Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 4 Article 48

8-15-2021

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Haitham M. Hussein Al-Ghraby
University of Kerbala, College of Engineering, Iraq, hythmalghraby304@gmail.com

Radhi A. Zaboon
Al-Mustansiriya University, College of Science, Department of Mathematics, Iraq, radhizaboon@gmail.com

Naseif J. Jawari
Al-Mustansiriya University, College of Science, Department of Mathematics, Iraq, nsaif642014@yahoo.com

Recommended Citation

Al-Ghraby, Haitham M. Hussein; Zaboon, Radhi A.; and Jawari, Naseif J. (2021) "Globally Null-Controllability and Complete stabilizability for Perturbed Linear Time-Varying System in Real Composition of Hilbert Spaces," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 4, Article 48.
DOI: 10.29350/qjps.2021.26.4.1320
Available at: https://qjps.researchcommons.org/home/vol26/iss4/48

Al-Qadisiyah Journal of Pure Science
QJPS
Al-Qadisiyah Journal of Pure Science
ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514
DOI: /10.29350/jops.
http://qu.edu.iq/journalsc/index.php/JOPS

Globally Null-Controllability and Complete stabilizability for Perturbed Linear Time-Varying System in Real Composition of Hilbert Spaces

Authors Names
a.Haitham M. Hussein
b.Radhi A. Zaboon
c.Naseif J. Al-Jawari

Article History

Received on: 3/6/2021
Revised on: 30/7/2021
Accepted on:30/8/2021

Keywords:
perturbed linear Time-Varying systems, Semigroup theory, Perturbation theory, Controllability,Stabilizability, Hilbert space, and Riccati equation.

DOI: https://doi.org/10.29350/jops.2021.26. 4.1320

ABSTRACT

This paper deals with the problem of controllability and stabilizability of the perturbed linear time-varying systems defined in some suitable real Hilbert space. The aim of this paper is to show that any globally null-controllable system is completely stabilizability and conversely, under some additional condition the complete stabilizability implies global null-controllability.

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[a]University of Kerbala, College of Engineering, Iraq Email: hythmalghraby304@gmail.com

[b]Al-Mustansiriya University, College of Science, Department of Mathematics, Iraq Email:radhizaboon@gmail.com

[c]Al-Mustansiriya University, College of Science, Department of Mathematics, Iraq Email:nsaif642014@yahoo.com

1. INTRODUCTION

Let $\mathbb{H}, U$ be a real Hilbert spaces, $\mathbb{H}^*, U^*$are the dual of the spaces $\mathbb{H}, U$, respectively, and $\mathbb{H}^m, U^m$ are the Hilbert product spaces doted.

Consider the perturbed linear time-varying nonlocal system of the form:

$$\left.\begin{array}{l}\begin{bmatrix}\frac{dz_1(t)}{dt}\\ \cdot\\ \cdot\\ \cdot\\ \frac{dz_m(t)}{dt}\end{bmatrix}=\left(\begin{bmatrix}\mathbb{A}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathbb{A}_m(t)\end{bmatrix}+\begin{bmatrix}\mathbb{B}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathbb{B}_m(t)\end{bmatrix}\right)\begin{bmatrix}z_1(t)\\ \cdot\\ \cdot\\ \cdot\\ z_m(t)\end{bmatrix}+\\ \qquad\begin{bmatrix}\mathbb{E}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathbb{E}_m(t)\end{bmatrix}\begin{bmatrix}u_1(t)\\ \cdot\\ \cdot\\ \cdot\\ u_m(t)\end{bmatrix},\forall t\geq 0\\ \begin{bmatrix}z_1(0)\\ \cdot\\ \cdot\\ \cdot\\ z_m(0)\end{bmatrix}=\begin{bmatrix}z_0\\ \cdot\\ \cdot\\ \cdot\\ z_0\end{bmatrix},\quad z_i(0)\in Z^i, i\in\{1,\dots,m\}\end{array}\right\}$$

Where, $A(t): D(A(t)) \subset \mathbb{H} \to \mathbb{H}$ be linear operator and $A(t) = diag(\mathbb{A}_1(t), \dots, \mathbb{A}_m(t))$ with $A_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq \mathbb{M}_i e^{\omega_i t}, t \geq 0$, for $\mathbb{M}_i \geq 1, \omega_i \in \mathbb{R}$, and all $i \in \{1, \dots, m\}$, and $B(t): \mathbb{H} \to \mathbb{H}$ with $B(t) = diag(\mathbb{B}_1(t), \dots, \mathbb{B}_m(t))$ are bounded linear operators i.e., there exists a real number c, such that $\|\mathbb{B}_i(t)\| \leq c$. Moreover, the operator $\mathbb{E}(t): U \to \mathbb{H}$ with $E(t) = diag(\mathbb{E}_1(t), \dots, \mathbb{E}_m(t))$ is bounded linear operator for all $i \in [1, m]$, $z_i(t) \in \mathbb{H}$ is the state, $u_i(t) \in U$ is the control, and $z_0 \in \mathbb{H}^m$, the inner product on the Hilbert product space doted defined by $\langle z_i, z_i\rangle_{\mathbb{H}^m}^{\frac{1}{2}} = \sum_{i=1}^{m}\langle z_i, z_i\rangle_{\mathbb{H}}^{\frac{1}{2}}$.

