4.12** Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s\}$. Define fuzzy soft closure operators $\ell_1, \ell_2: \text{FS}(\mathcal{M}, \mathcal{D}) \to \text{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_A) = \begin{cases} \bar{0}_{\mathcal{D}} & \text{if } \beta_A = \bar{0}_{\mathcal{D}}, \\ y_1^s & \text{if } \beta_A \subseteq \bar{y}_1^s, \\ x_i^{s+0.3} & \text{if } \beta_A \subseteq \{x_i^s, 0 < t < 0.7\}, \\ x_i^1 & \text{if } \beta_A \subseteq \{x_i^s, 0.7 \leq t \leq 1\}, \\ \ell_1(x_i^1) \cup \ell_1(y_k^s) & \text{if } \beta_A \subseteq \{(s, x_i \vee y_k^s) ; t, k \in I_0\}. \end{cases}$$

And

$$\ell_2(\beta_A) = \begin{cases} \bar{0}_{\mathcal{D}} & \text{if } \beta_A = \bar{0}_{\mathcal{D}}, \\ y_1^s & \text{if } \beta_A \subseteq \bar{y}_1^s, \\ x_i^1 & \text{if } \beta_A \subseteq \bar{x}_i^1, \\ \ell_2(x_i^1) \cup \ell_2(y_k^s) & \text{if } \beta_A \subseteq \{(s, x_i \vee y_k^s) ; t, k \in I_0\}. \end{cases}$$

Then, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is $P$-semi $T_2$-Cfs bicsp. Since for any $x_i^s, y_k^s$ are distinct fs-points, there exists an $\ell_1$-open fuzzy soft set $\beta_A = x_i^1$ such that $x_i^s \notin \beta_A$ and $y_k^{s'} \notin \ell_1(\beta_A) = x_i^s$.

**Proposition 4.13** Every $P$-semi $T_2$-Cfs bicsp is $PT_0$-Cfs bicsp.

**Proof.** Let $x_i^s$ and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. From $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is P-semi $T_2$-Cfs bicsp, there exists an $\ell_1$-open $\text{FSS } \beta_A$ such that $x_i^s \notin \beta_A$ and $y_k^{s'} \notin \ell_1(\beta_A)$ or there exists an $\ell_1$-open $\text{FSS } \mathcal{E}_B$ such that $y_k^{s'} \notin \mathcal{E}_B$ and $x_i^s \notin \ell_1(\mathcal{E}_B)$ for $i = 1$ or 2. This implies $x_i^s \notin \ell_1(x_i^1) \subseteq \ell_1(\beta_A)$ and $y_k^{s'} \notin \ell_1(y_k^s)$ or $y_k^{s'} \notin \ell_1(y_k^s) \subseteq \ell_1(\mathcal{E}_B)$ and $x_i^s \notin \ell_1(y_k^{s'})$ for $i = 1$ or 2. Hence, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a $PT_0$-Cfs bicsp. ■

The converse of above proposition is not true as seen in the next example.

**Example 4.14** In Example 3.4. $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is $PT_0$-Cfs bicsp. But $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not semi $T_2$-Cfs bicsp, since there exist $x_{0.5}^1$ and $y_{0.5}^s$ are distinct fs-points such that for any $\ell_1$-open $\text{FSS } \beta_A$, we have $x_{0.5}^1 \notin \beta_A$ and $y_{0.5}^s \notin \ell_1(\beta_A)$ and for any $\ell_1$-open $\text{FSS } \mathcal{E}_B$, we have $y_{0.5}^s \notin \mathcal{E}_B$ and $x_{0.5}^1 \notin \ell_1(\mathcal{E}_B)$.

**Theorem 4.15** Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a pairwise semi $T_2$-Cfs bicsp and let $(\mathcal{H}, \ell_{1\text{c}}, \ell_{2\text{c}}, \mathcal{D})$ be a closed Cfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then, $(\mathcal{H}, \ell_{1\text{c}}, \ell_{2\text{c}}, \mathcal{D})$ is a $P$-semi $T_2$-Cfs bi-csubsp.

