Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 4 Article 40

8-15-2021

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Mohamad Th. Hmood
University of Baghdad, Faculty of Education for Pure Sciences Ibn Al-Haitham, Department of Mathematics, Baghdad, Iraq

R. N. Majeed
University of Baghdad, Faculty of Education for Pure Sciences Ibn Al-Haitham, Department of Mathematics, Baghdad, Iraq

Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (Special issue num.4) (2021) pp. 405–422

Al–Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Authors Names

a. Mohamad Th. Hmood

b. R. N. Majeed

Article History

Received on: 30/6/2021

Revised on: 30 /7/2021

Accepted on: 15/8/2021

Keywords:

Fuzzy soft set, fuzzy soft bitopological space, pairwise T_0, pairwise T_1, pairwise T_2, pairwise semi T_2, pairwise pseudo T_2, pairwise Uryshon T_2.

DOI: https://doi.org/10.29350/ jops. 2021.26. 4.1414

ABSTRACT

The idea of Čech fuzzy soft bi-closure space (Čfs bicsp) $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a new one, and its basic features are defined and studied in [12]. In this paper, separation axioms, namely pairwise, $\mathrm{T_i}$ $(\mathrm{i} = 0,1,2)$, pairwise semi- (respectively, pairwise pseudo and pairwise Uryshon) $\mathrm{T_2}$-Čfs bicsp's are introduced and studied in both Čech fuzzy soft bi-closure space and their induced fuzzy soft bitopological spaces. It is shown that hereditary property is satisfied for $\mathrm{T_i}$, $\mathrm{i} = 0,1$ with respect to Čech fuzzy soft bi-closure space but for other mentioned types of separations axioms, hereditary property satisfies for closed subspaces of Čech fuzzy soft bi-closure space.

1.Introduction

In 1965, Zadeh [23] proposed the concept of a fuzzy set for the first time. Fuzzy sets were established to provide a mathematical representation for situations in which there are no definite membership requirements for ill-defined groups or collections of items. In 1999, Moldtsov [17] published a paper on the creation of soft set theory, a modern technique of showing complexity and confusion. The theory of soft sets has a wide range of potential applications, some of which Molodsov presented in [17]. In 2001 Maji et al. [11] combined fuzzy sets and soft sets to establish the concept of fuzzy soft sets. Later in 2011, Tanay and Kandemir [20] proposed a notion of topological structure based on fuzzy soft sets.

In 1966, Čech [2] presented the notion of $\check{C}$ech closure spaces $(X, \mathcal{C})$, where $\mathcal{C}: P(X) \to P(X)$ is a mapping fulfilling $\mathcal{C}(\emptyset) = \emptyset, V \subseteq \mathcal{C}(V)$ and $\mathcal{C}(V \cup F) = \mathcal{C}(V) \cup \mathcal{C}(F)$, the mapping termed $\check{C}$ech closure operator on X, which is analogous to a topological closure operator with the distinction that it does not have to be idempotent. That is, $\check{C}$ech satisfies just three of the four Kuratowski

a,b University of Baghdad, Faculty of Education for Pure Sciences Ibn Al-Haitham, Department of Mathematics, Baghdad, Iraq

closure axioms. Mashhour and Ghanim [14] introduced the idea of Čech fuzzy soft closure space 1985, when they replaced sets with fuzzy sets in the description of Čech closure space. Rao and Gowri [4] introduced the concept of biclosure space $(X, \mathcal{C}_1, \mathcal{C}_2)$ in 2006. Such space is equipped with two arbitrary $\check{C}$ech closure operators $\mathcal{C}_1$ and $\mathcal{C}_2$. Tapi and Navalakhe [21] later introduced the concept of fuzzy biclosure spaces in 2011. After the concept of soft theory appeared by Moldtsov [17], Gowri and Jegadeesan [5], and Krishnaveni and Sekar [7] in 2014 using the principle of soft sets, to introduce the concept of soft Čech closure spaces. Also, in the same year, Gowri and Jegadeesan [6] introduced the concept of soft biČech closure spaces.

Majeed [8] recently established the definition of Čech fuzzy soft closure spaces, which were motivated by Chang's concepts of fuzzy soft set and fuzzy soft topology [3]. Majeed and Maibed also studied the architecture of Čech fuzzy soft closure spaces, including separation axioms and connectedness [12, 13, 9, 10]. As a generalization to $\check{C}$ech fuzzy soft closure space [8], we recently presented and researched the concept of $\check{C}$ech fuzzy soft bi-closure spaces (Čfs bicsp's) [15] and give more study of $\check{C}$fs bicsp's in [16].

This work aims to introduce and study lower separation axioms in Čfs bicsp's. Section 3, devoted to defining pairwise T_0 and T_1 in Čfs bicsp's and their induced fs-bits's. We discuss the relation between PT_0 and PT_1, and study the hereditary property on Čfs bicsp's. Also, we give the relation between Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ and its induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ when induced fs-bits is PT_i, $i = 0,1$. In this section 4, pairwise T_2-Čfs bicsp and other types, namely, pairwise semi- (respectively, pairwise pseudo and pairwise Uryshon) T_2-Čfs bicsp's are introduced, the properties of each type are discussed as in Section 3. In addition, the relationships between separation axioms introduced in the current section and the previous section are obtained.

2. PRELIMINARIES

Throughout this work, M is the universal set, I=[0,1], I 0=(0,1), $\mathcal{D}$ is the set of parameters for $\mathcal{M}$, and $\mathcal{A}$ ba a nonempty subset of $\mathcal{D}$. A fuzzy set β is a mapping from $\mathcal{M}$ into I [23]. $I^{\mathcal{M}}$ refer to the family of all fuzzy sets of $\mathcal{M}$.

Definition 2.1 [22] A fuzzy soft set (fss) $\beta_{\mathcal{A}}$ on the universe set $\mathcal{M}$ is a mapping from $\mathcal{D}$ to $I^{\mathcal{M}}$, i.e., $\beta_{\mathcal{A}}: \mathcal{D} \to I^{\mathcal{M}}$, where $\beta_{\mathcal{A}}(d) \neq \bar{0}$ if $d \in \mathcal{A} \subseteq \mathcal{D}$ and $\beta_{\mathcal{A}}(d) = \bar{0}$ if $d \notin \mathcal{A}$, where $\bar{0}$ is the empty fuzzy set on $\mathcal{M}$. The family of all $fss's$ over $\mathcal{M}$ denoted by $FS(\mathcal{M}, \mathcal{D})$.

Definition 2.2 [22] Let $\beta_{\mathcal{A}}, \mu_{\mathcal{B}} \in FS(\mathcal{M}, \mathcal{D})$. Then,

1. $\beta_{\mathcal{A}} \subseteq \mu_{\mathcal{B}}$ iff $\beta_{\mathcal{A}}(d) \leq \mu_{\mathcal{B}}(d)$, for all $d \in \mathcal{D}$.
2. $\beta_{\mathcal{A}} = \mu_{\mathcal{B}}$ iff $\beta_{\mathcal{A}} \subseteq \mu_{\mathcal{B}}$ and $\mu_{\mathcal{B}} \subseteq \beta_{\mathcal{A}}$.
3. $\rho_{\mathcal{A}\cup\mathcal{B}} = \beta_{\mathcal{A}} \cup \mu_{\mathcal{B}}$ iff $\rho_{\mathcal{A}\cup\mathcal{B}}(d) = \beta_{\mathcal{A}}(d) \vee \mu_{\mathcal{B}}(d)$, for all $d \in \mathcal{D}$.
4. $\rho_{\mathcal{A}\cap\mathcal{B}} = \beta_{\mathcal{A}} \cap \mu_{\mathcal{B}}$ iff $\rho_{\mathcal{A}\cap\mathcal{B}}(d) = \beta_{\mathcal{A}}(d) \wedge \mu_{\mathcal{B}}(d)$, for all $d \in \mathcal{D}$.
5. The complement of $\beta_{\mathcal{A}}$ is denoted by $\beta_{\mathcal{A}}^c$ where $\beta_{\mathcal{A}}^c(d) = \bar{1} - \beta_{\mathcal{A}}(d)$, for all $d \in \mathcal{D}$, where $\bar{1}(x) = 1\ \forall x \in \mathcal{M}$.
6. β_s is called null fss, denoted $\tilde{0}_{\mathcal{D}}$, if $\beta_s(d) = \bar{0}$, for all $d \in \mathcal{D}$.
7. $\beta_{\mathcal{D}}$ is called universal fss, denoted $\tilde{1}_{\mathcal{D}}$, if $\beta_{\mathcal{D}}(d) = \bar{1}$, for all $d \in \mathcal{D}$.

Definition 2.3 [1] A fss $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$ is called fs-point (fs-point), denoted by x_t^s, if there exists $x \in \mathcal{M}$ and $s \in \mathcal{D}$ such that $\beta_{\mathcal{A}}(s)(x) = t$ $(0 < t \leq 1)$ and $\bar{0}$ otherwise for all $y \in \mathcal{M} - \{x\}$.

The fs-point x_t^s is said to belong to the fss $\beta_{\mathcal{A}}$, denoted by $x_t^s \widetilde{\in} \beta_{\mathcal{A}}$ if for the element $s \in \mathcal{M}, t \leq \beta_{\mathcal{A}}(s)(x)$.