The problem of controllability and stabilizability of the perturbed linear time-varying system(1), with $B(t) = 0$, and $m = 1$, for all $i \in \{1, \dots, m\}$ has been studied extensively, see; e.g. [9,11,15,17], and the reference therein. This problem related as an extension of the classical Kalman result [6] on controllability and stabilizability of linear control system is to find an admissible control $u(t)$ such that the corresponding solution $z(t)$ of the linear control system has desirable properties. Depending on the properties interested, one defines different qualitative problem. For instance, the null-controllability problem concerns the question of finding an admissible control $u(t)$ which steers an arbitrary state z_0 of the linear control system into zero; stabilizability problem is to find a control $u(t) = \mathbb{K}(t)z(t)$ such that the zero solution of the closed-loop system

$$d/_{dt}\, z(t) = [A(t) + B(t)\mathbb{K}(t)]z(t), t \geq 0$$

is asymptotically stable in the Lyapunov sense. Moreover, the linear control system is the stabilizable by the control $u(t) = \mathbb{K}(t)z(t)$, and it is the stabilizing feedback control of the linear control system. Different stabilizability concepts can be adopted to investigate the stability property of the linear control systems [4, 10, and 14]. Wonham [14] extended stability properties is the concept of the complete stabilizability, which relates to a strong exponentially stability of the linear control system. That is to say, the linear control system is completely stabilizability if for every number $\delta > 0$, there exists a feedback control $u(t) = \mathbb{K}(t)z(t)$ such that the solution $z(t, z_0)$ of the closed-loop system satisfies the following condition

$$\exists N > 0: \|z(t, z_0)\| \leq Ne^{-\delta t}\|z_0\|, \forall t \geq 0$$

This means that for every positive number $\delta > 0$, the system zero-input restraint of the closed-loop system decays faster than $e^{-\delta t}$. In other words, for any given in advance the decay rate $\delta > 0$, the linear control system can be δ-exponentially stabilizable. It is well known that, If a finite-dimensional time-invariant linear control system is globally null-controllable in finite time then it is stabilizable, but the converse is not true, studied by Kalman [6], and Wonham [15]. Nevertheless, if the system is completely stabilizable, then it is globally null-controllability in finite time discuss by Wonham [14].

However, in the infinite-dimensional control theory, investigations of controllability and stabilizability are more complicated and require more techniques that are sophisticated. The difficulties increase to the same extent as passing from time-invariant to time-varying system. Some extensions have been developed by Slemrod [13], Zabczyk [16] for time-invariant control systems in Hilbert spaces. For time-varying control systems in finite-dimensional spaces, using Kalman's decomposition method, [5] proved that the system is completely stabilizable if it is uniformly globally null-controllable and [12] extends some of [5] to time-varying control systems. P. Niamsup and V. N. Path. [1], develop the result in [12] on the relationship between the globally null-controllability and complete stabilizability for the linear time-varying control system in the real Hilbert space H.

From the all above, one can find a reasonable justification to accomplish the study of this paper. In this paper, we develop the result of P. Niamsup and V. N. Path. [1] on the relationship between the exact controllability and complete stabilizability for the perturbed linear time-varying system (1) in Hilbert space $\mathbb{H}$. We show that the system is complete stabilizable if it is globally null-controllable in finite time, and conversely under some additional growth condition on the evolution operator of the system, the system is globally null-controllable in finite time if it is complete stabilizable in finite time.

2. Concept and **Mathematical** Preliminaries

The following concepts will be used in this article.

1. All real numbers are represented by the letter $\mathbb{R}$, the set of all non-negative real numbers is denoted by $\mathbb{R}^+$

2. $\mathbb{H}, U$ denotes an infinite-dimensional real Hilbert spaces with the inner product $\langle .,.\rangle$, $\mathbb{H}^*, U^*$denotes the dual spaces of the spaces $\mathbb{H}, U$ respectively, $\mathbb{H}^m, U^m$ denotes the real Hilbert product spaces dotted with the inner product defined by $\langle z, z\rangle_{\mathbb{H}^m} = \sum_{i=1}^{m}\langle z_i, z_i\rangle_{\mathbb{H}}$.

3. $\mathcal{L}(\mathbb{H})$ denotes the Banach space of all linear bounded operators mapping $\mathbb{H}$ into $\mathbb{H}$, $L^2([t,s],\mathbb{H})$ denotes the set of all strongly measurable L^2-integrable and $\mathbb{H}$-valued functions on $[t,s]$.

4. $D(A), A^{-1}$ and A^*denote the domain, the inverse and the adjoint of the operator A respectively, $\overline{M}$ denotes the closure of a set M, I denotes the identity operator; the operator A is called self-adjoint if $A = A^*$, $\rho(A)$, and $R(\lambda, A)$ denotes the resolvent set and the resolvent of the operator A_i respectively.

5. An operator $\mathbb{Q}_i \in \mathcal{L}(\mathbb{H})$ is called non-negative definite ($\mathbb{Q}_i \geq 0$), if $\langle \mathbb{Q}_i z_i, z_i\rangle \geq 0$, for all $z_i \in \mathbb{H}$, and for all $i \in [1,m]$; $\mathcal{LO}(\mathbb{H})([t,+\infty],\mathbb{H}^+)$ denotes the set of all linear bounded self-adjoint non-negative definite operator-valued function on $[0,\infty)$, the operator$A(t)\colon D(A(t)) \subset \mathbb{H} \to \mathbb{H}$ be linear operator and $A(t) = diag(\mathbb{A}_1(t), \dots, \mathbb{A}_m(t))$ with $\mathbb{A}_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq \mathbb{M}_i e^{\omega_i t}, t \geq 0$, for $\mathbb{M}_i \geq 1, \omega_i \in \mathbb{R}$ for all $i \in \{1,\dots,m\}$ if:

$$\mathbb{A}_i(t)z_i = \lim_{t\to 0^+} \frac{1}{t}\left[\mathbb{T}_{t,i} - I_i\right]z_i \,, \forall z_i \in D\big(A_i(t)\big), i \in \{1,\dots,m\}$$

Where, $D\big(\mathbb{A}_i(t)\big) = \{z_i \in \mathbb{H} \colon \lim_{t\to 0^+}\frac{1}{t}\left[\mathbb{T}_{t,i} - I_i\right]z_i\}$ exists.

Consider the perturbed linear time-varying system (1), where $\mathbb{H}, U$ infinite-dimensional real Hilbert spaces and $\mathbb{H}^m, U^m$infinite-dimensional real Hilbert product spaces doted.

Assume the following assumptions satisfying.

