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The Arbitrage In Securities Market Model And Some There Properties

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ABSTRACT

The main objective of this paper is to prove the economic equilibrium of the securities market by relying on the concepts of functional analysis and building a financial market model (X, τ, K, M, π) where X is a real linear space and τ is a topology related to X and K represents the positive cone and M a subspace of X and π is a real linear function on M. Some important mathematical and economic definitions of the research project have been mentioned , such as the viability of the securities market and how to expand the securities market to include all securities and studying the necessary and sufficient condition to make the expanded securities market viable. Arbitrage has also been defined in the securities market and the study of its non-realization in the market and the knowledge of the necessary conditions for the non-realization of arbitrage , as the viability of securities market achieves non-arbitrage , and the necessary condition on which the price of any stock is calculated through arbitrage has been identified. The market arbitrage has been studied in the event that there are fixed transaction costs and in the absence of fixed costs.

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1. Introduction

Mathematical modeling of natural phenomena at the present time is one of the most important fields of scientific research, an activity, an acceleration, and an interest in growth and development, although this convergence between mathematics and various other sciences has been somewhat delayed if compared to the close relationship that linked mathematics with many other sciences such as physics, chemistry and engineering since its inception and its establishment as a distinguished research science. Perhaps this delay in the convergence between mathematics and economics is due to many reasons, including the lack of cognitive excitement that motivates the researcher on either side to bear the trouble of joint research in both fields. The scientific research between the mathematical and economic fields must have results of economic significance in order to gain the desired importance.

The applications of functional analysis in economics began to work since the eighties of the last century by providing theoretical studies related to the development and balance of financial

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markets through the use of the concepts of functional analysis and statistical concepts to make the market achieve growth and non-arbitrage.
Debreu (1959) introduced the basic theory of pricing origins and others such as Cox and Rose (1979), Harrison and Krebs 1979), Harrison and Pliska (1981), Duffy and Huang (1986) Morten and Willinger (1989), also Buck and Pliska (1990) and Delbaen add new results On the origins of pricing, and from these results, the existence of an equivalent Martingale measure is equivalent to arbitrage. The non-arbitrage and price growth rule for one group of securities can be achieved to market other securities which are a linear space and thus equal to the expected value of any equivalent Martingale measure.

2- *Basic concepts*

***Definition* (2.1) [8]**

A topological linear space (X,τ) is said an ordered topological linear space if X is an ordered linear space with positive cone K , we symbolizes by (X,K,τ) or $\mathcal{H}$.

***Definition* : (2.2) [2]**

A market model is an order quintile (X,τ,K,M,π) where X is $\mathcal{H}$ equipped with a locally convex topology τ , K is positive cone (with the origin deleted) , M is a subspace of X and a linear functional $\pi : M \to R$.

***Definition* : (2.3) [1]**

A price system is an order pair (M,π) where M is a subspace of X and $\pi : M \to R$ is a linear functional.

***Remark* :**

A linear functional $f : X \to R$ is said to be :

(1) K – positive if $f(x) \geq 0$ for all $x \in K$

(2) K- strictly positive if $f(x) > 0$ for all $x \in K$

***Remark* :**

Let Φ be the set of τ – continuous and K – positive linear functional on X and let Ψ be the set of τ – continuous and K – strictly positive linear functional on X , i.e

$\Phi = \{f \in X^* : f \text{ is K – positive}\}$,

$\Psi = \{f \in X^* : f \text{ is K – strictly positive}\}$

***Definition* (2.4) [7]**

A filtration on the probability space $(\Omega, \mathcal{F}, P)$ is a sequence $\{\mathcal{F}_t\}_{t\in I}$ of sub $\sigma -$ field of $\mathcal{F}$ such that for all $t \in I$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

***Definition* (2.5) [7]**

A real random variable is a measurable function $\chi : \Omega \to R$ on the probability space $(\Omega, \mathcal{F}, P)$, for each Borel set $A \in R$ and then $\chi^{-1}(A) = \{\omega: \chi(\omega) \in A\}$ is $\mathcal{F}$ measurable .

***Definition* (2.6) [7]**

A stochastic process can be defined as a collection of random variables denoted by $\{\chi_t\}_{t\in I}$ where $\subset R$.