**Proof.** Let $x_i^s$ and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{H}, \ell_{1\text{c}}, \ell_{2\text{c}}, \mathcal{D})$. Then, $x_i^s$ and $y_k^{s'}$ are distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Since $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is $P$-semi $T_2$-Cfs bicsp, then either there exists an $\ell_1$-open $\text{FSS } \beta_A$ such that $x_i^s \notin \beta_A$ and $y_k^{s'} \notin \ell_1(\beta_A)$ or there exists an $\ell_1$-open $\text{FSS } \mathcal{E}_B$ such that $y_k^{s'} \notin \mathcal{E}_B$ and $x_i^s \notin \ell_1(\mathcal{E}_B)$ for $i = 1$ or 2. Now, if $x_i^s \notin \beta_A$ and $y_k^{s'} \notin \ell_1(\beta_A)$, then by Lemma 4.5, $x_i^s \notin \beta_A$ and $y_k^{s'} \notin \ell_1(\beta_A)$, which is $\ell_{1\text{c}}$-open $\text{FSS } \mathcal{E}_B$ in $(\mathcal{H}, \ell_{1\text{c}}, \ell_{2\text{c}}, \mathcal{D})$. That is mean we find an $\ell_{1\text{c}}$-open $\text{FSS } \beta_A \cap \bar{H}_S$ in $(\mathcal{H}, \ell_{1\text{c}}, \ell_{2\text{c}}, \mathcal{D})$ contains $x_i^s$, for $i = 1$ or 2. To complete the proof, we must show $y_k^{s'} \notin \ell_{1\text{c}}(\beta_A \cap \bar{H}_D) = \bar{H}_S \cap \ell_{1\text{c}}(\beta_A \cap \bar{H}_D) \subseteq \bar{H}_S \cap \ell_{1\text{c}}(\beta_A) \cap \ell_{1\text{c}}(\mathcal{E}_B) = \bar{H}_D \cap \ell_{1\text{c}}(\beta_A)$. And since $y_k^{s'} \notin \ell_{1\text{c}}(\beta_A)$, then we have
Similarly, if there exists an \( \ell_{iN} \)-open \( \mathfrak{F} \) such that \( y_k^{s} \notin \mathfrak{F} \) and \( x_k^{s} \notin \ell.i(\mathfrak{F}) \). Hence, \((\mathcal{H}, \ell_{1N}, \ell_{2N}, \mathcal{D})\) is a P-semi \( T_{2,CFs} \) bi-csubsp of \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\).

**Definition 4.16** The induced \( fs \)-bits \((\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})\) of a \( Cfs \) bicsp \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is said to be P-semi \( T_2 \)-fs-bits, if for every distinct \( fs \)-point \( x_k^{s} \) and \( y_k^{s'} \), either there exists a \( \tau_{\ell_i} \)-open \( \mathfrak{F} 's \) \( \beta_{\ell} \) in \((\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})\) such that \( x_k^{s} \in \beta_{\ell} \) and \( y_k^{s'} \notin \tau_{\ell_i}cl(\beta_{\ell}) \), or there exists a \( \tau_{\ell_i} \)-open \( \mathfrak{F} 's \) \( \mathfrak{E} \) in \((\mathcal{U}, \tau_{\ell_1}, \tau_{\ell_2}, \mathcal{S})\) such that \( y_k^{s'} \in \mathfrak{E} \) and \( x_k^{s} \notin \tau_{\ell_i}cl(\mathfrak{E}) \), for \( i = 1 \) or \( 2 \).

**Theorem 4.17** If the induced \( fs \)-bits \((\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})\) is a P-semi \( T_2 \)-fs-bits, then \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is also P-semi \( T_2 \)-Cfs bicsp.