1. $A(t): D(A(t)) \subset \mathbb{H} \to \mathbb{H}$ be linear operator and $A(t) = diag(\mathbb{A}_1(t), \dots, \mathbb{A}_m(t))$ with $\mathbb{A}_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq \mathbb{M}_i e^{\omega_i t}, t \geq 0$, for $\mathbb{M}_i \geq 1, \omega_i \in \mathbb{R}$ for all $i \in \{1, \dots, m\}$.
2. $B(t): \mathbb{H} \to \mathbb{H}$ with $B(t) = diag(\mathbb{B}_1(t), \dots, \mathbb{B}_m(t))$ are bounded linear operators i.e., there exists a real number c_i, such that $\|\mathbb{B}_i(t)\| \leq c_i$, for all $i \in \{1, \dots, m\}$.
3. $E(t) = diag(\mathbb{E}_1(t), \dots, \mathbb{E}_m(t))$ is bounded linear operator for all $i \in [1, m]$, $z_i(t) \in \mathbb{H}$ is the state, $u_i(t) \in U$ is the control.

   Therefore, by "bounded perturbation theorem" [18], the operator $A(t) + B(t): D(A(t)) \subset \mathbb{H} \to \mathbb{H}$ with

$A(t) + B(t) = daig(\mathbb{A}_1(t) + \mathbb{B}_1(t), \dots, \mathbb{A}_m(t) + \mathbb{B}_m(t))$ is infinitesimal generator a strongly continuous semigroup $\mathbb{S}_{t,i}$ satisfies

$\|\mathbb{S}_{t,i}\| \leq \mathbb{M}_i e^{(\omega_i + \mathbb{M}_i c_i)t}$, for all $t \geq 0$, omits $B(t) \neq -A(t)$, and $B(t) \neq \alpha A(t)$, for all $\alpha > 1$, $i \in \{1, \dots, m\}$.

Moreover, the following conditions that guarantee the existence and uniqueness of the solution for the perturbed linear time-varying system (1).

4. The linear operators$[A(.) + B(.)]z(.)$, and $E(.)u(.)$ are continuous and bounded in $t \in \mathbb{R}^+$ for all $z(.) \in \mathbb{H}, u(.) \in U$, and for all $i \in \{1, \dots, m\}$.

5. $\overline{D(A(t) + B(t))} = \overline{D(A(t))} = \mathbb{H}, t \in \mathbb{R}^+$, $A(.) + B(.)$ generates a strongly continuous semigroup $\mathbb{S}_{t,i}$ on $\mathbb{H}$, and there is an evaluation operator $\mathbb{W}_{i,A+B}(t,s): \{(t,s): t \geq s \geq 0\} \to \mathcal{L}(\mathbb{H})$, such that $\mathbb{W}^*_{i,A+B}(t,s)$ is continuous in t, s and for each $z_i \in D(A_i(t))$, $\mathbb{W}_{i,A_i+B_i}(t,s)z_i \in D(A_i(t))$ for all $i \in \{1, \dots, m\}$.

(i) $\frac{d\mathbb{W}_{i,A+B}(t,s)z_i}{dt} = [A(t) + B(t)]\mathbb{W}_{i,A+B}(t,s)z_i$, $\mathbb{W}_{i,A+B}(s,s) = I$.

(ii) $\mathbb{W}_{i,A+B}(t,s) = \mathbb{W}_{i,A+B}(t,r)\mathbb{W}_{i,A+B}(r,s)$, for all $t \geq s \geq 0$, $i \in \{1, \dots, m\}$.

In this case, we say that $[A(t) + B(t)]$generates a strongly continuous evolution perturbation operator $\mathbb{W}_{i,A+B}(t,s)$, then for every initial state $z_0 \in \mathbb{H}$,and for every admissible control $u(t)$, the perturbed linear time-varying system (1)has a mild solution given by:

Let $z(\cdot) \in \mathbb{H}$ be a solution of the perturbed linear time-varying system (1), then by definition of evolution family we have $\mathbb{W}_{i,A+B}(t,s), z_i(s)$ is differentiable, that implies the $\mathbb{H}$-value function $g(s) = \mathbb{W}_{i,A+B}(t,s)z(s)$ is differentiable for $t \geq s \geq 0$; and :

$$\frac{\partial g(s)}{\partial s} = \mathbb{W}_{i,A+B}(t,s)\frac{\partial}{\partial s}z(s) + z(s)\frac{\partial}{\partial s}\mathbb{W}_{i,A+B}(t,s)$$

$$= \mathbb{W}_{i,A+B}(t,s)z(s)\big([A(t) + B(t)]z_i(s) + E(s)u(s)\big)z(s)\big(-[A(t) + B(t)]\mathbb{W}_{i,A+B}(t,s)Z(s)\big)$$

$$= \mathbb{W}_{i,A+B}(t,s)E(s)u(s),$$

Integration (2) *from* 0 *to* t, *yields:*

$$g(t) - g(0) = \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s)\,ds$$

Since:

$$g(s) = \mathbb{W}_{i,A+B}(t,s)z(s), t \geq s \geq 0$$

Then

$$\mathbb{W}_{i,A+B}(t,t)z(s) - \mathbb{W}_{i,A+B}(t,0)z(s) = \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s)\,ds$$

Hence,

$$\mathbb{W}_{i,A+B}(t,t)z(s) = \mathbb{W}_{i,A+B}(t,0)z(s) + \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s)\,ds$$

By definition of semigroup, we obtain the solution

$$\mathbb{W}_{i,A+B}(t,s)z(s) = z(t)$$
$$= \mathbb{W}_{i,A+B}(t,0)z_0$$
$$+ \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s)\,ds, \qquad (3)$$

The evaluation-perturbed operator $\mathbb{W}_i(t,s)$ is a natural extension to the strongly continuous semigroup of time-invariant perturbed linear systems. For example, if $A(t) + B(t) = A + B$ is independent of t then $\mathbb{W}_{i,A+B}(t,s)\mathbb{S}_{t,i}\mathbb{S}_{\tau,i}$, and the two parameter family of semigroup operators reduces to one parameter family $\mathbb{S}_{t,i}$, which is standard a strongly continuous semigroup generated by $A + B \in \mathcal{L}(\mathbb{H})$, $t \in \mathbb{R}^+$, then the semigroup evolution operators $\mathbb{W}_{i,A+B}(t,s)$ satisfying the above conditions always exists. However, if $A(t) + B(t)$, $t \in \mathbb{R}^+$is unbounded perturbed operator, then the evolution-perturbed operator $\mathbb{W}_{i,A+B}(t,s)$ exists provided additional assumptions [2,8].