Definition 2.4 [20] A triple $(\mathcal{M}, \mathcal{T}, \mathcal{D})$ is called a fuzzy soft topological space where $\mathcal{T}$ is a family of $fss's$ over $\mathcal{M}$ which satisfying the following properties.

1. $\tilde{0}_{\mathcal{D}}, \tilde{1}_{\mathcal{D}} \in \mathcal{T}$,
2. $\beta_{\mathcal{A}}, \mu_{\mathcal{B}} \in \mathcal{T} \Longrightarrow \beta_{\mathcal{A}} \cap \mu_{\mathcal{B}} \in \mathcal{T}$,
3. $(\beta_{\mathcal{A}})_i \in \mathcal{T}\ \forall i \Longrightarrow \cup_{i\in J} (\beta_{\mathcal{A}})_i \in \mathcal{T}$.

$\mathcal{T}$ is called a topology of $fss's$ on $\mathcal{M}$. Each member of $\mathcal{T}$ is called an $\mathcal{T}$-open fss. $\mu_{\mathcal{B}}$ is called a $\mathcal{T}$-closed fss in $(\mathcal{M}, \mathcal{T}, \mathcal{D})$ if $\mu_{\mathcal{B}}^c \in \mathcal{T}$.

Definition 2.5 [18] A quadruple $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{D})$ is said to be a fuzzy soft bi-topological space where $\mathcal{T}_1, \mathcal{T}_2$ are arbitrary fuzzy soft topologies on $\mathcal{M}$.

The following recall the concept of Čfs bicsp and its fundamental properties. For $i, j = 1,2$ where $i \neq j$. Other than that we will mention the value of i and j.

Definition 2.6 [15] A Čfs bicsp is a quadruple $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ where $\mathcal{M}$ is a nonempty set, and $\ell_1, \ell_2: FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ are two fuzzy soft closure operators on $\mathcal{M}$ which are correct according to the following axioms:

(A_1) $\ell_i(\tilde{0}_{\mathcal{D}}) = \tilde{0}_{\mathcal{D}}$,
(A_2) $\beta_{\mathcal{A}} \sqsubseteq \ell_i(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$,
(A_3) $\ell_i(\beta_{\mathcal{A}} \cup \mu_{\mathcal{B}}) = \ell_i(\beta_{\mathcal{A}}) \cup \ell_i(\mu_{\mathcal{B}})$ for all $\beta_{\mathcal{A}}, \mu_{\mathcal{B}} \in FS(\mathcal{M}, \mathcal{D})$.

Definition 2.7 [15] A fss $\beta_{\mathcal{A}}$ of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be ℓ_i-closed (ℓ_i-open, respectively) fss if $\ell_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$(respectively, $\ell_i(\beta_{\mathcal{A}}^c)$=$\beta_{\mathcal{A}}^c$) And, it is called a closed fss if $\ell_i(\ell_j(\beta_{\mathcal{A}})) = \beta_{\mathcal{A}}$. For $i, j = 1$ or 2 where $i \neq j$. The complement of a closed fss is called an open fss.

Proposition 2.8 [15] Let $\beta_{\mathcal{A}}$ be a fss of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then, $\beta_{\mathcal{A}}$ is a closed fss in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ if and only if $\beta_{\mathcal{A}}$ is ℓ_j-closed fss.

Definition 2.9 [15] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $\mathcal{H} \sqsubseteq \mathcal{M}$. The quadruple $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is called a Čech fuzzy soft bi-closure subspace (Čfs bi-csubsp) of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$, where $\ell_{i_{\mathcal{H}}}: FS(\mathcal{H}, \mathcal{D}) \to FS(\mathcal{H}, \mathcal{D})$ defind by $\ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}}) = \widetilde{\mathcal{H}}_S \cap \ell_i(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{H}, \mathcal{D})$. The Čfs bi-csubsp $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is said to be a closed (respectively, open) subspace if $\widetilde{\mathcal{H}}_{\mathcal{D}}$ is a closed (resp. open) fss over $\mathcal{M}$.

Proposition 2.10 [15] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Let $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. Then, $\beta_{\mathcal{A}}$ is a closed fss over $\mathcal{H}$ if and only if $\ell_{j_{\mathcal{H}}}(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$.

Definition 2.11 [16] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp, the induced fuzzy soft bitopological space (induced fs-bits, for short) of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$, denoted by $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ where $\mathcal{T}_{\ell_i} = \{\beta_{\mathcal{A}}^c : \ell_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}\}$.

Definition 2.12 [16] Let $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ be the induced fs-bits of the Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ and $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. $\beta_{\mathcal{A}}$ is called an $\mathcal{T}_{\ell_i}$-open fss in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$, if $\beta_{\mathcal{A}} \in \mathcal{T}_{\ell_i}$. The complement of an

$\mathcal{T}_{\ell_i}$-open fss $\beta_{\mathcal{A}}$ is a $\mathcal{T}_{\ell_i}$-closed fss, and $\beta_{\mathcal{A}}$ is called an open fss in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$, if $\beta_{\mathcal{A}}$ is an $\mathcal{T}_{\ell_i}$-open, for $i = 1,2$.

Definition 2.13 [16] Let $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ be the induced fs-bits of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ and let $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. The fuzzy soft closure of $\beta_{\mathcal{A}}$ for ℓ_i and ℓ_j, denoted by $\mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}$-$cl(\beta_{\mathcal{A}})$, is the intersection of all closed fuzzy soft supersets of $\beta_{\mathcal{A}}$. i.e., $\mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}$-$cl(\beta_{\mathcal{A}}) = \cap \{ \rho_{\mathcal{C}} : \beta_{\mathcal{A}} \subseteq \rho_{\mathcal{C}}$ and $\ell_i(\ell_j(\rho_{\mathcal{C}})) = \rho_{\mathcal{C}} \}$.

Theorem 2.14 [16] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$be Čfs bicsp and $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ be the induced fs-bits of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then for any $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$.

$$\mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}int(\beta_{\mathcal{A}}) \subseteq Int_i\left(Int_j(\beta_{\mathcal{A}})\right) \subseteq \beta_{\mathcal{A}} \subseteq \ell_i\left(\ell_j(\beta_{\mathcal{A}})\right) \subseteq \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(\beta_{\mathcal{A}}).$$

proposition 2.15 [16] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be Čfs bicsp and $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ be the induced fs-bits of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then for any $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$.

$$\mathcal{T}_{\ell_i}\text{-}int(\beta_{\mathcal{A}}) \subseteq Int_i(\beta_{\mathcal{A}}) \subseteq \beta_{\mathcal{A}} \subseteq \mathcal{L}_i(\beta_{\mathcal{A}}) \subseteq \mathcal{T}_{\ell_i}\text{-}cl(\beta_{\mathcal{A}}).$$

Definition 2.16 [16] Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ and $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ be two Čfs bicsp's on $\mathcal{M}$. Then ℓ_i is said to finer than ℓ_i^*, or equivalently ℓ_i^* is coarser than ℓ_i, if for each $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$, $\ell_i(\beta_{\mathcal{A}}) \subseteq \ell_i^*(\beta_{\mathcal{A}})$, for $i = 0,1$.

3. Pairwise T_i- Čech fuzzy soft bi-closure spaces, $i = 0, 1$

Definition 3.1 A Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise T_0-Čfs bicsp (PT_0-Čfs bicsp), if for every pair of distinct fs-points x_t^s and $y_k^{s'}$, either $x_t^s \widetilde{\notin} \ell_i(y_k^{s'})$ or $y_k^{s'} \widetilde{\notin} \ell_i(x_t^s)$, for $i = 1$ or 2.

Now we give some examples to illustrate Definition 3.1.

Example 3.2 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be the discrete Čfs bicsp (i.e., $\ell_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$), then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_0-Čfs bicsp.

Example 3.3 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be the trivial Čfs bicsp (i.e., for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$, $\ell_i(\beta_{\mathcal{A}}) = \tilde{0}_{\mathcal{D}}$ if $\beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}$ and $\ell_i(\beta_{\mathcal{A}}) = \tilde{1}_{\mathcal{D}}$ otherwise). Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not PT_0-Čfs bicsp because for any distinct fs-points x_t^s and $y_k^{s'}$, we have $x_t^s \widetilde{\in} \tilde{1}_{\mathcal{D}} = \ell_i(y_k^{s'})$ and $y_k^{s'} \widetilde{\in} \tilde{1}_{\mathcal{D}} = \ell_i(x_t^s)$.

Example 3.4 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})$ such that $\Gamma = \{(s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2});\ t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ t_1 \in I_0\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_{0.6}^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ 0 < k_1 < 0.6\}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ 0.6 \le k_1 \le 1\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if\beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ x_{0.3}^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ 0 < t_1 < 0.3\}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ 0.3 \le t_1 \le 1\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ k_1 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_2(x_{t_1}^{s_1}) \cup \ell_2(y_{k_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & if\ \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a Čfs bicsp. To show $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$is PT_0-Čfs bicsp, we have three cases for distinct fs-points in $\mathcal{M}$.

