[Al-Qadisiyah Journal of Pure Science](https://qjps.researchcommons.org/home)

Volume 26 | Number 4 Article 47

8-15-2021

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Recommended Citation
Yalçın, Sibel and Bayram, Hasan (2021) "On (p,q)-Starlike Harmonic Functions Defined By Subordination," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 4, Article 47.
DOI: 10.29350/qjps.2021.26.3.1364
Available at: https://qjps.researchcommons.org/home/vol26/iss4/47

Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (Special issue num.4) (2021) pp. 491–500

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Article History

Received on: 17/6/2021
Revised on: 20/7/2021
Accepted on: 22/8/2021

Keywords:
Complex harmonic functions
univalent functions
(p,q)-calculus

DOI: https://doi.org/10.29350/jops.2021.26. 4.1364

ABSTRACT

We introduce and investigate classes of (p,q)-starlike harmonic univalent functions defined by subordination. We first obtained a coefficient characterization of these functions. We give necessary and sufficient convolution conditions, distortion bounds, compactness and extreme points for the (p,q)-starlike harmonic univalent with negative coefficients.

1. Introduction

Let H denote the class of continuous complex-valued harmonic functions which are harmonic in the open unit disk $\mathbb{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$ and let $\mathcal{A}$ be the subclass of H consisting of functions which are analytic in $\mathbb{U}$. A function harmonic in $\mathbb{U}$ may be written as $f=\mathfrak{h}+\overline{\mathfrak{g}}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are analytic in $\mathbb{U}$. We call $\mathfrak{h}$ the analytic part and $\mathfrak{g}$ is the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{U}$ is that $|\mathfrak{h}'(z)|>|\mathfrak{g}'(z)|$ (see [5]). To this end, whitout loss of generality, we may write

$$\mathfrak{h}(z)=z+\sum_{k=2}^{\infty} a_k z^k \text{ and } \mathfrak{g}(z)=\sum_{k=1}^{\infty} b_k z^k \tag{1}$$

Let $\mathcal{SH}$ denote the class of functions $f=\mathfrak{h}+\overline{\mathfrak{g}}$ which are harmonic, univalent and sense-preserving in $\mathbb{U}$ for which $\mathfrak{h}(0)=\mathfrak{h}'(0)-1=0=\mathfrak{g}(0)$. One shows easily that the sense-preserving property implies that $|b_1|<1$. Harmonic functions and its subclasses have been studied by many researchers in the past (see [20-25]).

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Clunie and Sheil-Small [5] investigated the class $\mathcal{SH}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{SH}$ and its subclasses. The theory of quantum calculus known as q-calculus is equivalent to traditional infinitesimal calculus without the notion of limits. The q-calculus was started by Euler and Jacobi, who found many interesting applications in various areas of mathematics, physics, and engineering sciences. In the theory of special functions by Sahai and Yadav [15], quantum calculus based on two parameters (p, q) was quoted.

Indeed generalization of q-calculus is the post-quantum calculus, denoted (p, q)-calculus. The (p, q)-integer was introduced in order to give a generalization or to unify several forms of q-oscillator algebras, well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [4]. Throughout this article, we will use basic notations and definitions of the (p, q)- calculus as follows: Let $p > 0, q > 0$. For any non-negative integer k, the (p, q)-integer number k, denoted by $[k]_{p,q}$ is

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}, \qquad (k = 1,2,3, \dots), \qquad [0]_{p,q} = 0.$$

The twin-basic number is a natural generalization of the q-number, that is,

$$[k]_q = \frac{1 - q^k}{1 - q}, \qquad (k = 1,2,3, \dots), \qquad q \neq 1.$$

Similarly, the (p, q)-differential operator of a function f, analytic in $\mathbb{U}$ is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \qquad p \neq q, \qquad z \in \mathbb{U}.$$

One can easily show that $D_{p,q}f(z) \to f(z)$ as $p \to 1^-$ and $q \to 1^-$. It is clear that q-integer and (p, q)-integers differs, that is, we cannot obtain (p, q)-integers just by replacing q by q/p in the definition of q-integers. Also, clearly $\lim_{q\to 1^-} \quad \lim_{p\to 1^-}[k]_{p,q} = k$. For details on q-calculus and (p,q)-calculus, one can refer to [15, 12].