1. Definition

The perturbed linear time-varying system (1), is called globally null-controllability (GNC) in finite time if, for every $z_0 \in \mathbb{H}$, there exist a number $T > 0$ and an admissible control $u(t)$ such that

$$\mathbb{W}_{i,A+B}(t,0)z_0 + \int_0^T \mathbb{W}_{i,A+B}(T,s)E(s)u(s)\,ds = 0$$

2. Remark [2,3]

The perturbed linear time-varying system (1) is called globally null-controllability if and only if $\exists T > 0, \mathbb{b}_i > 0$ such that

$$\int_0^T \left\|E^*(s)\mathbb{W}^*_{i,A+B}(T,s)z^*\right\|^2 ds \geq \mathbb{b}_i \left\|\mathbb{W}^*_{i,A+B}(T,0)z^*\right\|^2, \forall z^* \in \mathbb{H}^*$$

3. Definition

The perturbed linear time-varying system (1) is called completely stabilizability (CZs) if every number $(\omega_i + \mathbb{M}_i c_i) > 0$ there exists a feedback control $u(t) = \mathbb{K}(t)z(t)$, where $\mathbb{K}(t) \in \mathcal{L}(\mathbb{H}, U)$ is bounded on $\mathbb{R}^+$, such that the solution $z(t, z_0)$ of the closed-loop nonlocal system

$$\begin{bmatrix} \frac{dz_1(t)}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dz_m(t)}{dt} \end{bmatrix} = \left(\left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) + \begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{K}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{K}_m(t) \end{bmatrix} \right) \begin{bmatrix} z_1(t) \\ \cdot \\ \cdot \\ \cdot \\ z_1(t) \end{bmatrix}$$

$$\begin{bmatrix} z_1(0) \\ \cdot \\ \cdot \\ \cdot \\ z_1(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ \cdot \\ \cdot \\ \cdot \\ z_0 \end{bmatrix}$$

Satisfies

$\exists N > 0: \|z_i(t, z_0)\| \leq N e^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\|, \forall t \geq 0, i \in [1, m]$. The solution for the stabilizability nonlocal problem involves a perturbed Riccati operator equation (PROE) of the form:

$$\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dP_m(t)}{dt} \end{bmatrix} + \left(\begin{bmatrix} \mathbb{A}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m^*(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix}$$

$$+ \begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right)$$

$$- \begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{E}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad (4)$$

Where, $\mathbb{Q}_i(t) \geq 0$, is given self-adjoint operators for all $i \in \{1, \dots, m\}$, it is not clear a priori what a solution of (PROE) means. We will defined the solution of (PROE)(4) as follows.

4. Definition

The (mild) solution of (PROE)(4) is a linear operator $P_i(t) \in \mathcal{L}(\mathbb{H})$ satisfying the following conditions:

1) The scalar function $\langle P_i(t)z_i, y_i \rangle$ is differentiable on $[0, \infty)$ for every $z_i, y_i \in D\big(A(t) + B(t)\big) = D(A(t))$, $i \in \{1, \dots, m\}$.
2) For all $z_i, y_i \in D\big(A(t) + B(t)\big) = D(A(t))$, $t \in \mathbb{R}^+$, $i \in \{1, \dots, m\}$:

$$\frac{d}{dt}\langle P_i(t)z_i, y_i \rangle + \langle P_i(t)z_i, \big(A(t) + B(t)\big)y_i \rangle + \langle P_i(t)\big(A(t) + B(t)\big)z_i, y_i \rangle - \langle P_i(t)E(t)E^*(t)P_i(t)z_i, y_i \rangle + \langle \mathbb{Q}_i(t)z_i, y_i \rangle = 0.$$

We state the following sufficient condition, which guarantees the existence of a bounded solution $P_i(t)$ of (PROE)(4).

5. Definition .

Let $\mathbb{Q}_i(t) \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$. The perturbed linear time-varying system (1) is called $\mathbb{Q}_i(t)$-stabilizable if for every initial state z_0, there is a control $u(t) \in L^2([0, \infty), U)$ such that the cost function:

$$\begin{bmatrix} J_1(u_1) \\ \cdot \\ \cdot \\ \cdot \\ J_m(u_i) \end{bmatrix} = \begin{bmatrix} \int_0^\infty [\|u_1(t)\|^2 + \langle \mathbb{Q}_1(t) z_1(t,z_0), z_1(t,z_0)\rangle]\, dt \\ \cdot \\ \cdot \\ \cdot \\ \int_0^\infty [\|u_1(t)\|^2 + \langle \mathbb{Q}_1(t) z_1(t,z_0), z_1(t,z_0)\rangle]\, dt \end{bmatrix} \tag{5}$$

Exists and is finite for all $i \in \{1, \dots, m\}$.

6. Remark [2]

If the perturbed linear time-varying system (1) is $\mathbb{Q}_i(t)$-stabilizable, then the (PROE)(4) has the solution $P_i(t) \in \mathcal{LO}([0,\infty), \mathbb{H}^+)$ bounded on $\mathbb{R}^+$.