Case (1): If $x \neq y$ and $s_1 = s_2$, then we have $x_{t_1}^{s_1}$ and $y_{k_1}^{s_1}$ are distinct fs-points. It is clear that $y_{k_1}^{s_1} \tilde{\notin} \ell_1(x_{t_1}^{s_1})$ because $k_1 > (x_1^{s_1})(s_1)(y)$=$(x_1)(y) = 0$. Similarly, $x_{t_2}^{s_2}$ and $y_{k_2}^{s_2}$ are distinct fs-points and $y_{k_2}^{s_2} \tilde{\notin} \ell_1(x_{t_2}^{s_2})$.

Case (2): If $x = y$ and $s_1 \neq s_2$, then $x_{t_1}^{s_1}$ and $x_{t_2}^{s_2}$ are distinct fs-points. It is clear that $x_{t_1}^{s_1} \tilde{\notin} \ell_1(x_{t_2}^{s_2})$ because $t_1 > \ell_1(x_1^{s_2})(s_1)(x)$=$\bar{0}(x) = 0$. Similarly, $y_{k_1}^{s_1}$ and $y_{k_2}^{s_2}$ are distinct fs-points and $y_{k_2}^{s_2} \tilde{\notin} \ell_1(y_{k_1}^{s_1})$.

Case (3): If $x \neq y$ and $s_1 \neq s_2$, then we have $x_{t_1}^{s_1}$ and $y_{k_2}^{s_2}$ are distinct fs-points such that $y_{k_2}^{s_2} \tilde{\notin} \ell_1(x_{t_1}^{s_1})$. Similarly, $x_{t_2}^{s_2}$ and $y_{k_1}^{s_1}$ are distinct fs-points and $y_{k_1}^{s_1} \tilde{\notin} \ell_1(x_{t_2}^{s_2})$. Hence, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_0-Čfs bicsp.

Theorem 3.5 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a PT_0-Čfs bicsp, then for any two distinct fs-points x_t^s and $y_k^{s'}$, $\ell_i(x_t^s) \neq \ell_i(y_k^{s'})$, for $i = 1\ or\ 2$.

Proof: Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a PT_0-Čfs bicsp, and let x_t^s and $y_k^{s'}$ be any two distinct fs-points. Suppose that $\ell_i(x_t^s) = \ell_i(y_k^{s'})$ for $i = 1\ and\ 2$. Since $x_t^s \tilde{\in} \ell_i(x_t^s)$ and $y_k^{s'} \tilde{\in} \ell_i(y_k^{s'})$ for $i = 1\ and\ 2$. Then from the hypothesis, $x_t^s \tilde{\in} \ell_i(y_k^{s'})$ and $y_k^{s'} \tilde{\in} \ell_i(x_t^s)$ for $i = 1\ and\ 2$. This implies $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not PT_0-Čfs bicsp, which is a contradiction. Hence, $\ell_i(x_t^s) \neq \ell_i(y_k^{s'})$ for $i = 1\ or\ 2$.■

The converse of the above theorem is not true, as the following example show.

Example 3.6 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})$ such that Γ =$\{(s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2});\ t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, x_{0.2} \vee y_{0.1})\} & if\ \beta_{\mathcal{A}} \subseteq x_{0.2}^{s_1}, \\ \{(s_1, x_{0.1} \vee y_{0.2})\} & if\ \beta_{\mathcal{A}} \subseteq y_{0.2}^{s_1}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ 0.2 < t_1 \le 1\} \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1}; 0.2 < k_1 \le 1\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, x_{0.7} \vee y_{0.5})\} & if\ \beta_{\mathcal{A}} \subseteq x_{0.7}^{s_1}, \\ \{(s_1, x_{0.5} \vee y_{0.7})\} & if\ \beta_{\mathcal{A}} \subseteq y_{0.7}^{s_1}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ 0.7 < t_1 \le 1\}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1}; 0.7 < k_1 \le 1\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_2(x_{t_1}^{s_1}) \cup \ell_2(y_{k_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & if\ \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a Čfs bicsp. It is clear that for any two distinct fs-points x_t^s and $y_k^{s'}$, where $s, s' \in \{s_1, s_2\}$ and $t, k \in I_0$ we have $\ell_i(x_t^s) \neq \ell_i(y_k^{s'})$ for $i = 1\ or\ 2$. However, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not PT_0-Čfs bicsp, since there exist two distinct fs-point $x_{0.1}^{s_1}$ and $y_{0.1}^{s_1}$ such that $x_{0.1}^{s_1} \tilde{\in} \ell_i(y_{0.1}^{s_1})$ and $y_{0.1}^{s_1} \tilde{\in} \ell_i(x_{0.1}^{s_1})$ for $i = 1,2$.

Next, we show that PT_0 is a hereditary property on Čfs bi-csubsp.

Theorem 3.7 Every Čfs bi-csubsp of a PT_0- Čfs bicsp, is a PT_0-Čfs bi-csubsp.

Proof Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a PT_0-Čfs bicsp and $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$. Since $FS(\mathcal{H}, \mathcal{D}) \subseteq FS(\mathcal{M}, \mathcal{D})$, then x_t^s and $y_k^{s'}$ are distinct fs-points in $FS(\mathcal{M}, \mathcal{D})$. Since $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_0-Čfs bicsp. Then, $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ or $y_k^{s'} \tilde{\notin} \ell_i(x_t^s)$ for $i = 1\ or\ 2$. This implies either $x_t^s \tilde{\notin} \ell_i(y_k^{s'}) \cap \tilde{\mathcal{H}}_S$ or $y_k^{s'} \tilde{\notin} \ell_i(x_t^s) \cap \tilde{\mathcal{H}}_S$. Then $x_t^s \tilde{\notin} \ell_{i_{\mathcal{H}}}(y_k^{s'})$ or $y_k^{s'} \tilde{\notin} \ell_{i_{\mathcal{H}}}(x_t^s)$. Hence $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is PT_0-Čfs bi-csubsp. ■

Definition 3.8 The induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise T_0-fsbits (PT_0-fsbits, for short), if for every two distinct fs-points x_t^s and $y_k^{s'}$, either $x_t^s \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(y_k^{s'})$ or $y_k^{s'} \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(x_t^s)$, for $i, j = 1\ or\ 2$.

To give the relationship between the induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ which is PT_0-fs-bits and Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ we need first to introduce the following lemma.

Lemma 3.9 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp. Then for any $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$, $\beta_{\mathcal{A}} \subseteq \ell_i(\beta_{\mathcal{A}}) \subseteq \ell_i\left(\ \ell_j(\beta_{\mathcal{A}})\right)$.

Proof. Obvious. ■

Theorem 3.10 If $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is a PT_0-fs-bits, then $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is also PT_0-Čfs bicsp.

Proof: Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. Since $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is a PT_0-fs-bits, then $x_t^s \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(y_k^{s'})$ or $y_k^{s'} \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(x_t^s)$, for $i,j = 1\ or\ 2$. By Theorem 2.14 and Lamma 3.9, we get, $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ or $y_k^{s'} \tilde{\notin} \ell_i(x_t^s)$. This implies $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_0-Čfs bicsp.■

The converse Theorem 3.10 is not true, as we shown in the following example.

Example 3.11 Let $\mathcal{M} = \{x,y\}$, $\mathcal{D} = \{s_1,s_2\}$ and. Define fuzzy soft closure operators $\ell_1,\ell_2: FS(\mathcal{M},\mathcal{E}) \to FS(\mathcal{M},\mathcal{E})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ \{(s_1,x_1),(s_2,y_1)\} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_1}, \\ \{(s_1,y_1),(s_2,x_1)\} & if\ \beta_{\mathcal{A}} \subseteq y_1^{s_1}, \\ \{(s_1,x_1),(s_2,x_1)\} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_2}, \\ \{(s_1,y_1),(s_2,y_1)\} & if\ \beta_{\mathcal{A}} \subseteq y_1^{s_2}, \\ \{(s_1,x_1),(s_2,x_1 \vee y_1)\} & if\ \beta_{\mathcal{A}} \subseteq \{(s_1,x_1),(s_2,x_1)\};\ \beta_{\mathcal{A}} \notin \{x_t^{s_i}, i = 1,2, t \in I_0\}, \\ \{(s_1,x_1 \vee y_1),(s_2,y_1)\} & if\ \beta_{\mathcal{A}} \subseteq \{(s_1,x_1),(s_2,y_1)\};\ \beta_{\mathcal{A}} \notin \{x_t^{s_1}, y_k^{s_2};\ t,k \in I_0\}, \\ \{(s_1,x_1 \vee y_1),(s_2,x_1)\} & if\ \beta_{\mathcal{A}} \subseteq \{(s_1,y_1),(s_2,x_1)\};\ \beta_{\mathcal{A}} \notin \{x_t^{s_2}, y_k^{s_1};\ t,k \in I_0\}, \\ \{(s_1,y_1),(s_2,x_1 \vee y_1)\} & if\ \beta_{\mathcal{A}} \subseteq \{(s_1,y_1),(s_2,y_1)\};\ \beta_{\mathcal{A}} \notin \{y_k^{s_i}; i = 1,2, k \in I_0\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And $\ell_2(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M},\mathcal{E})$. Then, $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$is PT_0-Čfs bicsp. But $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is not PT_0-fs-bits, because for any two distinct fs-points x_t^s and $y_k^{s'}$, where $s,s' \in \{s_1,s_2\}$ and $t,k \in I_0$ we have $x_t^s \tilde{\in} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(y_k^{s'}) = \tilde{1}_{\mathcal{D}}$ and $y_k^{s'} \tilde{\in} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(x_t^s) = \tilde{1}_{\mathcal{D}}$ for $i,j = 1\ or\ 2$.