In 1990, Ismail et. al. [11] used q-calculus, in the theory of analytic univalent functions by defining a class of complex valued functions that are analytic on the open unit disk $\mathbb{U}$ with the normalizations $f(0) = 0,\ f'(0) = 1$, and $|f(qz)| \leq |f(z)|$ on $\mathbb{U}$ for every q, $q \in (0,1)$. Motivated by these authors, several researches used the theory of analytic univalent functions and q-calculus; for example see [16] and [3]. The q-difference operator of analytic functions $\mathfrak{h}$ and $\mathfrak{g}$ given by (1) are by definition, given as follows [12]

$$D_{p,q}\mathfrak{h}(z) = \begin{cases} \dfrac{\mathfrak{h}(pz) - \mathfrak{h}(qz)}{(p - q)z} & ; z \neq 0 \\ \mathfrak{h}'(0) & ; z = 0 \end{cases} \text{ and } D_{p,q}\mathfrak{g}(z) = \begin{cases} \dfrac{\mathfrak{g}(pz) - \mathfrak{g}(qz)}{(p - q)z} & ; z \neq 0 \\ \mathfrak{g}'(0) & ; z = 0 \end{cases} \qquad (2).$$

Thus, for the function $\mathfrak{h}$ and $\mathfrak{g}$ of the form (1), we have

$$D_{p,q}\mathfrak{h}(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1} \text{ and } D_{p,q}\mathfrak{g}(z) = \sum_{k=1}^{\infty} [k]_{p,q} b_k z^{k-1}.$$

A harmonic function $f = \mathfrak{h} + \bar{\mathfrak{g}}$ defined by (1) is said to be (p, q)-harmonic, locally univalent and sense-preserving in $\mathbb{U}$ denoted by $\mathcal{SH}_{p,q}$, if and only if the second dilatation $w_{p,q}$ satisfies the condition

$$|w_{p,q}(z)| = \left|\frac{D_{p,q}\mathfrak{g}(z)}{D_{p,q}\mathfrak{h}(z)}\right| < 1 \tag{3}$$

where $0 < p, q < 1, p \neq q$ and $z \in \mathbb{U}$. Note that as $q \to 1^-$, $\mathcal{SH}_q$, reduces to the family $\mathcal{SH}$ (see [2]).

We say that an analytic function f is subordinate to an analytic function F and write $f \prec F$, if there exists a complex valued function w which maps $\mathbb{U}$ into oneself with $w(0) = 0$, such that $f(z) = F(w(z)) \quad (z \in \mathbb{U})$.

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \quad and \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Denote by $\mathcal{SH}_{p,q}(A,B)$ the subclass of $\mathcal{SH}_{p,q}$ consisting of functions f of the form (1) that satisfy the condition

$$\frac{zD_{p,q}\mathfrak{h}(z) - \overline{zD_{p,q}\mathfrak{g}(z)}}{\mathfrak{h}(z) + \overline{\mathfrak{g}(z)}} \prec \frac{1+Az}{1+Bz} \tag{4}$$

($0 < p, q < 1, p \neq q$, $z \in \mathbb{U}$ and $-B \leq A < B \leq 1$) where is $D_{p,q}\mathfrak{h}(z)$ and $D_{p,q}\mathfrak{g}(z)$ are defined by (2). We also let the subclass $\mathcal{TSH}_{p,q}(A,B)$ consist of harmonic functions $f = \mathfrak{h} + \bar{\mathfrak{g}}$ in $\mathcal{SH}_{p,q}(A,B)$ so that $\mathfrak{h}$ and $\mathfrak{g}$ are of the form

$$\mathfrak{h}(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \text{ and } \mathfrak{g}(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{5}$$

By suitably specializing the parameters, the classes $\mathcal{SH}_{p,q}(A,B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $\mathcal{SH}_{p,q}(A,B) = \mathcal{SH}_q(A,B)$ for $p \to 1^-$ ([19]),

(ii) $\mathcal{SH}_{p,q}(A,B) = \mathcal{SH}^0(A,B)$ for $p \to 1^-$ and $q \to 1^-$ ([6]),

(iii) $\mathcal{SH}_{p,q}(-1,1) = \mathcal{H}_q^0(0)$ for $p \to 1^-$ ([14], [1]),

(iv) $\mathcal{SH}_{p,q}(2\alpha - 1,1) = \mathcal{H}_q^0(\alpha)$ for $p \to 1^-$ ([14], [1]),

(v) $\mathcal{SH}_{p,q}(-1,1) = \mathcal{S}_H^*$ for $p \to 1^-$ and $q \to 1^-$ ([13], [17], [18]),

(vi) $\mathcal{SH}_{p,q}(2\alpha - 1,1) = \mathcal{S}_H^*(\alpha)$ for $p \to 1^-$ and $q \to 1^-$ ([13]).