3. MAIN RESULT

Consider the perturbed linear time-varying system (1) in real Hilbert space. As we have already hinted at introduction, for time-invariant control system in finite-dimensional spaces Wonham [13] studied the equivalence of complete stabilizability and global null-controllability and for the case of infinite-dimensional systems, assuming a compactness property on the semigroup Slemrod [11] showed that the time-invariant control system in Hilbert spaces is completely stabilizable if and only if it is globally null-controllability in finite time. P. Niamsup and V. N. Path. [1], develop the result in [13] on the relationship between the globally null-controllability and complete stabilizability for the linear time-varying control system in the real Hilbert space $\mathbb{H}$. In this paper, we develop the result in [1] on the relationship between the exact controllability and complete stabilizability for the perturbed linear time-varying nonlocal system (1) in Hilbert space $\mathbb{H}$. We show that the system is complete stabilizable if it is globally null-controllable in finite time, and conversely under some additional growth condition on the evolution operator of the system, the system is globally null-controllable in finite time if it is complete stabilizable in finite time. In this section we us the following growth condition on the perturbed evolution$\mathbb{W}_{i,A_{i+}B_i}(t,s)$:

(H) $\exists M_i > 0, \gamma_i + M_i c_i > 0$: $\|\mathbb{W}_{i,A+B}(t,s)\| \le M_i e^{(\gamma_i + M_i c_i)|t-s|}, \forall t,s \ge 0, i \in \{1,\dots,m\}$.

From [8] that the growth condition $\mathbb{H}$ hold for time-invariant system when $m = 1, B_1(t) = 0$, for, and $A(t) \in \mathcal{L}(\mathbb{H})$ is a linear continuous constant operator as well as for time-varying system when $A(t)$ is a matrix function uniformly bounded in $t \in \mathbb{R}^+$.

Proposition 3.1

The perturbed linear time-varying system (1) is completely stabilizable (CS_z) if it is globally null-controllability (GNC) in finite time.

Proof:

Assume that the perturbed linear time-varying system (1) is globally null-controllability (GNC) in finite time. Let $(\omega_i + \mathbb{M}_i c_i) > 0$, for all $i \in \{1, \dots, m\}$ be any given number. We take a change of the state variable $y_i(t) = e^{(\omega_i + \mathbb{M}_i c_i)t} z_i(t)$, then the perturbed linear time-varying system (1) is transformed to the system given by:

$$\begin{bmatrix} \frac{d}{dt}y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \frac{d}{dt}y_m(t) \end{bmatrix} = \left(\begin{bmatrix} \widetilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \begin{bmatrix} y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \cdot \\ \cdot \\ \cdot \\ u_m(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ \cdot \\ \cdot \\ \cdot \\ y_m(0) \end{bmatrix} = \begin{bmatrix} y_0 z_0 \\ \cdot \\ \cdot \\ \cdot \\ y_0 z_0 \end{bmatrix}, \forall i \in [1, m] = \{1, \dots, m\}, \quad 6)$$

Where,

$$\left(\begin{bmatrix} \widetilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t) \end{bmatrix} \right) = \left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) \times \begin{bmatrix} e^{(\omega_1 + \mathbb{M}_1 c_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(\omega_m + \mathbb{M}_m c_m)} \end{bmatrix} \times I$$

and,

$$\begin{bmatrix} \widetilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{E}}_m(t) \end{bmatrix} = \begin{bmatrix} e^{(\omega_1 + \mathbb{M}_1 c_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(\omega_m + \mathbb{M}_m c_m)} \end{bmatrix} \begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix}$$

We choose an operator function $\mathbb{Q}_i(t) \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$ bounded on $\mathbb{R}^+$, such that

$$\begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} \geq 2 \left(\begin{bmatrix} \widetilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t) \end{bmatrix} \right) + \left(\begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{E}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m^*(t) \end{bmatrix} \right), \quad (7)$$

We now show that the perturbed linear time-varying system

$\left[\widetilde{\mathbb{A}}_i(t) + \widetilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)\right]$:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \frac{d}{dt}x_m(t) \end{bmatrix} = \left(\begin{bmatrix} \widetilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \cdot \\ \cdot \\ \cdot \\ u_m(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ \cdot \\ \cdot \\ \cdot \\ x_m(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \cdot \\ \cdot \\ \cdot \\ x_0 \end{bmatrix}, \forall i \in \{1, \dots, m\},$$

is globally null-controllability in finite time.

Moreover, by the globally null-controllability of previous system $\left[\left(\mathbb{A}_i(t) + \mathbb{B}_i(t)\right), \mathbb{E}_i(t)\right]$, for every $x_0 \in \mathbb{H}$ there exists a time $T > 0$, and admissible control $u(t) \in L^2([0, T], U)$ such that:

$$\mathbb{W}_{i,A+B}(T, 0)x_0 + \int_0^T \mathbb{W}_{i,A+B}(T, s)\mathbb{E}_i(s)u_i(s)\, ds = 0, \forall i \in \{1, \dots, m\}, \quad (8)$$

Multiplying both sides of equation (8) with $e^{(\omega_i + \mathbb{M}_i c_i)T}$ and observing that

$$\mathbb{W}_{i,\tilde{A}+\tilde{B}}(t, s) = e^{(\omega_i + \mathbb{M}_i c_i)|t-s|}\mathbb{W}_{i,A+B}(t, s)$$

We find

$$\mathbb{W}_{i,\tilde{A}+\tilde{B}}(T, 0)x_0 + \int_0^t \mathbb{W}_{i,\tilde{A}+\tilde{B}}(T, s)\mathbb{E}_i(s)\tilde{u}_i(s)\, ds = 0, \forall i \in \{1, \dots, m\}$$

Where $\tilde{u}_i(s) = e^{(\omega_i + \mathbb{M}_i c_i)s}u_i(s)$,

This implies that the perturbed linear time-varying system $\left[\widetilde{\mathbb{A}}_i(t) + \widetilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)\right]$ is globally null-controllability in finite time. Let $u_{i,x_i}(t)$ be an admissible control as stated by to the solution $x_i(t)$ for all $i \in \{1, \dots, \mathrm{m}\}$ of the perturbed linear time-varying nonlocal system $\left[\widetilde{\mathbb{A}}_i(t) + \widetilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)\right]$ steers $x_0 \in \mathbb{H}$ into zero in time T.

For every initial state $x_0 \in \mathbb{H}$ there is an admissible control $u_{i,x_i}(t) \in L^2([0, T], U)$ such that the solution $x_i(t)$ of the system as maintained by the control $u_{i,x_i}(t)$ satisfies $x_i(0) = x_0, x_i(T) = 0$, for all $i \in \{1, \dots, \mathrm{m}\}$.