Now, we introduce the notion of pairwise T_1-Čfs bicsp's.

Definition 3.12 A Čfs bicsp $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is said to be pairwise T_1-Čfs bicsp (PT_1-Čfs bicsp), if for every two distinct fs-points x_t^s and $y_k^{s'}$ we have $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ and $y_k^{s'} \tilde{\notin} \ell_j(x_t^s)$.

Next, we give an example of a PT_1-Čfs bicsp.

Example 3.13 Let $\mathcal{M} = \{x,y\}$, $\mathcal{D} = \{s_1,s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M},\mathcal{D})$ such that $\Gamma = \{(s_1, x_{t_1} \vee y_{k_1}),(s_2,x_{t_2} \vee y_{k_2});\quad t_1,t_2,k_1,k_2 \in I_0\}$. Define fuzzy soft closure operators $\ell_1,\ell_2: FS(\mathcal{M},\mathcal{D}) \to FS(\mathcal{M},\mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ t_1 \in I_0\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ k_1 \in I_0\}, \\ y_{0.8}^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ 0 < k_2 < 0.8\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ 0.8 \le k_2 \le 1\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if\ \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ t_1 \in I_0\}, \\ x_{0.9}^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ 0 < t_2 < 0.9\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ 0.9 \le t_2 \le 1\}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ k_1 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_2(x_{t_1}^{s_1}) \cup \ell_2(y_{k_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_1-Čfs bicsp. Since for any two distinct fs-points $x_t^{s_1}$ and $y_k^{s_2}$ we have $[x_t^{s_1} \tilde{\notin} \ell_1(y_k^{s_2})$ and $y_k^{s_2} \tilde{\notin} \ell_2(x_t^{s_1})]$ and $[x_t^{s_1} \tilde{\notin} \ell_2(y_k^{s_2})$ and $y_k^{s_2} \tilde{\notin} \ell_1(x_t^{s_1})]$.

In the next, we give the relation between PT_1 and PT_0-Čfs bicsp's.

Proposition 3.14 Every PT_1-Čfs bicsp is PT_0-Čfs bicsp.
Proof: Follows directly from the definition of PT_1- Čfs bicsp. ■

The converse of Proposition 3.14 is not true, as seen in the following example.

Example 3.15 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})$ such that $\Gamma = \{(s_1, x_{t_1}), (s_2, x_{t_2} \vee y_{k_2});\ \ t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 \colon FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\,, \\ x_{0.7}^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1}, 0 < t_1 < 0.7\}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1}, 0.7 \le t_1 \le 1\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_1(x_{t_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ t_1 \in I_0\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_{0.4}^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2}, 0 < k_2 < 0.4\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2}, 0.4 \le k_2 \le 1\}, \\ \{\ell_2(x_{t_1}^{s_1}), \ell_2(x_{t_2}^{s_2}) \cup \ell_2(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_0-Čfs bicsp. However, it is not PT_1-Čfs bicsp because there exist $x_{0.5}^{s_1}$ and $y_{0.2}^{s_1}$ are two distinct fs-points such that $x_{0.5}^{s_1} \tilde{\in} \ell_1(y_{0.2}^{s_1})$=$\tilde{1}_{\mathcal{D}}$.

Theorem 3.16 If every fs-point in a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a closed fss, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_1-Čfs bicsp.

Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. From hypothesis, we have $\ell_i(x_t^s)$= x_t^s and $\ell_i(y_k^{s'}) = y_k^{s'}$ for $i = 1, 2$. This implies $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ and $y_k^{s'} \tilde{\notin} \ell_j(x_t^s)$. Thus, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_1-Čfs bicsp. ■

The converse of the above theorem is not true in general as we have seen in the following example.

Example 3.17 In Example 3.13, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_1-Čfs bicsp, yet there exists fs-point $y_{0.6}^{h_2}$ such that $\ell_1(y_{0.6}^{s_2}) = y_{0.8}^{s_2}$ and $x_{0.3}^{s_2}$ such that $\ell_2(x_{0.3}^{s_2}) = x_{0.9}^{s_2}$.

Theorem 3.18 Every Čfs bi-csubsp of PT_1-Čfs bicsp is PT_1- Čfs bi-csubsp.

Proof: Similar to the proof of Theorem 3.7. ■

In the next definition the concept of PT_1 in the induced fs-bits is given.

Definition 3.19 The induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise T_1-fs-bits (PT_1-fs-bits, for short) if for every two distinct fs-points x_t^s and $y_k^{s'}$, we have $x_t^s \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(y_k^{s'})$ and $y_k^{s'} \tilde{\notin} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(x_t^s)$, for $i, j = 1\ or\ 2, i \neq j$.

The next theorem gives the relationship between the induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ which is PT_1-fs-bits and Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$.

Theorem 3.20 If $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is a PT_1-fs-bits, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is also PT_1- Čfs bicsp.

Proof: Similar to the proof of Theorem 3.19. ■

Example 3.21 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})$ such that Γ =$\{(s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2});\ t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \rightarrow FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{x_{t_1}^{s_1};\ t_1 \in I_0\}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{x_{t_2}^{s_2};\ t_2 \in I_0\}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \in \{y_{k_1}^{s_1};\ k_1 \in I_0\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ k_2 \in I_0\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

And $\ell_2(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_1-Čfs bicsp. But $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is not PT_1-fs-bits, because for any two distinct fs-points x_t^s and $y_k^{s'}$, where $s, s' \in \{s_1, s_2\}$ and $t, k \in I_0$ we have $x_t^s \tilde{\in} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(y_k^{s'}) = \tilde{1}_{\mathcal{D}}$ and $y_k^{s'} \tilde{\in} \mathcal{T}_{\ell_i}\mathcal{T}_{\ell_j}\text{-}cl(x_t^s) = \tilde{1}_{\mathcal{D}}$ for $i, j = 1$ *or* 2.

Proposition 3.22 If $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_i-Čfs bicsp and ℓ_1^*, ℓ_2^* are Č-fsco's on $\mathcal{U}$ such that ℓ_1^* and ℓ_2^* is finer than ℓ_1 and ℓ_2 respectively, then $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is PT_i-Čfs bicsp for $i = 1,2$.
Proof: We prove the proposition when $i = 1$ and the proof is similar for $i = 0$. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$. From hypothesis $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_1-Čfs bicsp, this yield $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ and $y_k^{s'} \tilde{\notin} \ell_j(x_t^s)$, for $i, j = 1, 2, i \neq j$. Since ℓ_1^* and ℓ_2^* is finer than ℓ_1 and ℓ_2 respectively, that means $\ell_1^*(\beta_{\mathcal{A}}) \subseteq \ell_1(\beta_{\mathcal{A}})$ and $\ell_2^*(\beta_{\mathcal{A}}) \subseteq \ell_2(\beta_{\mathcal{A}})$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. This implies, $x_t^s \tilde{\notin} \ell_i^*(y_k^{s'})$ and $y_k^{s'} \tilde{\notin} \ell_j^*(x_t^s)$. Hence, $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is PT_1-Čfs bicsp. ■

4. Pairwise T_2- Čech fuzzy soft bi-closure spaces.

Definition 4.1 A Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise T_2-Čfs bicsp (PT_2-Čfs bicsp), if for every two distinct fs-points x_t^s and $y_k^{s'}$, there exist disjoint ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$.

Next, we give an example of PT_2-Čfs bicsp.
Example 4.2 Let $\mathcal{M}$={x, y, z}, $\mathcal{D}$={s}. Define fuzzy soft closure operators $\ell_1, \ell_2: FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ \{(s, x_{t+0.2} \vee y_{t+0.2})\} & if\ \beta_{\mathcal{A}} \in \{x_t^s;\ 0 < t < 0.8\}, \\ \{(s, x_1 \vee y_1)\} & if\ \beta_{\mathcal{A}} \in \{x_t^s;\ 0.8 \leq t \leq 1\}, \\ \{(s, y_{k+0.2})\} & if\ \beta_{\mathcal{A}} \in \{y_k^s;\ 0 < k < 0.8\}, \\ \{(s, y_1)\} & if\ \beta_{\mathcal{A}} \in \{y_k^s;\ 0.8 \leq k \leq 1\}, \\ \{(s, z_{r+0.2})\} & if\ \beta_{\mathcal{A}} \in \{z_r^s;\ 0 < r < 0.8\}, \\ \{(s, z_1)\} & if\ \beta_{\mathcal{A}} \in \{z_r^s; 0.8 \leq r \leq 1\}, \\ \ell_1(x_t^s) \cup \ell_1(y_k^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k); t, k \in I_0\}, \\ \ell_1(x_t^s) \cup \ell_1(z_r^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee z_r); t, r \in I_0\}, \\ \ell_1(y_k^s) \cup \ell_1(z_r^s) & if\ \beta_{\mathcal{A}} \in \{(s, y_k \vee z_r); k, r \in I_0\}, \\ \ell_1(x_t^s) \cup \ell_1(y_k^s) \cup \ell_1(z_r^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k \vee z_r); t, k, r \in I_0\}. \end{cases}$$