**Proof.** Let \( x_k^{s} \) and \( y_k^{s'} \) be any two distinct \( fs \)-points in \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\). From hypothesis, either there exists a \( \tau_{\ell_i} \)-open \( \mathfrak{F} \) \( \beta_{\ell} \) in \((\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})\) such that \( x_k^{s} \notin \beta_{\ell} \) and \( y_k^{s'} \in \tau_{\ell_i}cl(\beta_{\ell}) \), or there exists a \( \tau_{\ell_i} \)-open \( \mathfrak{F} \) \( \mathfrak{E} \) in \((\mathcal{U}, \tau_{\ell_1}, \tau_{\ell_2}, \mathcal{S})\) such that \( y_k^{s'} \in \mathfrak{E} \) and \( x_k^{s} \notin \tau_{\ell_i}cl(\mathfrak{E}) \). By proposition 2.20, we have the following, either there exists an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \beta_{\ell} \) such that \( x_k^{s} \in \beta_{\ell} \) and \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \) or there exists an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \mathfrak{E} \) such that \( y_k^{s'} \in \mathfrak{E} \) and \( x_k^{s} \notin \ell_{i}(\mathfrak{E}) \), \( i = 1 \) or \( 2 \). Thus, \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is a P-semi \( T_2 \)-Cfs bicsp.

**Proposition 4.18** If \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is a P-semi \( T_2 \)-Cfs bicsp and \( \ell_1^{\circ} \) and \( \ell_2^{\circ} \) are \( Cfs \)’s on \( U \) such that \( \ell_1^{\circ} \) \( \ell_2^{\circ} \) is finer than \( \ell_1 \) respectively, then \((\mathcal{M}, \ell_1^{\circ}, \ell_2^{\circ}, \mathcal{D})\) is P-semi \( T_2 \)-Cfs bicsp.

**Proof.** Let \( x_k^{s} \) and \( y_k^{s'} \) be any two distinct \( fs \)-points in \( \mathcal{M} \). Since \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is P-semi \( T_2 \)-Cfs bicsp, then either there exist an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \beta_{\ell} \) such that \( x_k^{s} \notin \beta_{\ell} \) and \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \), or there exist an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \mathfrak{E} \) such that \( y_k^{s'} \in \mathfrak{E} \) and \( x_k^{s} \notin \ell_{i}(\mathfrak{E}) \), \( i = 1 \) or \( 2 \). Suppose, if there exists an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \beta_{\ell} \) such that \( x_k^{s} \notin \beta_{\ell} \) and \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \). On the other hand since \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \), then \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \). This implies \((\mathcal{M}, \ell_1^{\circ}, \ell_2^{\circ}, \mathcal{D})\) is P-semi \( T_2 \)-Cfs bicsp. Similarly, if there exists an \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \mathfrak{E} \) such that \( y_k^{s'} \in \mathfrak{E} \) and \( x_k^{s} \notin \ell_{i}(\mathfrak{E}) \). By the same way, we can prove the proposition when \( i = 2 \).

**Definition 4.19** A \( Cfs \) bicsp \((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})\) is said to be pairwise pseudo \( T_2 \)-\( Cfs \) bicsp (P-pseudo \( T_2 \)-\( Cfs \) bicsp, for short), if for every two distinct \( fs \)-points \( x_k^{s} \) and \( y_k^{s'} \), there exist \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \beta_{\ell} \) and \( \ell_{iN} \)-open \( \mathfrak{F} \) \( \mathfrak{E} \) such that \( x_k^{s} \notin \beta_{\ell} \), \( y_k^{s'} \notin \ell_{i}(\beta_{\ell}) \) and \( y_k^{s'} \in \mathfrak{E} \), \( x_k^{s} \notin \ell_{i}(\mathfrak{E}) \).

An example of pairwise pseudo \( T_2 \)-\( Cfs \) bicsp in given in the following.