Defined the admissible control $\tilde{u}_{i,x_i}(t) \in L^2([0, T], U), t \geq 0$ by

$$\tilde{u}_{i,x_i}(t) = \begin{Bmatrix} u_{i,x_i}(t), if\ t \in [0, T] \\ 0, \ if\ t > T \end{Bmatrix}$$

Therefore, we have

$$J_i\left(\tilde{u}_{i,x_i}(t)\right) = \int_0^\infty \left[\left\|\tilde{u}_{i,x_i}(t)\right\|^2 + \langle \mathbb{Q}_i(t)x_i(t), x_i(t)\rangle\right] dt$$

$$= \int_0^T \left[\left\|\tilde{u}_{i,x_i}(t)\right\|^2 + \langle \mathbb{Q}_i(t)x_i(t), x_i(t)\rangle\right] dt < +\infty$$

Therefore, the perturbed linear time-varying nonlocal system $\left[\widetilde{\mathbb{A}}_i(t) + \widetilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)\right]$ is $\mathbb{Q}_i(t)$-stabilizable. By using remark 2 to the cost function (5), we can find an operator function $P_i \in \mathcal{LO}([0,\infty), \mathbb{H}^+)$ which a solution of the following PROE

$$\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dP_m(t)}{dt} \end{bmatrix} + \left(\begin{bmatrix} \widetilde{\mathbb{A}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m^*(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{B}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m^*(t) \end{bmatrix}\right)\begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix}\begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix}\left(\begin{bmatrix} \widetilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t) \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} \widetilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t) \end{bmatrix}\right) - \begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix}\left(\begin{bmatrix} \widetilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{E}}_m(t) \end{bmatrix}\begin{bmatrix} \widetilde{\mathbb{E}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\mathbb{E}}_m^*(t) \end{bmatrix}\right)\begin{bmatrix} P_1(t) \\ \cdot \\ \cdot \\ \cdot \\ P_m(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix}\frac{dP_1(t)}{dt}\\ \cdot\\ \cdot\\ \cdot\\ \frac{dP_m(t)}{dt}\end{bmatrix}+\left(\begin{bmatrix}\widetilde{\mathbb{A}}_1^*(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{A}}_m^*(t)\end{bmatrix}+\begin{bmatrix}\widetilde{\mathbb{B}}_1^*(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{B}}_m^*(t)\end{bmatrix}\right)\begin{bmatrix}P_1(t)\\ \cdot\\ \cdot\\ \cdot\\ P_m(t)\end{bmatrix}$$

$$+\begin{bmatrix}P_1(t)\\ \cdot\\ \cdot\\ \cdot\\ P_m(t)\end{bmatrix}\left(\begin{bmatrix}\widetilde{\mathbb{A}}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{A}}_m(t)\end{bmatrix}+\begin{bmatrix}\widetilde{\mathbb{B}}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{B}}_m(t)\end{bmatrix}\right)$$

$$-2e^{(\omega_i+\mathbb{M}_ic_i)t}\begin{bmatrix}P_1(t)\\ \cdot\\ \cdot\\ \cdot\\ P_m(t)\end{bmatrix}\left(\begin{bmatrix}\widetilde{\mathbb{E}}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{E}}_m(t)\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{E}}_1^*(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \widetilde{\mathbb{E}}_m^*(t)\end{bmatrix}\right)\begin{bmatrix}P_1(t)\\ \cdot\\ \cdot\\ \cdot\\ P_m(t)\end{bmatrix}$$

$$+\begin{bmatrix}\mathbb{Q}_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathbb{Q}_m(t)\end{bmatrix}=\begin{bmatrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix}, \qquad (9)$$

We consider a Lyapunov-like function

$$\mathbb{W}_{i,A+B}(t,y_i)=\langle P_i(t)y_i,y_i\rangle+\|y_i\|^2$$

and construct a feedback control of the form

$$u_i(t)=-\frac{1}{2}e^{-2(\omega_i+\mathbb{M}_ic_i)t}\widetilde{\mathbb{E}}_i(t)[P_i(t)-I]y_i(t), \qquad (10)$$

By taking the derivative of $W_{i,\mathrm{A+B}}(.)$ in t along the solution $y_i(t)$ of the system (6), and using the chosen feedback control and the PROE (9), we have

$$\frac{d}{dt}W_{iA+B,}(t,y_i)=\langle\frac{d}{dt}P_i(t)y_i(t),y_i(t)\rangle+2\langle P(t)\frac{d}{dt}y_i(t),y_i(t)\rangle+2\langle\frac{d}{dt}y_i(t),y_i(t)\rangle,$$

$$=\langle\left(-\left(\widetilde{\mathbb{A}}_i^*+\widetilde{\mathbb{B}}_i^*\right)P_i-P_i\left(\widetilde{\mathbb{A}}_i+\widetilde{\mathbb{B}}_i\right)+2e^{-2(\omega_i+\mathbb{M}_ic_i)t}P_i\widetilde{\mathbb{E}}_i\widetilde{\mathbb{E}}_i^*\mathbb{Q}_i\right)y_i,y_i\rangle$$

$$+2\langle P_i\left[\left(\widetilde{\mathbb{A}}_i^*+\widetilde{\mathbb{B}}_i^*\right)y_i+\widetilde{\mathbb{E}}_iu_i\right],y_i\rangle+2\langle\left(\widetilde{\mathbb{A}}_i^*+\widetilde{\mathbb{B}}_i^*\right)y_i+\widetilde{\mathbb{E}}_iu_i,y_i\rangle,$$

$$=e^{-2(\omega_i+\mathbb{M}_ic_i)t}\langle P_i\widetilde{\mathbb{E}}_i\widetilde{\mathbb{E}}_i^*P_iy_i,y_i\rangle+2\langle P_i\widetilde{\mathbb{E}}_iu_i,y_i\rangle+2\langle\left(\widetilde{\mathbb{A}}_i+\widetilde{\mathbb{B}}_i\right)y_i,y_i\rangle+2\langle\widetilde{\mathbb{E}}_iu_i,y_i\rangle-\langle\mathbb{Q}_iy_i,y_i\rangle$$