And $\ell_2(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ for all $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is Čfs bicsp which is PT_2-Čfs bicsp. To explain that we have three cases for distinct fs-points as follows:

Case (1): x_t^s, y_k^s are distinct fs-points, there exist ℓ_1-open fss $\beta_{\mathcal{A}}$={(s, x_1)} and there exist ℓ_2-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, y_1)$} such that $x_t^s \widetilde{\in} \beta_{\mathcal{A}}$ and $y_k^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. And there exist ℓ_2-open fss $\beta_{\mathcal{A}}$={(s, x_1)} and there exist ℓ_1-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, y_1)$} such that $x_t^s \widetilde{\in} \beta_{\mathcal{A}}$ and $y_k^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$.
Case (2): x_t^s, z_r^s are distinct fs-points, there exist ℓ_1-open fss $\beta_{\mathcal{A}}$={(s, x_1)} and there exist ℓ_2-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, z_1)$} such that $x_t^s \widetilde{\in} \beta_{\mathcal{A}}$ and $z_r^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. And there exist ℓ_2-open fss $\beta_{\mathcal{A}}$={(s, x_1)} and there exist ℓ_1-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, z_1)$} such that $x_t^s \widetilde{\in} \beta_{\mathcal{A}}$ and $z_r^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$.
Case (3): x_t^s, z_r^s are distinct fs-points, there exist ℓ_1-open fss $\beta_{\mathcal{A}}$={(s, y_1)} and there exist ℓ_2-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, z_1)$} such that $y_k^s \widetilde{\in} \beta_{\mathcal{A}}$ and $z_r^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. And there exist ℓ_2-open fss $\beta_{\mathcal{A}}$={(s, y_1)} and there exist ℓ_1-open fss $\mathcal{E}_{\mathcal{B}}$={$(s, z_1)$} such that $y_k^s \widetilde{\in} \beta_{\mathcal{A}}$ and $z_r^s \widetilde{\in} \mathcal{E}_{\mathcal{B}}$, and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$.

Remark 4.3 If $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_2-Čfs bicsp, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ need not to be PT_1-Čfs bicsp. To see that, in Example 4.2, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is PT_2-Čfs bicsp but it is not PT_1-Čfs bicsp. Since there exist $x_{0.3}^s$ and $y_{0.4}^s$ are distinct fs-points, and $y_{0.4}^s \widetilde{\in} \ell_1(x_{0.3}^s) = \{(s, x_{0.5} \vee y_{0.5})\}$.

In order to study the hereditary property in PT_2-Čfs bicsp's, we need first to give the following two lemmas.

Lemma 4.4 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$.Then for any $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$, we have $\beta_{\mathcal{A}}^c \cap \widetilde{\mathcal{H}}_{\mathcal{D}} = \widetilde{\mathcal{H}}_{\mathcal{D}} - \left(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)$

Proof. Let $\beta_{\mathcal{A}} \in FS(\mathcal{M}, \mathcal{D})$. From the definition of $\widetilde{\mathcal{H}}_{\mathcal{D}}$, it is clear that for any $s \in \mathcal{D}$ and $x \in \mathcal{H}$, $\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}} = \beta_{\mathcal{A}}$. Now, for any $s \in \mathcal{D}$ and $x \in \mathcal{H}$.
$\left[\widetilde{\mathcal{H}}_{\mathcal{D}} - \left(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)\right](s)$=$(\widetilde{\mathcal{H}}_{\mathcal{D}} - \beta_{\mathcal{A}})\ (s) \in I^{\mathcal{M}}$. Now,

$$(\widetilde{\mathcal{H}}_{\mathcal{D}} - \beta_{\mathcal{A}})\ (s)\ (x) = (\widetilde{\mathcal{H}}_{\mathcal{D}}(s) - \beta_{\mathcal{A}}(s))(x) = ((\beta_{\mathcal{A}}^c(s) \cap \widetilde{\mathcal{H}}_{\mathcal{D}}(s))(x) = (\widetilde{\mathcal{H}}_{\mathcal{D}} - \beta_{\mathcal{A}}) \cap \widetilde{\mathcal{H}}_{\mathcal{D}}.$$ ■

Lemma 4.5 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a Čfs bicsp and $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. If $\beta_{\mathcal{A}}$ is an $\mathcal{L}_i$-open fss of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$, then $\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_S$ is an $\ell_{i_{\mathcal{H}}}$-open fss in $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$, for $i = 1$ *or* 2.

Proof. We proof the lemma when $i = 1$ and the prove is similar for $i = 2$. Let $\beta_{\mathcal{A}}$ be an ℓ_1-open fss in $((\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then, $\beta_{\mathcal{A}}^c$ is a ℓ_1-closed fss in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. To prove $\widetilde{\mathcal{H}}_{\mathcal{D}} \cap \beta_{\mathcal{A}}$ is an $\ell_{1_{\mathcal{H}}}$-open in $\mathcal{H}$. That means we must prove $\ell_{1_{\mathcal{H}}}(\widetilde{\mathcal{H}}_{\mathcal{D}} - (\widetilde{\mathcal{H}}_{\mathcal{D}} \cap \beta_{\mathcal{A}})) = \widetilde{\mathcal{H}}_{\mathcal{D}} - (\widetilde{\mathcal{H}}_{\mathcal{D}} \cap \beta_{\mathcal{A}})$. Now,

$$\begin{aligned}
\ell_{1_{\mathcal{H}}}\left(\widetilde{\mathcal{H}}_{\mathcal{D}} - \left(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)\right) &= \widetilde{\mathcal{H}}_{\mathcal{D}} \cap \ell_1(\widetilde{\mathcal{H}}_{\mathcal{D}} - (\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}})) && \text{(By definition 2.9)} \\
&\sqsubseteq \widetilde{\mathcal{H}}_{\mathcal{D}} \cap \ell_1(\beta_{\mathcal{A}}^c) && (\text{By } \widetilde{\mathcal{H}}_{\mathcal{D}} - (\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqsubseteq \beta_{\mathcal{A}}^c) \\
&= \widetilde{\mathcal{H}}_{\mathcal{D}} \cap \beta_{\mathcal{A}}^c && (\text{By } \beta_{\mathcal{A}}^c \text{ is } \ell_1\text{-closed}) \\
&= \widetilde{\mathcal{H}}_{\mathcal{D}} - (\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_S) && \text{(By Lemma (4.4))}.
\end{aligned}$$

On the other hand, $\widetilde{\mathcal{H}}_{\mathcal{D}} - (\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqsubseteq \ell_{1_{\mathcal{H}}}\left(\widetilde{\mathcal{H}}_{\mathcal{D}} - \left(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)\right)$. Then, $\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ is an ℓ_1-open fss in $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$. ■

Now, we are ready to discuss the hereditary property in PT_2- Čfs bicsp's.

Theorem 4.6 Let $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ be a PT_2- Čfs bicsp and let $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. Then, $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ is a PT_2-Čfs bi-csubsp of $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$.

Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$. Then, x_t^s and $y_k^{s'}$ are distinct fs-point in $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. Since $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is a PT_2-Čfs bicsp, there exist two disjoint open $fss's$ $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ in ℓ_i and ℓ_j respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$. Consequently, $x_t^s \tilde{\in} \beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) \cap (\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) = \tilde{0}_{\mathcal{D}}$. By Lemma 4.5, $\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ are open $fss's$ in $\ell_{i_{\mathcal{H}}}$ and $\ell_{j_{\mathcal{H}}}$ respectively. Hence $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ is a PT_2-Čfs bi-csubsp of $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. ■

Definition 4.7 The induced fs-bits $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ of a Čfs bicsp $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is said to be PT_2-fs-bits, if for every two distinct fs-points x_t^s and $y_k^{s'}$, there exist an open $fss's$ $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ in τ_{ℓ_i} and τ_{ℓ_j} respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}=\tilde{0}_{\mathcal{D}}$.

Theorem 4.8 The induced fs-bits $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is PT_2- fs-bits of $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ if and only if $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_2-Čfs bicsp.

Proof. Suppose $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is PT_2-fs-bits and let x_t^s and $y_k^{s'}$ be two distinct fs-points in $\mathcal{M}$. Since $(\mathcal{M},\mathcal{T}_{\ell_1},\mathcal{T}_{\ell_2},\mathcal{D})$ is PT_2-fs-bits, there exist $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ open $fss's$ in τ_{ℓ_i} and τ_{ℓ_j} respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}=\tilde{0}_{\mathcal{D}}$. Since $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ are open $fss's$ in τ_{ℓ_i} and τ_{ℓ_j} respectively, then τ_{ℓ_i}-$int(\beta_{\mathcal{A}})$=$\beta_{\mathcal{A}}$ and τ_{ℓ_j}-$int(\mathcal{E}_{\mathcal{B}})$= $\mathcal{E}_{\mathcal{B}}$, for $i,j = 1,2, i \neq j$ From proposition 2.15, we have $Int_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ and $Int_j(\mathcal{E}_{\mathcal{B}}) = \mathcal{E}_{\mathcal{B}}$. That means there exist $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ are open $fss's$ in ℓ_i and ℓ_j respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\beta_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. Hence, $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_2-Čfs bicsp.