**Example 4.20** Let \( \mathcal{M} = \{x, y\} \), \( \mathcal{D} = \{s\} \). Define fuzzy soft closure operators \( \ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \rightarrow FS(\mathcal{M}, \mathcal{D}) \) as follows:

\[
\ell_1(\beta_{\ell}) = \begin{cases} 
\tilde{0}_\mathcal{D} & \text{if } \beta_{\ell} = \tilde{0}_\mathcal{D}, \\
\beta_{\ell} & \text{if } \beta_{\ell} \in \{(s, x_{1-t} \lor y_{1}); 0 \leq t < 1\}, \\
\beta_{\ell} & \text{if } \beta_{\ell} \in \{(s, x_{1} \lor y_{1-k}); 0 \leq k < 1\}, \\
\beta_{\ell} & \text{if } \beta_{\ell} \in \{(s, x_{t}); 0 < t \leq 1\}, \\
\beta_{\ell} & \text{if } \beta_{\ell} \in \{(s, y_{k}); 0 < k \leq 1\}, \\
\ell_1(x_k^{s}) \cup \ell_1(y_k^{s'}) & \text{if } \beta_{\ell} \in \{(s, x_{t} \lor y_{k}); t, k \in l_0\}.
\end{cases}
\]
Proposition 4.21 Every P-pseudo $T_2^*$-Čfs bicsp is P-semi $T_2^*$-Čfs bicsp.

Proof. Follows directly from the definition of P-pseudo $T_2^*$-Čfs bicsp.

Proposition 4.22 Every P-pseudo $T_2^*$-Čfs bicsp is $PT_1^*$-Čfs bicsp.

Proof. Let $x_{t_1}$ and $y_{k_1}$ be any two distinct fs-points in $M$. From hypothesis $(M, \ell_1, \ell_2, D)$ is P-pseudo $T_2^*$-Čfs bicsp, then exist $\ell_1$-open fss $\beta$ and $\ell_2$-open fss $\beta$ such that $x_{t_1} \in \beta$, $y_{k_1} \in \beta$, and $x_{t_1} \notin \beta$ and $y_{k_1} \notin \beta$. Since $x_{t_1} \notin \beta$ and $y_{k_1} \notin \beta$, then it follows that $x_{t_1}$ is a $\beta$-Čfs bicsp. Hence $(M, \ell_1, \ell_2, D)$ is a $PT_1^*$-Čfs bicsp.

The next example shows that converse of the above proposition is not true.

Example 4.23 Let $M = \{x, y\}$, $D = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(M, D)$ such that $\Gamma = \{s_1, x_{t_1} \vee y_{k_1}, s_2, x_{t_2} \vee y_{k_2} \}; t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $L_1, L_2 : FS(M, D) \rightarrow FS(M, D)$ as follows:

$$
\ell_1(\beta) = \begin{cases} 
0D & \text{if } \beta = 0D, \\
x_{s_1}^1 & \text{if } \beta \subseteq x_{s_1}^1, \\
x_{s_2}^1 & \text{if } \beta \subseteq x_{s_2}^1, \\
y_{s_1}^1 & \text{if } \beta \subseteq y_{s_1}^1, \\
y_{s_2}^1 & \text{if } \beta \subseteq y_{s_2}^1, \\
\{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & \text{if } \beta \subseteq \Gamma.
\end{cases}
$$

And

$$
\ell_2(\beta) = \begin{cases} 
0D & \text{if } \beta = 0D, \\
x_{s_1}^1 & \text{if } \beta \subseteq x_{s_1}^1, \\
x_{s_2}^1 & \text{if } \beta \subseteq x_{s_2}^1, \\
y_{s_1}^1 & \text{if } \beta \subseteq y_{s_1}^1, \\
y_{s_2}^1 & \text{if } \beta \subseteq y_{s_2}^1, \\
\{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & \text{if } \beta \subseteq \Gamma.
\end{cases}
$$