Making use of the techniques and methodology used by Dziok (see [6], [7] and [8]), Dziok et al. (see [9] and [10]), in this paper we find necessary and suficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class

$\mathcal{TSH}_{p,q}(A,B)$. In this paper we find necessary and sufficient conditions, distortion bounds, extreme points for the above defined class $\mathcal{TSH}_{p,q}(A,B)$.

2. Main Results

For functions f_1 and $f_2 \in \mathcal{H}$ of the form

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{j,k} z^k + \sum_{k=1}^{\infty} \overline{b_{j,k} z^k}, \quad (j = 1,2,\dots),$$

we define the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k} a_{2,k} z^k + \sum_{k=1}^{\infty} \overline{b_{1,k} b_{2,k} z^k}, \quad z \in \mathbb{U}.$$

The first theorem we introduce a sufficient coefficient bound for harmonic functions in $\mathcal{SH}_{p,q}(A,B)$.

Theorem 1. Let $f \in \mathcal{SH}_{p,q}$. Then $f \in \mathcal{SH}_{p,q}(A,B)$ if and only if

$$f(z) * \Phi_{p,q}(z;\zeta) \neq 0, \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{U}\backslash\{0\}),$$

where

$$\Phi_{p,q}(z;\zeta) = z\frac{(A-B)\zeta + [1 + B\zeta - (1 + A\zeta)(p+q)]z + pq(1 + A\zeta)z^2}{(1-pz)(1-qz)(1-z)} + \bar{z}\frac{[2 + (A+B)\zeta] - [1 + B\zeta + (p+q)(1 + A\zeta)]\bar{z} + pq(1 + A\zeta)\bar{z}^2}{(1-p\bar{z})(1-q\bar{z})(1-\bar{z})}.$$

Proof. Let $f \in \mathcal{SH}_{p,q}$ be of the form (1). Then $f \in \mathcal{SH}_{p,q}(A,B)$ if and only if it satisfies (4) or equivalently

$$\frac{zD_{p,q}\mathfrak{h}(z) - \overline{zD_{p,q}\mathfrak{g}(z)}}{\mathfrak{h}(z) + \overline{\mathfrak{g}(z)}} \neq \frac{1 + A\zeta}{1 + B\zeta}, \tag{6}$$

where $\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{U}\backslash\{0\}$. Since

$$\mathfrak{h}(z) = \mathfrak{h}(z) * \frac{z}{1-z}, \quad \mathfrak{g}(z) = \mathfrak{g}(z) * \frac{z}{1-z},$$

and

$$zD_{p,q}\mathfrak{h}(z) = \mathfrak{h}(z) * \frac{z}{(1-pz)(1-qz)}, \qquad zD_{p,q}\mathfrak{g}(z) = \mathfrak{g}(z) * \frac{z}{(1-pz)(1-qz)},$$

the inequality (6) yields

$$(1 + A\zeta)\left[\mathfrak{h}(z) + \overline{\mathfrak{g}(z)}\right] - (1 + B\zeta)\left[zD_{p,q}\mathfrak{h}(z) - \overline{zD_{p,q}\mathfrak{g}(z)}\right]$$

$$= \mathfrak{h}(z) * \left\{(1+A\zeta)\frac{z}{1-z} - (1+B\zeta)\frac{z}{(1-pz)(1-qz)}\right\} + \overline{\mathfrak{g}(z)} * \left\{(1+B\zeta)\frac{\bar{z}}{(1-p\bar{z})(1-q\bar{z})} + (1+A\zeta)\frac{\bar{z}}{1-\bar{z}}\right\}$$

$$= h(z) * z\frac{(A-B)\zeta + [1+B\zeta-(1+A\zeta)(p+q)]z + pq(1+A\zeta)z^2}{(1-pz)(1-qz)(1-z)}$$