$$=-\langle\left[\mathbb{Q}_i(t)-2\left(\widetilde{\mathbb{A}}_i+\widetilde{\mathbb{B}}_i\right)(t)-e^{-2(\omega_i+\mathbb{M}_ic_i)t}\widetilde{\mathbb{E}}_i(t)\widetilde{\mathbb{E}}_i^*(t)\right]y_i(t),y_i(t)\rangle$$

$$=-\langle\left[\mathbb{Q}_i(t)-2\left(\widetilde{\mathbb{A}}_i+\widetilde{\mathbb{B}}_i\right)(t)-\mathbb{E}_i(t)\mathbb{E}_i^*(t)\right]y_i(t),y_i(t)\rangle$$

By choosing $\mathbb{Q}_i(t)$ from (7), we find $\left[\mathbb{Q}_i(t) - 2\left(\widetilde{\mathbb{A}}_i + \widetilde{\mathbb{B}}_i\right)(t) - \mathbb{E}_i(t)\mathbb{E}_i^*(t)\right] \geq 0$, and hence:

$$\frac{d}{dt}\mathbb{W}_{i,A+B_i}(t, y_i) \leq 0, \forall t \geq 0, i \in \{1, \dots, M\}, \tag{11}$$

This inequality shows the property boundedness of the solution $y_i(t)$ for all $i \in \{1, \dots, m\}$ for the nonlocal system (6).

Actually, by integrating the inequality (11) from zero to t, we obtain that:

$$\mathbb{W}_{i,A+B}(t, y_i) - \mathbb{W}_{i,A+B}(0, y_0) \leq 0,$$

and hence

$$\langle P_i(t)y_i(t), y_i(t)\rangle - \|y_i(t)\|^2 \leq \langle P_i(0)y_0, y_0\rangle - \|y_0\|^2.$$

Since $P_i(t) \geq 0$, for all $t \in \mathbb{R}^+, i \in \{1, \dots, m\}$,

$$\|y_i(t)\| \leq \langle P_i(0)y_0, y_0\rangle - \|y_0\|^2, \forall t \geq 0, i \in \{1, \dots, m\},$$

We can find a number $N > 0$ such that:

$$\|y_i(t)\| \leq N\|y_0\|, \forall t \geq 0, i \in \{1, \dots, m\}.$$

Therefore, by returning to the solution $z_i(t, z_0)$ of the perturbed linear time-varying system(1), we finally obtain that:

$$\|z_i(t, z_0)\| \leq Ne^{-(\omega_i + \mathbb{M}_i c_i)}\|z_0\|, \forall t \geq 0, i \in \{1, \dots, m\}$$

Then, the perturbed linear time-varying nonlocal system(1) is completely stabilizable by the feedback control(10) trasfromed in the state $z_i(t)$ as

$$u_i(t) = -\frac{1}{2}e^{-2(\omega_i + \mathbb{M}_i c_i)t}\widetilde{\mathbb{E}}_i(t)[P_i(t) - I]y_i(t) = \mathbb{K}_i(t)z_i(t),$$

Where $\mathbb{K}_i(t) = -\frac{1}{2}\widetilde{\mathbb{E}}_i(t)[P_i(t) - I]$, which is bounded on $\mathbb{R}^+$.

Then, the globally null-controllability implies completely stabilizable. ■

Proposition 3.2

If the perturbed linear time-varying nonlocal system (1) is globally null-controllability (GNC) with condition $\mathbb{H}$, then it is completely stabilizable (CS_z) in finite time.

Proof:

Suppose that perturbed linear time-varying system (1) completely stabilizable. By property of the perturbed evolution operator we have:

$$\exists M_i > 0, \gamma_i + M_i c_i > 0: \|\mathbb{W}_i^*(t, s)\| \leq M_i e^{(\gamma_i + M_i c_i)|t-s|}, \forall t, s \geq 0, i \in \{1, \dots, m\}$$

For chosen number $\omega_i + \mathbb{M}_i c_i > \gamma_i + M_i c_i > 0$, there is an operator$\mathbb{K}_i \in \mathcal{L}(\mathbb{H}, U)$, which is bounded on $\mathbb{R}^+$ such that the solution $z_i(t, z_0) = \mathbb{U}_{i,A_{i+}B_i+\mathbb{K}_i}(t,0)z_0$, where $\mathbb{U}_{i,A_{i+}B_i+\mathbb{K}_i}(t,0)z_0$ is the evolution operator generated by the operator $\left[\left(\mathbb{A}_i(t) + \mathbb{B}_i(t)\right) + \mathbb{K}_i(t)\mathbb{E}_i(t)\right]$, satisfies

$\exists N > 0$ such that

$$\|z_i(t,z_0)\| = \left\|\mathbb{U}_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(t,0)z_0\right\| \le Ne^{-(\omega_i+\mathbb{M}_i c_i)t}\|z_0\|, \forall t \ge 0, i \in \{1,\dots,m\} \tag{12}$$

Furthermore, for every $z_0 \in \mathbb{H}$ feedback control $u_i(t) = \mathbb{K}_i(t)z_i(t)$, the solution $z(t,z_0)$ of the perturbed linear time-varying system (1) is defined as

$$z_i(t,z_0) = \mathbb{W}_{i,\mathbb{A}_{i+}\mathbb{B}_i+}(t,0)z_0 + \int_0^t \mathbb{U}_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(t,s)\mathbb{B}_i(s)u_i(s)\,ds.$$

Therefore,

$$\mathbb{W}_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,0)z_0 = \mathbb{U}_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(t,0)z_0 - \int_0^t \mathbb{W}_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,s)\mathbb{E}_i(s)\,\mathbb{K}_i(s)\mathbb{U}_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(t,0)z_0,$$

For all $t \in \mathbb{R}^+$, and $i \in \{1,\dots,m\}$.