Then, $(M, \ell_1, \ell_2, D)$ is a $PT_1^*$-Čfs bicsp. But $(M, \ell_1, \ell_2, D)$ is not P-pseudo $T_2^*$-Čfs bicsp. To show that consider $x_{t_1}^{s_1}$ and $y_{k_1}^{s_2}$ are distinct fs-points. The open fss $\beta$ such that $x_{t_1}^{s_1} \in \beta$ are:
1. \( \beta_A = 1_D \), implies \( y^{s_A}_{0,7} \notin \ell_1(1_D) \).
2. \( \beta_A = \{ (s_1 x_1 \lor y_1), (s_2 y_1) \} \), implies \( y^{s_A}_{0,7} \notin \ell_1(\beta_A) \).
3. \( \beta_A = \{ (s_1 x_1), (s_2 x_1 \lor y_1) \} \), implies \( y^{s_A}_{0,7} \notin \ell_1(\beta_A) \).

Hence, for all open \( fss \) \( \beta_A \) such that \( x^{s_A}_{0,5} \notin \beta_A \), we have \( b^{h_A}_{0,7} \notin \ell_1(\beta_A) \). Thus, \((M, \ell_1, \ell_2, D)\) is not \( P\) pseudo \( T_2\)-\( \tilde{C}fs \) bicsp.

**Theorem 4.24** Let \((M, \ell_1, \ell_2, D)\) be a \( P\)-pseudo \( T_2\)-\( \tilde{C}fs \) bicsp and let \((\mathcal{H}, \ell_{1x'}, \ell_{2x'}, D)\) be a closed \( \tilde{C}fs \) bi-csubsp of \((M, \ell_1, \ell_2, D)\). Then, \((\mathcal{H}, \ell_{1x'}, \ell_{2x'}, D)\) is a pairwise pseudo \( T_2\)-\( \tilde{C}fs \) bi-csubsp.

**Proof.** Similar of Theorem 4.15.

**Definition 4.25** The induced \( fss\) bits \((\mathcal{M}, \tau_{\ell_1}, \tau_{\ell_2}, D)\) of \((M, \ell_1, \ell_2, D)\) is said to be \( P\)-pseudo \( T_2\)-\( fss\) bits, if for every distinct \( fss\) points \( x^s_i \) and \( y^{s_A}_{k} \) there exist \( T_{\ell_i}\)-open \( fss \) \( \beta_A \) and \( T_{\ell_j}\)-open \( fss \) \( E_B \) in \((\mathcal{M}, T_{\ell_1}, T_{\ell_2}, D)\) such that \( x^{s_i} \notin \beta_A \), \( y^{s_A}_{k} \notin T_{\ell_i}\)-cl(\( \beta_A \)) and \( y^{s_A}_{k} \notin E_B \) and \( x^{s_i} \notin T_{\ell_i}\)-cl(\( E_B \)).

**Theorem 4.26** If the induced \( fss\) bits \((\mathcal{M}, T_{\ell_1}, T_{\ell_2}, D)\) is a \( P\)-pseudo \( T_2\)-\( fss\) bits, then \((M, \ell_1, \ell_2, D)\) is also \( P\)-pseudo \( T_2\)-\( \tilde{C}fs \) bicsp.

**Proof.** Similar of Theorem 4.17.

**Proposition 4.27** If \((\mathcal{M}, \ell_1, \ell_2, D)\) be a \( P\)-pseudo \( T_2\)-\( \tilde{C}fs \) bicsp and \( \ell_1^* \) and \( \ell_2^* \) are \( \tilde{C}fs\) co's on \( \mathcal{M} \) such that \( \ell_1^* \) is finer than \( \ell_1 \) and \( \ell_2 \) respectively, then \((\mathcal{M}, \ell_1^*, \ell_2^*, D)\) is \( P\)-pseudo \( T_2\)-\( \tilde{C}fs \) bicsp, for \( i = 1, 2 \).

**Proof.** Similar of Proposition 4.18.

**Definition 4.28** A \( \tilde{C}fs \) bicsp \((\mathcal{M}, \ell_1, \ell_2, D)\) is said to be pairwise Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp (P-Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp), if for every two distinct \( fss\) points \( x^s_i \) and \( y^{s_A}_{k} \) there exist \( \ell_i\)-open \( fss \) \( \beta_A \) and \( \ell_j\)-open \( fss \) \( E_B \) such that \( x^{s_i} \notin \beta_A \), \( y^{s_A}_{k} \notin E_B \) and \( \ell_i(\beta_A) \cap \ell_j(E_B) = \emptyset_D \).