$$+\overline{\mathfrak{g}(z)} * \bar{z}\frac{[2+(A+B)\zeta] - [1+B\zeta+(p+q)(1+A\zeta)]\bar{z} + pq(1+A\zeta)\bar{z}^2}{(1-p\bar{z})(1-q\bar{z})(1-\bar{z})}$$

$$f(z) * \Phi_{p,q}(z;\zeta) \neq 0.$$

Theorem 2. Let $f = \mathfrak{h} + \bar{\mathfrak{g}}$ be given by (1). If

$$\sum_{k=1}^{\infty}\left\{\left([k]_{p,q}(1+B)-1-A\right)|a_k| + \left([k]_{p,q}(1+B)+1+A\right)|b_k|\right\} \leq 2(B-A), \qquad (7)$$

where $a_1 = 1,\ 0 < p,q < 1,\ -B \leq A < B \leq 1$, then f is sense-preserving and locally univalent in $\mathbb{U}$, and $f \in \mathcal{SH}_{p,q}(A,B)$.

Proof. Since

$$\left|D_{p,q}\mathfrak{h}(z)\right| \geq 1 - \sum_{k=2}^{\infty}[k]_{p,q}|a_k||z|^{k-1}$$

$$> 1 - \sum_{k=2}^{\infty}\frac{(1+B)[k]_{p,q}-1-A}{B-A}|a_k|$$

$$\geq \sum_{k=1}^{\infty}\frac{(1+B)[k]_{p,q}+1+A}{B-A}|b_k|$$

$$> \sum_{k=1}^{\infty}\frac{(1+B)[k]_{p,q}+1+A}{B-A}|b_k||z|^{k-1}$$

$$\geq \sum_{k=1}^{\infty}[k]_{p,q}|b_k||z|^{k-1} \geq \left|D_{p,q}\mathfrak{g}(z)\right|,$$

it follows that $f \in \mathcal{SH}_{p,q}$. On the other hand, $f \in \mathcal{SH}_{p,q}(A,B)$ if and only if there exists a complex valued function $w; w(0) = 0, |w(z)| < 1\ (z \in \mathbb{U})$ such that

$$\frac{zD_{p,q}\mathfrak{h}(z) - \overline{zD_{p,q}\mathfrak{g}(z)}}{\mathfrak{h}(z) + \overline{\mathfrak{g}(z)}} = \frac{1+Aw(z)}{1+Bw(z)}$$

or equivalently

$$\left|\frac{zD_{p,q}\mathfrak{h}(z)-\mathfrak{h}(z)-\overline{zD_{p,q}\mathfrak{g}(z)}-\overline{\mathfrak{g}(z)}}{BzD_{p,q}\mathfrak{h}(z)-B\overline{zD_{p,q}\mathfrak{g}(z)}-A\mathfrak{h}(z)-A\overline{\mathfrak{g}(z)}}\right|<1, \tag{8}$$

The above inequality (7) holds, since for $|z|=r \ (0<r<1)$ we obtain

$$\begin{aligned}
&\left|zD_{p,q}\mathfrak{h}(z)-\mathfrak{h}(z)-\overline{zD_{p,q}\mathfrak{g}(z)}-\overline{\mathfrak{g}(z)}\right|-\left|BzD_{p,q}\mathfrak{h}(z)-B\overline{zD_{p,q}\mathfrak{g}(z)}-A\mathfrak{h}(z)-A\overline{\mathfrak{g}(z)}\right| \\
&=\left|\sum_{k=2}^{\infty}\big([k]_{p,q}-1\big)a_k z^k-\sum_{k=1}^{\infty}\big([k]_{p,q}+1\big)\overline{b_k z^k}\right| \\
&-\left|(B-A)z+\sum_{k=2}^{\infty}\big([k]_{p,q}B-A\big)a_k z^k-\sum_{k=1}^{\infty}\big([k]_{p,q}B+A\big)\overline{b_k z^k}\right| \\
&\le \sum_{k=2}^{\infty}\big([k]_{p,q}-1\big)|a_k|r^k+\sum_{k=1}^{\infty}\big([k]_{p,q}+1\big)|b_k|r^k-(B-A)r \\
&+\sum_{k=2}^{\infty}\big([k]_{p,q}B-A\big)|a_k|r^k+\sum_{k=1}^{\infty}\big([k]_{p,q}B+A\big)|b_k|r^k \\
&\le r\left\{\sum_{k=2}^{\infty}\big([k]_{p,q}(1+B)-1-A\big)|a_k|r^{k-1}+\sum_{k=2}^{\infty}\big([k]_{p,q}(1+B)+1+A\big)|b_k|r^{k-1}\right. \\
&\left.-(B-A)\right\}<0.
\end{aligned}$$