Since the above relation holds for every $z_0 \in \mathbb{H}$, the following estimate holds for every $z_i^* \in \mathbb{H}^*$, we have

$$\left\|\mathbb{W}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,0)z_i^*\right\| \le \left\|\mathbb{U}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(t,0)z_i^*\right\| + \int_0^t \left\|\mathbb{U}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i+\mathbb{K}_i}(s,0)\mathbb{K}_i^*(s)\mathbb{E}_i^*(s)\mathbb{W}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,s)z_i^*\right\|ds$$

By using condition (12), we obtain that:

$$\begin{aligned}\left\|\mathbb{W}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,0)z_i^*\right\| &\le Ne^{-(\omega_i+\mathbb{M}_i c_i)t}\|z_0\| + NK\int_0^t e^{-(\omega_i+\mathbb{M}_i c_i)t}\left\|\mathbb{E}_i^*(s)\mathbb{W}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,s)\right\|ds \\ &\le Ne^{-(\omega_i+\mathbb{M}_i c_i)t}\|z_0\| + NK\left(\int_0^t e^{-(\omega_i+\mathbb{M}_i c_i)t}\right)^{\frac{1}{2}} \\ &\times\left(\int_0^t \left\|\mathbb{E}_i^*(s)\mathbb{W}^*_{i,\mathbb{A}_{i+}\mathbb{B}_i}(t,s)\right\|^2 ds\right)^{\frac{1}{2}},\end{aligned} \qquad 13)$$

Where $K = \sup\{\mathbb{K}_i(s): s \in [0,\infty)\}$. Setting $\mu_i(t) = \left(\int_0^t e^{-(\omega_i+\mathbb{M}_i c_i)t}\right)^{\frac{1}{2}}$, we obtain that:

$$\mu_i(t) = \left(\frac{1}{2(\omega_i + \mathbb{M}_i c_i)} - \frac{1}{2(\omega_i + \mathbb{M}_i c_i)}e^{-2(\omega_i+\mathbb{M}_i c_i)t}\right)^{\frac{1}{2}}, \tag{14}$$

We prove the globally null-controllable of the perturbed linear time-varyingl system(1), by contradiction. Suppose that the system(1) is not globally null-controllable in any finite time $t > 0$. Then, by using remark 2, for every $t > 0$, and for some chosen number $\mathbb{b}_i > 0, \boldsymbol{\kappa} \in (0,1)$, satisfying

$$\mathbb{b}_i < \left[\frac{(1-\boldsymbol{\kappa})\sqrt{2(\omega_i + \mathbb{M}_i c_i)}}{NK}\right]^2, \tag{15}$$

there is $z_0^* \in \mathbb{H}^*$, such that:

$$\int_0^t \left\|\mathbb{B}_i^*(s)\mathbb{W}_{i,A_i+B_i}^*(t,s)z_0^*\right\| ds \le \mathbb{b}_i \left\|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\right\|^2, \tag{16}$$

This means that, the inequality (16) shows that $z_0^* \neq 0$, we may assume that the inequality (16) satisfying for all $z_0 : \|z_0^*\| = 1$; otherwise we can take $z_i^* = \frac{z_0^*}{\|z_0^*\|}$ for all $i \in [1,m] = \{1, \dots, m\}$. Therefore, by using inequality(13), (16), we obtain that:

$$\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\| < Ne^{-(\omega_i+\mathbb{M}_i c_i)t} + \sqrt{\kappa}NK\mu_i(t)\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\|, \tag{17}$$

On other hand, we observe that:

$$1 = \|z_0^*\| = \left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\| \le \left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)\right\|\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\|,$$

Which, due to (12), and $\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\| \neq 0$, we have:

$$\frac{1}{\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\|} \le \left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)\right\| < M_i e^{(\gamma_i + M_i c_i)t}, \tag{18}$$

Thus, by using (17) and (18), we obtain that:

$$1 < \frac{N_i e^{-(\omega_i+\mathbb{M}_i c_i)t}}{\left\|\mathbb{W}_{i,\mathbb{A}_i+\mathbb{B}_i}^*(t,0)z_0^*\right\|} + \sqrt{\kappa}NK\mu_i(t) < NM_i e^{-[(\omega_i+\mathbb{M}_i c_i)-(\gamma_i+M_i c_i)]t} + \sqrt{\kappa}NK\mu_i(t)$$

Hence

$$1 - \sqrt{\kappa}NK\mu_i(t) < NM_i e^{-[(\omega_i+\mathbb{M}_i c_i)-(\gamma_i+M_i c_i)]t}, \forall t > 0$$

This means that, the above relation does not depend on z_0^*, we can let t go to infinity and observation from (14) that:

$\mu_i(t) \to \left(1\Big/\sqrt{2(\omega_i + \mathbb{M}_i c_i)}\right)$, the right hand-side goes to zero because of $(\omega_i + \mathbb{M}_i c_i) > (\gamma_i + M_i c_i)$, we obtain that:

$$1 - \sqrt{\kappa}NK\frac{1}{\sqrt{2(\omega_i + \mathbb{M}_i c_i)}} \le 0.$$

Then, by using (15) it follows the condition

$$\kappa < 1 - \sqrt{\kappa} NK \frac{1}{\sqrt{2(\omega_i + \mathbb{M}_i c_i)}} \leq 0$$

Which is contradiction. This means that the perturbed linear time-varying nonlocal system(1) is globally null-controllability in finite time.

Theorem 3.1

The perturbed linear time-varying system (1) is completely stabilizable (CS_z) iff it is globally null-controllability (GNC) in finite time. Conversely, assume the condition $\mathbb{H}$, the perturbed linear time-varying nonlocal system (1) is globally null-controllability (GNC) if it is completely stabilizable (CS_z) in finite time.

Proof:

The proof is direct way on proposition 3.1, and proposition 3.2. ■

4. Conclusions

In this paper, we have established the equivalence of globally null controllability and complete stabilizability for perturbed linear time-varying control systems in real Hilbert space. The result obtained by extends existing results in the literature to infinite-dimensional and time-varying perturbed control systems.

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