The next example explains the concept of P-Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp.

**Example 4.29** Let \( \mathcal{M} = \{ x, y \} \), \( D = \{ s \} \). Define fuzzy soft closure operators \( \ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{E}) \rightarrow FS(\mathcal{M}, \mathcal{E}) \) as follows:

\[
\ell_1(\beta_A) = \ell_2(\beta_A) = \begin{cases} 
0_D & \text{if } \beta_A = 0_D, \\
x^s_i & \text{if } \beta_A \subseteq x^s_i, \\
y^{s_A}_{1} & \text{if } \beta_A \subseteq y^{s_A}_{1}, \\
1_D & \text{other wise.}
\end{cases}
\]

Then \((\mathcal{M}, \ell_1, \ell_2, D)\) P-Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp. Since for any \( x^s_i \) and \( y^{s_A}_{k} \) are distinct \( fss\) points, there exist \( \ell_i\)-open \( fss \) \( \beta_A = x^s_i \) and \( \ell_j\)-open \( fss \) \( E_B = y^{s_A}_{1} \) such that \( x^{s_i} \notin \beta_A \), \( y^{s_A}_{k} \notin E_B \) and \( \ell_i(\beta_A) \cap \ell_j(E_B) = \emptyset_D \) for \( i, j = 1, 2 \), \( i \neq j \).

**Proposition 4.30** Every P-Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp is \( P\)-pesudo \( PT_2\)-\( \tilde{C}fs \) bicsp.

**Proof.** Let \( x^s_i \) and \( y^{s_A}_{k} \) be any two distinct \( fss\) points in \((\mathcal{M}, \ell_1, \ell_2, D)\). Since \((\mathcal{M}, \ell_1, \ell_2, D)\) is P-Uryshon \( T_2\)-\( \tilde{C}fs \) bicsp, there exist \( \ell_i\)-open \( fss \) \( \beta_A \) and \( \ell_j\)-open \( fss \) \( E_B \) such that \( x^{s_i} \notin \beta_A \), \( y^{s_A}_{k} \notin E_B \) and \( \ell_i(\beta_A) \cap \ell_j(E_B) = \emptyset_D \). This implies \( x^{s_i} \notin \ell_j(E_B) \) and \( y^{s_A}_{k} \notin \ell_i(\beta_A) \). Therefore, \((\mathcal{M}, \ell_1, \ell_2, D)\) is \( P\)-pesudo \( T_2\)-\( \tilde{C}fs \) bicsp.
**Proposition 4.31** Every P-Uryshon $T_2$-Čfs bicsp is $PT_2$-Čfs bicsp.

**Proof.** The proof follows immediately from the definition of P-Uryshon $T_2$-Čfs bicsp and the property $\beta_{\ell} \subseteq \ell_i(\beta_{\ell})$ for any fuzzy soft set $\beta_{\ell}$. □

The hereditary property in P-Uryshon $T_2$-Čfs bicsp’s are studied in the next theorem.

**Theorem 4.32** Let $(M, \ell_1, \ell_2, D)$ be a P-Uryshon $T_2$-Čfs bicsp and let $(H, \ell_1, \ell_2, D)$ be a closed Čfs bi-csubsp of $(M, \ell_1, \ell_2, D)$. Then $(H, \ell_1, \ell_2, D)$ is a P-Uryshon $PT_2$-Čfs bi-csubsp.