The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(B-A)x_k}{[k]_{p,q}(1+B)-1-A}z^k+\sum_{k=1}^{\infty}\frac{(B-A)y_k}{[k]_{p,q}(1+B)+1+A}\overline{z^k} \tag{9}$$

where

$$\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$$

shows that the coefficient bound given by (7) is sharp. The functions of the form (9) are in $\mathcal{SH}_{p,q}(A,B)$ because

$$\sum_{k=1}^{\infty}\frac{(1+B)[k]_{p,q}-1-A}{2(B-A)}|a_k|+\frac{(1+B)[k]_{p,q}+1+A}{2(B-A)}|b_k|=\sum_{k=1}^{\infty}(|x_k|+|y_k|)=1.$$

Theorem 3. Let $f=\mathfrak{h}+\bar{\mathfrak{g}}$ be defined by (5). Then $f\in\mathcal{TSH}_{p,q}(A,B)$ if and only if the condition (4) holds.

Proof. The "if" part follows from Theorem 1. For the "only-if" part, assume that $f\in\mathcal{TSH}_{p,q}(A,B)$, then by (7) we have

$$\left|\frac{\sum_{k=2}^{\infty}\left([k]_{p,q}-1\right)|a_k|z^k+\sum_{k=1}^{\infty}\left([k]_{p,q}+1\right)|b_k|\bar{z}^k}{(B-A)z-\sum_{k=2}^{\infty}\left([k]_{p,q}B-A\right)|a_k|z^k-\sum_{k=1}^{\infty}\left([k]_{p,q}B+A\right)|b_k|\bar{z}^k}\right|<1$$

For $z=r<1$ we obtain

$$\frac{\sum_{k=2}^{\infty}\left([k]_{p,q}-1\right)|a_k|r^{k-1}+\sum_{k=1}^{\infty}\left([k]_{p,q}+1\right)|b_k|r^{k-1}}{(B-A)-\sum_{k=2}^{\infty}\left([k]_{p,q}B-A\right)r^k-\sum_{k=1}^{\infty}\left([k]_{p,q}B+A\right)r^{k-1}}<1.$$

This gives the condition (7).

Theorem 4. Let f be given by (5). Then $f\in\mathcal{TSH}_{p,q}(A,B)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}\left(X_k\mathfrak{h}_k(z)+Y_k\mathfrak{g}_k(z)\right) \tag{10}$$

where

$$\mathfrak{h}_1(z)=z,$$

$$\mathfrak{h}_k(z)=z-\frac{B-A}{(1+B)[k]_{p,q}-1-A}z^k \quad (k=2,3,4,\dots),$$

$$\mathfrak{g}_k(z)=z+\frac{B-A}{(1+B)[k]_{p,q}+1+A}\bar{z}^k \quad (k=1,2,3,\dots),$$

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1, \qquad X_k\geq 0, Y_k\geq 0.$$

In particular, the extreme points of $f\in\mathcal{TSH}_{p,q}(A,B)$ are $\{\mathfrak{h}_k\}$ and $\{\mathfrak{g}_k\}$.

Proof. For functions f of the form (10) we have

$$f(z)=\sum_{k=1}^{\infty}\left(X_k\mathfrak{h}_k(z)+Y_k\mathfrak{g}_k(z)\right)$$

$$=\sum_{k=1}^{\infty}(X_k+Y_k)\,z-\sum_{k=2}^{\infty}\frac{(B-A)x_k}{[k]_{p,q}(1+B)-1-A}X_kz^k+\sum_{k=1}^{\infty}\frac{(B-A)y_k}{[k]_{p,q}(1+B)+1+A}Y_k\overline{z^k}.$$