Conversely, similar to the first direction. ■

Lemma 4.9 Let $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ and $(\mathcal{M},\ell_1^*,\ell_2^*,\mathcal{D})$ be Čfs bicsp's, For any $\beta_{\mathcal{A}} \in FS(\mathcal{M},\mathcal{D})$, if $\ell_i(\beta_{\mathcal{A}}) \subseteq \ell_i^*(\beta_{\mathcal{A}})$, then $Int_i^*(\beta_{\mathcal{A}}) \subseteq Int_i(\beta_{\mathcal{A}})$, for $i = 1$ or 2.

Proof. We proof the lemma when $i = 1$ and the proof is similar for $i = 2$. Let $\beta_{\mathcal{A}} \in FS(\mathcal{M},\mathcal{D})$. From hypothesis, $\ell_1(\beta_{\mathcal{A}}^c) \subseteq \ell_1^*(\beta_{\mathcal{A}}^c)$, implies $(\ell_1^*(\beta_{\mathcal{A}}^c))^c \subseteq (\ell_1(\beta_{\mathcal{A}}^c))^c$. Therefore, $Int_1^*(\beta_{\mathcal{A}}) \subseteq Int_1(\beta_{\mathcal{A}})$. ■

Proposition 4.10 If $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_2-Čfs bicsp and ℓ_1^* and ℓ_2^* are Č-fsco's on $\mathcal{U}$ such that ℓ_1^*,ℓ_2^* is finer than ℓ_1,ℓ_2 respectivly, then $(\mathcal{M},\ell_1^*,\ell_2^*,\mathcal{D})$ is PT_2-Čfs bicsp, for $i = 1,2$.

Proof. Let x_t^s and $y_k^{s'}$be any two distinct fs-points in $\mathcal{M}$. Since $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_2-Čfs bicsp, then there exist two disjoint open $fss's$ $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ in ℓ_i and ℓ_j respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$. That is mean $Int_i(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ and $Int_j(\mathcal{E}_{\mathcal{B}}) = \mathcal{E}_{\mathcal{B}}$. Since ℓ_i^* is finer than ℓ_i, this yields by Lemma 4.9, $\beta_{\mathcal{A}} \subseteq Int_i(\beta_{\mathcal{A}}) \subseteq Int_i^*(\beta_{\mathcal{A}}) \subseteq \beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}} \sqsubseteq Int_j(\mathcal{E}_{\mathcal{B}}) \subseteq Int_j^*(\mathcal{E}_{\mathcal{B}}) \subseteq \mathcal{E}_{\mathcal{B}}$. Therefore, there exist two disjoint open $fss's$ $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ in ℓ_i^* and ℓ_j^* respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$. Hence, $(\mathcal{M},\ell_1^*,\ell_2^*,\mathcal{D})$is PT_2-Čfs bicsp. ■

Definition 4.11 A Čfs bicsp $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is said to be pairwise semi T_2-Čfs bicsp (P- semi T_2-Čfs bicsp), if for every two distinct fs-points x_t^s and $y_k^{s'}$, either there exists an ℓ_i-open fss $\beta_{\mathcal{A}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$ or there exists an ℓ_i-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$, for $i = 1$ or 2.

Example 4.12 Let $\mathcal{M}$={x, y}, $\mathcal{D}$={s}. Define fuzzy soft closure operators $\ell_1, \ell_2: FS(\mathcal{M},\mathcal{D}) \to FS(\mathcal{M},\mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ y_1^s & if\ \beta_{\mathcal{A}} \subseteq y_1^s, \\ x_{t+0.3}^s & if\ \beta_{\mathcal{A}} \in \{x_t^s, 0 < t < 0.7\}, \\ x_1^s & if\ \beta_{\mathcal{A}} \in \{x_t^s\ , 0.7 \leq t \leq 1\}, \\ \ell_1(x_t^s) \cup \ell_1(y_k^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k)\ ;\ t,k \in I_0\}. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ y_1^s & if\ \beta_{\mathcal{A}} \subseteq y_1^s, \\ x_1^s & if\ \beta_{\mathcal{A}} \subseteq x_1^s, \\ \ell_2(x_t^s) \cup \ell_2(y_k^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k)\ ;\ t,k \in I_0\}. \end{cases}$$

Then, $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is P-semi T_2-Čfs bicsp. Since for any x_t^s, y_k^s are distinct fs-points, there exists an ℓ_1-open fuzzy soft set $\beta_{\mathcal{A}} = x_1^s$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^s \tilde{\notin} \ell_1(\beta_{\mathcal{A}})$= x_t^s.

Proposition 4.13 Every P-semi T_2-Čfs bicsp is PT_0-Čfs bicsp.

Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. From $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is P-semi T_2-Čfs bicsp, there exists an ℓ_i-open fss $\beta_{\mathcal{A}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$ or there exists an ℓ_i-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$ for $i = 1$ or 2. This implies $x_t^s \tilde{\in} \ell_i(x_t^s) \subseteq \ell_i(\beta_{\mathcal{A}})$ and $y_k^{s'} \tilde{\notin} \ell_i(x_t^s)$ or $y_k^{s'} \tilde{\in} \ell_i(y_k^{s'}) \subseteq \ell_i(\mathcal{E}_{\mathcal{B}})$ and $x_t^s \tilde{\notin} \ell_i(y_k^{s'})$ for $i = 1$ or 2. Hence, $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is a PT_0-Čfs bicsp. ■

The converse of above proposition is not true as seen in the next example.

Example 4.14 In Example 3.4. $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is PT_0-Čfs bicsp. But $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is not semi T_2-Čfs bicsp, since there exist $x_{0.5}^{s_1}$ and $y_{0.5}^{s_2}$ are distinct fs-points such that for any ℓ_1-open fss $\beta_{\mathcal{A}}$, we have $x_{0.5}^{s_1} \tilde{\in} \beta_{\mathcal{A}}$ and $y_{0.5}^{s_2} \tilde{\in} \ell_1(\beta_{\mathcal{A}})$ and for any ℓ_1-open fss $\mathcal{E}_{\mathcal{B}}$, we have $y_{0.5}^{s_2} \tilde{\in} \mathcal{E}_{\mathcal{B}}$, $x_{0.5}^{s_1} \tilde{\in} \ell_1(\mathcal{E}_{\mathcal{B}})$.

Theorem 4.15 Let $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ be a pairwise semi T_2-Čfs bicsp and let $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ be a closed Čfs bi-csubsp of $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. Then, $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ is a P-semi T_2-Čfs bi-csubsp.

Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$. Then, x_t^s and $y_k^{s'}$ are distinct fs-points in $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$. Since $(\mathcal{M},\ell_1,\ell_2,\mathcal{D})$ is P-semi T_2-Čfs bicsp, then either there exists an ℓ_i-open fss $\beta_{\mathcal{A}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$ or there exists an ℓ_i-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$, $i = 1$ or 2. Now, if $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$, then by Lemma 4.5, $x_t^s \tilde{\in} \beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_S$ which is $\mathcal{L}_{i_{\mathcal{V}}}$-open fss in $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$. That is mean we find an $\ell_{i_{\mathcal{H}}}$-open fss $\beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_S$ in $(\mathcal{H},\ell_{1_{\mathcal{H}}},\ell_{2_{\mathcal{H}}},\mathcal{D})$ contains x_t^s, for $i = 1$ or 2. To complete the proof, we must show $y_k^{s'} \tilde{\notin} \ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_{\mathcal{D}})$. It is clear that from the definition of $\ell_{i_{\mathcal{H}}}$ we have, $\ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_{\mathcal{D}})$= $\tilde{\mathcal{H}}_S \cap \ell_i(\beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_{\mathcal{D}}) \subseteq \tilde{\mathcal{H}}_S \cap \ell_i(\beta_{\mathcal{A}}) \cap \ell_i(\tilde{\mathcal{H}}_{\mathcal{D}}) = \tilde{\mathcal{H}}_{\mathcal{D}} \cap \ell_i(\beta_{\mathcal{A}})$. And since $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$, then we have

$y_k^{s'} \tilde{\notin} \ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}} \cap \tilde{\mathcal{H}}_{\mathcal{D}})$. Similarly, if there exists an $\ell_{i_{\mathcal{H}}}$-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$. Hence, $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is a P-semi T_2-Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. ■

Definition 4.16 The induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be P-semi T_2-fs-bits, if for every distinct fs-point x_t^s and $y_k^{s'}$, either there exists a τ_{ℓ_i}-open $fss's$ $\beta_{\mathcal{A}}$ in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \tau_{\ell_i}\text{-}cl(\beta_{\mathcal{A}})$, or there exists a τ_{ℓ_i}-open $fss's$ $\mathcal{E}_{\mathcal{B}}$ in $(\mathcal{U}, \tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}, S)$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \tau_{\ell_i}\text{-}cl(\mathcal{E}_{\mathcal{B}})$, for $i = 1$ or 2.

Theorem 4.17 If the induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is a P-semi T_2-fs-bits, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is also P-semi T_2-Čfs bicsp.

Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. From hypothesis, either there exists a τ_{ℓ_i}-open fss $\beta_{\mathcal{A}}$ in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \tau_{\ell_i}\text{-}cl(\beta_{\mathcal{A}})$, or there exists a τ_{ℓ_i}-open fss $\mathcal{E}_{\mathcal{B}}$ in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \tau_{\ell_i}\text{-}cl(\mathcal{E}_{\mathcal{B}})$. By proposition 2.20, we have the following, either there exists an ℓ_i-open fss $\beta_{\mathcal{A}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$ or there exists an ℓ_i-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$, $i = 1$ or 2. Thus, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a P-semi T_2- Čfs bicsp . ■

Proposition 4.18 If $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a P-semi T_2-Čfs bicsp and ℓ_1^* and ℓ_2^* are Č-fsco's on $\mathcal{U}$ such that ℓ_1^*, ℓ_2^* is finer than ℓ_1, ℓ_2 respectivly, then $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is P-semi T_2-Čfs bicsp.

Proof. Let x_t^s and $y_k^{s'}$be any two distinct fs-points in $\mathcal{M}$. Since $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is P-semi T_2-Čfs bicsp, then either there exist an ℓ_i-open fss $\beta_{\mathcal{A}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$, or there exist an ℓ_i-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mu_B$ and $x_t^s \tilde{\notin} \ell_i(\mathcal{E}_{\mathcal{B}})$, $i = 1, 2$. Suppose, if there exists an ℓ_1-open fss $\beta_{\mathcal{A}}$ in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ that is mean $Int_1(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$. Since ℓ_1^* finer than ℓ_1, then by Lemma 4.9, we have $Int_1(\beta_{\mathcal{A}}) \subseteq Int_1^*(\beta_{\mathcal{A}})$ and that is mean there exists an ℓ_1^*-open fss $\beta_{\mathcal{A}}$ in $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$. On the other hand since $y_k^{s'} \tilde{\notin} \ell_1(\beta_{\mathcal{A}})$ then $y_k^{s'} \tilde{\notin} \ell_1^*(\beta_{\mathcal{A}})$. This implies $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is P-semi T_2-Čfs bicsp. Similarly, if there exists an ℓ_1^*-open fss $\mathcal{E}_{\mathcal{B}}$ such that $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \ell_1^*(\mathcal{E}_{\mathcal{B}})$. By the same way, we can prove the proposition when $i = 2$. ■

Definition 4.19 A Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise pseudo T_2- Čfs bicsp (P-pseudo T_2- Čfs bicsp, for short), if for every two distinct fs-points x_t^s and $y_k^{s'}$, there exist ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}, y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}, x_t^s \tilde{\notin} \ell_j(\mathcal{E}_{\mathcal{B}})$.

An example of pairwise pseudo T_2- Čfs bicsp in given in the following.

Example 4.20 Let $\mathcal{M}$=$\{x, y\}$, $\mathcal{D}$=$\{s\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{(s, x_{1-t} \vee y_1); 0 \le t < 1\}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{(s, x_1 \vee y_{1-k});\ 0 \le k < 1\}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{(s, x_t); 0 < t \le 1\}, \\ \beta_{\mathcal{A}} & if\ \beta_{\mathcal{A}} \in \{(s, y_k); 0 < k \le 1\}, \\ \ell_1(x_t^s) \cup \ell_1(y_k^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k)\ ;\ t, k \in I_0\}. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ x_1^s & if\ \beta_{\mathcal{A}} \subseteq x_1^s, \\ y_1^s & if\ \beta_{\mathcal{A}} \subseteq y_1^s, \\ \ell_2(x_t^s) \cup \ell_2(y_k^s) & if\ \beta_{\mathcal{A}} \in \{(s, x_t \vee y_k)\ ;\ t, k \in I_0\}. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a P-pseudo T_2- Čfs bicsp, Since for any x_t^s, y_k^s are distinct fs-points there exist ℓ_1-open fss $\beta_{\mathcal{A}} = \{(s, x_t)\}$ and ℓ_2-open fss $\mathcal{E}_{\mathcal{B}} = \{(s, y_1)\}$ such that $x_t^s \,\tilde{\in}\, \beta_{\mathcal{A}} = \{(s, x_t)\}$, $y_k^s \,\tilde{\notin}\, \ell_1(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ and $y_k^s \,\tilde{\in}\, \mathcal{E}_{\mathcal{B}} = \{(s, y_1)\}$, $x_t^s \,\tilde{\notin}\, \ell_2(\mathcal{E}_{\mathcal{B}}) = \mathcal{E}_{\mathcal{B}}$, and there exist ℓ_2-open fss $\beta_{\mathcal{A}} = \{(s, x_1)\}$ and ℓ_1-open fss $\mathcal{E}_{\mathcal{B}} = \{(s, y_k)\}$ such that $x_t^s \,\tilde{\in}\, \beta_{\mathcal{A}} = \{(s, x_1)\}$, $y_k^s \,\tilde{\notin}\, \ell_2(\beta_{\mathcal{A}}) = \beta_{\mathcal{A}}$ and $y_k^s \,\tilde{\in}\, \mathcal{E}_{\mathcal{B}} = \{(s, y_k)\}$, $x_t^s \,\tilde{\notin}\, \ell_1(\mathcal{E}_{\mathcal{B}}) = \mathcal{E}_{\mathcal{B}}$.

Proposition 4.21 Every P-pseudo T_2- Čfs bicsp is P-semi T_2-Čfs bicsp.
Proof. Follows directly from the definition of P-pseudo T_2- Čfs bicsp. ■

Proposition 4.22 Every P-pseudo T_2- Čfs bicsp is PT_1-Čfs bicsp.
Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $\mathcal{M}$. From hypothesis $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is P-pseudo T_2-Čfs bicsp, then there exist ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$ such that $x_t^s \,\tilde{\in}\, \beta_{\mathcal{A}}$, $y_k^{s'} \,\tilde{\notin}\, \ell_i(\beta_{\mathcal{A}})$ and $y_k^{s'} \,\tilde{\in}\, \mathcal{E}_{\mathcal{B}}$, $x_t^s \,\tilde{\notin}\, \ell_j(\mathcal{E}_{\mathcal{B}})$. Since $x_t^s \,\tilde{\in}\, \beta_{\mathcal{A}}$ and $y_k^{s'} \,\tilde{\in}\, \mathcal{E}_{\mathcal{B}}$, then it follows $x_t^s \,\tilde{\in}\, \ell_i(x_t^{s_1}) \subseteq \ell_i(\beta_{\mathcal{A}})$ and $y_k^{s'} \,\tilde{\in}\, \ell_j(y_k^{s'}) \subseteq \ell_j(\mathcal{E}_{\mathcal{B}})$. And since $y_k^{s'} \,\tilde{\notin}\, \ell_i(\beta_{\mathcal{A}})$ and $x_t^s \,\tilde{\notin}\, \ell_i(\mathcal{E}_{\mathcal{B}})$, then $y_k^{s'} \,\tilde{\notin}\, \ell_i(x_t^s)$ and $x_t^s \,\tilde{\notin}\, \ell_j(y_k^{s'})$. Hence $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_1-Čfs bicsp. ■

The next example shows that converse of the above proposition is not true.

Example 4.23 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $\Gamma \subseteq FS(\mathcal{M}, \mathcal{D})$ such that $\Gamma = \{(s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2});\ t_1, t_2, k_1, k_2 \in I_0\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : FS(\mathcal{M}, \mathcal{D}) \to FS(\mathcal{M}, \mathcal{D})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_1}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_2}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \subseteq y_1^{s_1}, \\ y_{0.8}^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ 0 < k_2 < 0.8\}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \in \{y_{k_2}^{s_2};\ 0.8 \le k_2 \le 1\}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

And

$$\ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}, \\ x_1^{s_1} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_1}, \\ x_1^{s_2} & if\ \beta_{\mathcal{A}} \subseteq x_1^{s_2}, \\ y_1^{s_1} & if\ \beta_{\mathcal{A}} \subseteq y_1^{s_1}, \\ y_1^{s_2} & if\ \beta_{\mathcal{A}} \subseteq y_1^{s_2}, \\ \{\ell_1(x_{t_1}^{s_1}) \cup \ell_1(y_{k_1}^{s_1}), \ell_1(x_{t_2}^{s_2}) \cup \ell_1(y_{k_2}^{s_2})\} & if \beta_{\mathcal{A}} \in \Gamma. \end{cases}$$

Then, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a PT_1-Čfs bicsp. But $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not P-pseudo T_2-Čfs bicsp. To show that consider $x_{0.5}^{s_1}$ and $y_{0,7}^{s_2}$ are distinct fs-points. The open $fss's$ $\beta_{\mathcal{A}}$ such that $x_{0.5}^{s_1} \,\tilde{\in}\, \beta_{\mathcal{A}}$ are:

1. $\beta_{\mathcal{A}} = \tilde{1}_{\mathcal{D}}$, implies $y_{0,7}^{s_2} \tilde{\in} \ell_1(\tilde{1}_{\mathcal{D}})$.
2. $\beta_{\mathcal{A}} = \{(s_1, x_1 \vee y_1), (s_2, y_1)\}$, implies $y_{0,7}^{s_2} \tilde{\in} \ell_1(\beta_{\mathcal{A}})$.
3. $\beta_{\mathcal{A}} = \{(s_1, x_1), (s_2, x_1 \vee y_1)\}$, implies$y_{0,7}^{s_2} \tilde{\in} \ell_1(\beta_{\mathcal{A}})$.