**Proof.** Let $x_i^s$ and $y_k^s$ be any two distinct fs-points in $(H, \ell_1, \ell_2, D)$. Then, $x_i^s$ and $y_k^s$ are distinct fs-points in $(M, \ell_1, \ell_2, D)$. Since $(M, \ell_1, \ell_2, D)$ is a P-Uryshon $T_2$-Čfs bicsp, it follows there exist $\ell_i$-open fss $\beta_{\ell}$ and $\ell_j$-open fss $E_B$ such that $x_i^s \in \beta_{\ell}$, $y_k^s \in E_B$ and $\ell_i(\beta_{\ell}) \cap \ell_j(E_B) = \emptyset_D$. By Lemma 4.5, $\beta_{\ell} \cap H_D$ and $E_B \cap H_D$ are open fss’s in $\ell_1$ and $\ell_2$ respectively such that $x_i^s \in \beta_{\ell} \cap H_D$, $y_k^s \in E_B \cap H_D$. Next, we must show that $\ell_i(\beta_{\ell} \cap H_D) \cap \ell_j(E_B \cap H_D) = \emptyset_D$.

Now, from the definition of $\ell_1$ and $\ell_2$ we get,$$
\ell_i(\beta_{\ell} \cap H_D) \cap \ell_j(E_B \cap H_D) = [H_D \cap \ell_i(\beta_{\ell} \cap H_D)] \cap [H_D \cap \ell_j(E_B \cap H_D)]
= [\ell_i(\beta_{\ell}) \cap \ell_j(E_B)] \cap H_D
= \emptyset_D.
$$

Therefore, $(H, \ell_1, \ell_2, D)$ is a P-Uryshon $T_2$-Čfs bi-csubsp of $(M, \ell_1, \ell_2, D)$. □

**Definition 4.33** The induced fs-bits $(M, \ell_1, \ell_2, D)$ of $(M, \ell_1, \ell_2, D)$ is said to be P-Uryshon $T_2$-fs-bits, if for every distinct fs-points $x_i^s$ and $y_k^s$, there exist $\tau_{\ell_i}$-open fss $\beta_{\ell}$ and $\tau_{\ell_j}$-open fss $E_B$ such that $x_i^s \in \beta_{\ell}$, $y_k^s \in E_B$ and $T_{\ell_i}$-cl$(\beta_{\ell}) \cap T_{\ell_j}$-cl$(E_B) = \emptyset_D$.

**Theorem 4.34** If the induced fs-bits $(M, \ell_1, \ell_2, D)$ is a P-Uryshon $T_2$-fs-bits, then $(M, \ell_1, \ell_2, D)$ is also P-Uryshon $T_2$-Čfs bicsp.

**Proof.** Let $x_i^s$ and $y_k^s$ be any two distinct fs-points in $M$. Since $(M, \ell_1, \ell_2, D)$ is an P-Uryshon $T_2$-fs-bits, then there exist $T_{\ell_i}$-open fss $\beta_{\ell}$ and $T_{\ell_j}$-open fss $E_B$ such that $x_i^s \notin \beta_{\ell}$, $y_k^s \notin E_B$ and $T_{\ell_i}$-cl$(\beta_{\ell}) \cap T_{\ell_j}$-cl$(E_B) = \emptyset_D$. By proposition 2.20, we obtain $\beta_{\ell}$ and $E_B$ are open fuzzy soft sets in $\ell_1$ and $\ell_2$ respectively such that $x_i^s \notin \beta_{\ell}$ and $y_k^s \notin E_B$ and $\ell_i(\beta_{\ell}) \cap \ell_j(E_B) = \emptyset_D$. Hence, $(M, \ell_1, \ell_2, D)$ is a P-Uryshon $T_2$-Čfs bicsp. □

**Proposition 4.35** If $(M, \ell_1, \ell_2, D)$ be a P-Uryshon $T_2$-Čfs bicsp and $\ell_1^*, \ell_2^*$ are Č-fsc’s on $M$ such that $\ell_1^*$ and $\ell_2^*$ is finer than $\ell_1$ and $\ell_2$ respectively, then $(M, \ell_1^*, \ell_2^*, D)$ is pairwise for $i = 1, 2$.

**Proof.** Similar to the proof of Proposition 4.18. □

**References**