Then

$$\sum_{k=2}^{\infty}\frac{(1+B)[k]_{p,q}-1-A}{(B-A)}|a_k|+\sum_{k=1}^{\infty}\frac{(1+B)[k]_{p,q}+1+A}{(B-A)}|b_k|=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1$$

and so $f\in\mathcal{TSH}_{p,q}(A,B)$. Conversely, suppose that $f\in\mathcal{TSH}_{p,q}(A,B)$. Setting

$$X_k=\frac{(1+B)[k]_{p,q}-1-A}{B-A}|a_k| \qquad (k=2,3,4,\dots),$$

$$Y_k = \frac{(1+B)[k]_{p,q} + 1 + A}{B-A}|b_k| \qquad (k = 1,2,3,\dots),$$

where $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, we obtain

$$f(z) = \sum_{k=1}^{\infty}\left(X_k \mathfrak{h}_k(z) + Y_k \mathfrak{g}_k(z)\right)$$

As required. This last expression is non-negative by (7), and so the proof is complete. We have $f \in \mathcal{TSH}_{p,q}(A,B)$.

Finally, we give the distortion bounds for functions in $\mathcal{TSH}_{p,q}(A,B)$ which yields a covering result for this class.

Theorem 5. Let $f \in \mathcal{TSH}_{p,q}(A,B)$. Then for $|z| = r < 1$ and $|b_1| < \frac{B-A}{2+A+B}$,

we have

$$|f(z)| \leq (1+|b_1|)r + \frac{B-A}{(p+q)(1+B)-1-A}\left(1 - \frac{2+A+B}{B-A}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{B-A}{(p+q)(1+B)-1-A}\left(1 - \frac{2+A+B}{B-A}|b_1|\right)r^2.$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f \in \mathcal{TSH}_{p,q}(A,B)$. Taking the absolute value of f we obtain

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)\, r^k \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)\, r^2$$

$$\leq (1+|b_1|)r + \frac{B-A}{[2]_{p,q}(1+B)-1-A}\sum_{k=2}^{\infty}\left\{\frac{(1+B)[k]_{p,q} - 1 - A}{B-A}|a_k| + \frac{(1+B)[k]_{p,q} + 1 + A}{B-A}|b_k|\right\}r^2$$

$$\leq (1+|b_1|)r + \frac{B-A}{(p+q)(1+B)-1-A}\left(1 - \frac{2+A+B}{B-A}|b_1|\right)r^2.$$

Corollary 6. Let $f = \mathfrak{h} + \overline{\mathfrak{g}}$ with $\mathfrak{h}$ and $\mathfrak{g}$ of the form (6). If $f \in \mathcal{TSH}_{p,q}(A,B)$ then

$$\left\{w: |w| < \frac{(p+q-1)(1+B) - [(1+B)(p+q-1) - 2(1+A)]|b_1|}{(p+q)(1+B)-1-A}\right\} \subset f(\mathbb{U}), p+q \geq 1.$$

Theorem 7. The family $\mathcal{TSH}_{p,q}(A,B)$ is closed under convex combination.

Proof. For $j = 1,2,\dots$ suppose that

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{j_k}| z^k + \sum_{k=1}^{\infty} |b_{j_k}| \bar{z}^k.$$

Then by Theorem 2,

$$\sum_{k=2}^{\infty} \frac{(1+B)[k]_{p,q} - 1 - A}{B - A} |a_{j_k}| + \sum_{k=1}^{\infty} \frac{(1+B)[k]_{p,q} + 1 + A}{B - A} |b_{j_k}| \leq 1. \qquad (11)$$

For $\sum_{j=1}^{\infty} t_j = 1$, $0 \leq t_j \leq 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j |a_{j_k}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} t_j |b_{j_k}| \right) \bar{z}^k.$$

Then by (11),

$$\sum_{k=1}^{\infty} \left\{ \frac{(1+B)[k]_{p,q} - 1 - A}{B - A} \sum_{j=1}^{\infty} t_j |a_{j_k}| + \frac{(1+B)[k]_{p,q} + 1 + A}{B - A} \sum_{j=1}^{\infty} t_j |b_{j_k}| \right\}$$

$$= \sum_{j=1}^{\infty} t_j \sum_{k=1}^{\infty} \left(\frac{(1+B)[k]_{p,q} - 1 - A}{B - A} |a_{j_k}| + \frac{(1+B)[k]_{p,q} + 1 + A}{B - A} |b_{j_k}| \right)$$

$$\leq \sum_{j=1}^{\infty} t_j = 1$$

hence

$$\sum_{j=1}^{\infty} t_j f_j(z) \in \mathcal{TSH}_{p,q}(A, B).$$

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