Hence, for all open fss $\beta_{\mathcal{A}}$ such that $x_{0.5}^{s_1} \tilde{\in} \beta_{\mathcal{A}}$, we have $b_{0,7}^{h_2} \tilde{\in} \ell_1(\beta_{\mathcal{A}})$. Thus, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is not P-pseudo T_2-Čfs bicsp.

Theorem 4.24 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a P-pseudo T_2-Čfs bicsp and let $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a closed Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then, $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is a pairwise pseudo T_2-Čfs bi-csubsp.
Proof. Similar of Theorem 4.15. ■

Definition 4.25 The induced fs-bits $(\mathcal{M}, \tau_{\ell_1}, \tau_{\ell_2}, \mathcal{D})$ of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be P- pseudo T_2-fs-bits, if for every distinct fs-points x_t^s and $y_k^{s'}$ there exist $\mathcal{T}_{\ell_i}$-open fss $\beta_{\mathcal{A}}$ and $\mathcal{T}_{\ell_j}$-open fss $\mathcal{E}_{\mathcal{B}}$ in $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\notin} \mathcal{T}_{\ell_i}\text{-}cl(\beta_{\mathcal{A}})$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $x_t^s \tilde{\notin} \mathcal{T}_{\ell_i}\text{-}cl(\mathcal{E}_{\mathcal{B}})$.

Theorem 4.26 If the induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is a P-pseudo T_2-fs-bits, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is also P-pseudo T_2-Čfs bicsp.
Proof. Similar of Theorem 4.17. ■

Proposition 4.27 If $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a P-pseudo T_2-Čfs bicsp and ℓ_1^* and ℓ_2^* are Č-fsco's on $\mathcal{M}$ such that ℓ_1^*, ℓ_2^* is finer than ℓ_1, ℓ_2 respectivly, then $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is P-pseudo T_2-Čfs bicsp, for $i = 1,2$.
Proof. Similar of Proposition 4.18. ■

Definition 4.28 A Čfs bicsp $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be pairwise Uryshon T_2-Čfs bicsp (P-Uryshon T_2-Čfs bicsp), if for every two distinct fs-points x_t^s and $y_k^{s'}$, there exist ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})$=$\tilde{0}_{\mathcal{D}}$.

The next example explains the concept of P-Uryshon T_2-Čfs bicsp.
Example 4.29 Let $\mathcal{M}$=$\{x, y\}$, $\mathcal{D}$=$\{s\}$. Define fuzzy soft closure operators $\ell_1, \ell_2 : FS(\mathcal{M}, \mathcal{E}) \to FS(\mathcal{M}, \mathcal{E})$ as follows:

$$\ell_1(\beta_{\mathcal{A}}) = \ell_2(\beta_{\mathcal{A}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \beta_{\mathcal{A}} = \tilde{0}_{\mathcal{D}}\ , \\ x_1^s & if\ \beta_{\mathcal{A}} \subseteq x_1^s, \\ y_1^s & if\ \beta_{\mathcal{A}} \subseteq y_1^s, \\ \tilde{1}_{\mathcal{D}} & other\ wise. \end{cases}$$

Then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ P-Uryshon T_2-Čfs bicsp. Since for any x_t^s and y_k^s are distinct fs-points, there exist ℓ_i-open fss $\beta_{\mathcal{A}}$=x_1^s and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$=$y_1^s$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^s \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})$= $x_1^s \cap y_1^s$=$\tilde{0}_{\mathcal{D}}$ for $i, j = 1,2$, $i \neq j$.

Proposition 4. 30 Every P-Uryshon T_2-Čfs bicsp is P-pesudo PT_2-Čfs bicsp.
Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Since $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is P-Uryshon T_2-Čfs bicsp, there exist ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open fss $\mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})$=$\tilde{0}_{\mathcal{D}}$. This implies $x_t^s \tilde{\notin} \ell_j(\mathcal{E}_{\mathcal{B}})$ and $y_k^{s'} \tilde{\notin} \ell_i(\beta_{\mathcal{A}})$. Therefore, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is P-pesudo T_2-Čfs bicsp. ■

Proposition 4.31 Every P-Uryshon T_2-Čfs bicsp is PT_2-Čfs bicsp.
Proof. The proof follows immediately from the definition of P-Uryshon T_2-Čfs bicsp and the property $\beta_{\mathcal{A}} \subseteq \ell_i(\beta_{\mathcal{A}})$ for any fuzzy soft set $\beta_{\mathcal{A}}$. ■

The hereditary property in P-Uryshon T_2-Čfs bicsp's are studied in the next theorem.
Theorem 4.32 Let $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a P-Uryshon T_2-Čfs bicsp and let $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ be a closed Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Then $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is a P-Uryshon PT_2- Čfs bi-csubsp.
Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$. Then, x_t^s and $y_k^{s'}$ are distinct fs-points in $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. Since $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a P-Uryshon T_2-Čfs bicsp, it follows there exist ℓ_i-open fss $\beta_{\mathcal{A}}$ and ℓ_j-open $fss\, \mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})=\tilde{0}_{\mathcal{D}}$. By Lemma 4.5, $\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$ are open $fss's$ in $\ell_{i_{\mathcal{H}}}$ and $\ell_{j_{\mathcal{H}}}$ sespectivly such that $x_t^s \tilde{\in} \beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}$. Next, we must show that $\ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \cap \ell_{j_{\mathcal{H}}}(\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) = \tilde{0}_{\mathcal{D}}$. Now, from the definition of $\ell_{i_{\mathcal{H}}}$ and $\ell_{j_{\mathcal{H}}}$ we get,

$$\begin{aligned}\ell_{i_{\mathcal{H}}}(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) \cap \ell_{j_{\mathcal{H}}}(\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}}) &= [\widetilde{\mathcal{H}}_{\mathcal{D}} \cap \ell_i(\beta_{\mathcal{A}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}})] \cap [\widetilde{\mathcal{H}}_{\mathcal{D}} \cap \ell_j(\mathcal{E}_{\mathcal{B}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}})] \\ &\subseteq [\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})] \cap \widetilde{\mathcal{H}}_{\mathcal{D}} \\ &= \tilde{0}_{\mathcal{D}} \cap \widetilde{\mathcal{H}}_{\mathcal{D}} \\ &= \tilde{0}_{\mathcal{D}}.\end{aligned}$$

Therefore, $(\mathcal{H}, \ell_{1_{\mathcal{H}}}, \ell_{2_{\mathcal{H}}}, \mathcal{D})$ is a P-Uryshon T_2-Čfs bi-csubsp of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$. ■

Definition 4.33 The induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ of $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is said to be P-Uryshon T_2-fs-bits, if for every distinct fs-points x_t^s and $y_k^{s'}$, there exist τ_{ℓ_i}-open fss $\beta_{\mathcal{A}}$ and τ_{ℓ_j}-open $fss\, \mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\mathcal{T}_{\ell_i}$-$cl(\beta_{\mathcal{A}}) \cap \mathcal{T}_{\ell_j}$-$cl(\mathcal{E}_{\mathcal{B}}) = \tilde{0}_{\mathcal{D}}$.

Theorem 4.34 If the induced fs-bits $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is a P-Uryshon T_2-fs-bits, then $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is also P-Uryshon T_2-Čfs bicsp.
Proof. Let x_t^s and $y_k^{s'}$ be any two distinct fs-points in $\mathcal{M}$. Since $(\mathcal{M}, \mathcal{T}_{\ell_1}, \mathcal{T}_{\ell_2}, \mathcal{D})$ is an P-Uryshon T_2-fs-bits, then there exist $\mathcal{T}_{\ell_i}$-open fss $\beta_{\mathcal{A}}$ and τ_{ℓ_j}-open $fss\, \mathcal{E}_{\mathcal{B}}$ such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$, $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\mathcal{T}_{\ell_i}$-$cl(\beta_{\mathcal{A}}) \cap \mathcal{T}_{\ell_j}$-$cl(\mathcal{E}_{\mathcal{B}}) = \tilde{0}_{\mathcal{D}}$. By proposition 2.20, we obtain $\beta_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ are open fuzzy soft sets in ℓ_i and ℓ_j respectively such that $x_t^s \tilde{\in} \beta_{\mathcal{A}}$ and $y_k^{s'} \tilde{\in} \mathcal{E}_{\mathcal{B}}$ and $\ell_i(\beta_{\mathcal{A}}) \cap \ell_j(\mathcal{E}_{\mathcal{B}})=\tilde{0}_{\mathcal{D}}$. Hence, $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ is a P-Uryshon T_2-Čfs bicsp. ■

Proposition 4.35 If $(\mathcal{M}, \ell_1, \ell_2, \mathcal{D})$ be a P-Uryshon T_2-Čfs bicsp and ℓ_1^*, ℓ_2^* are Č-fsco's on $\mathcal{M}$ such that ℓ_1^* and ℓ_2^* is finer than ℓ_1 and ℓ_2 respectively, then $(\mathcal{M}, \ell_1^*, \ell_2^*, \mathcal{D})$ is pairwise for $i = 1,2$.

Proof. Similar to the proof of Proposition 4.18. ■

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