***Definition* (2.7) [7]**

A stochastic process $\chi = \{\chi_t\}_{t\in I}$ is called in adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if for all $t \in I$ the random variable χ_t is $\mathcal{F}_t$ −measurable.

***Remark* :**

A price process is a stochastic process denoted by $Z = \{Z_t\}_{t\in I}$ which is adapted with filtration $\{\mathcal{F}_t\}_{t\in I}$, i.e Z_t is $\mathcal{F}_t$ − measurable.

***Definition* (2.8) [3]**

A simple strategy is $\mathbb{R}^{d+1}$ − valued stochastic process $\theta = (\theta_t)_{t\in I}$ that satisfies the following conditions :

1- θ is denoted to $\mathrm{F} = \{\mathcal{F}_t\}_{t\in I}$, i.e $\theta_t \in \mathcal{F}_t$ for each $t \in I$.

2- $E\left(\left(\theta_t^i Z_t^i\right)^2\right) < \infty$, for each $t \in I$ and $= 0,1,2, \dots \dots . d$.

3- there exists a finite integer k and a sequence of dates

$0 = t_0 < t_1 < \cdots \dots \dots \dots . . < t_k = T$ such that $t_n \in I$ and $\theta_t(\omega)$ is a constant over the interval $t_{n-1} < t < t_n$ i.e $[t_{n-1}, t_n)$ for every state $\omega (n = 1,2, \dots . . , k)$

***Remark* : assumption $\mathfrak{I}$**

$\mathfrak{I}$: family of preference relations $(\geq^*)$ satisfy three properties (convex , $\tau -$ continuos , $K -$ increasing)

***Definition* (2.9) [1][2]**

The market model (X, τ, K, M, π) is viable if there exists some $\geq^* \in \mathfrak{I}$ and some $m^* \in M$ such that :

(1) $\pi(m^*) \leq 0$ **(2)** $m^* \geq^* m$ for all $m \in M$ with $\pi(m) \leq 0$

The equivalent condition to make (M, π) viable that if the linear functional π has extension property i.e there exists ψ such that $\psi \setminus M = \pi$ where ψ is strictly positive linear functional .

Suppose that a market model (X, τ, K, M, π) is given and $\notin M$, then x can be bought and sold in same other market by the price p , so we can define :

$\mathrm{E} = [M \cup \{x\}] = \{m + \lambda x : m \in M , \lambda \in R\}$ and define :

$\Pi : \mathrm{E} \to \mathrm{R}$ by $\Pi(m + \lambda x) = \pi(m) + \lambda p$

***Definition* (2.10) [6]**

We say that a price p is consistent with (X, τ, K, M, π) if the extended market model $(X, \tau, K, \mathrm{E}, \Pi)$ is viable.

Remark :

We denote by :

- $C(x)$ the set of all consistent price of x , i.e :

$C(x) = \{p \in R : p\ is$ price of $x\ consistent\ with (X, \tau, K, M, \pi)\}$

- $\Sigma(x)$ the set of all price of x , i.e :

$\Sigma(x) = \{\psi(x) : \psi \in \Psi\ and\ \psi \setminus M = \pi\}$

***Theorem* (2.11)**

If (X, τ, K, M, π) is viable , then $C(x) = \Sigma(x)$ for all $x \in X$.

Proof :

Let $x \in X$, $x \notin M$
Since (X,τ,K,M,π) is viable , then there exists $\psi \in \Psi$ such that $\psi \setminus M = \pi$
So : $\psi\,(m+\lambda x) = \psi\,(m) + \lambda\psi\,(x)$ (since ψ linear functional)
$= \psi\,(m) + \lambda p$ (where $\psi\,(x) = p$)
$\Longrightarrow \psi\,(m+\lambda x) = \Pi\,(m+\lambda x)$
$\Longrightarrow \psi \setminus \mathrm{E} = \Pi$ $\Longrightarrow (X,\tau,K,\mathrm{E},\Pi)$ is viable
$\Longrightarrow p = \psi\,(x)$ is price for x consistent with(X,τ,K,M,π)
$\Longrightarrow p \in C(x)$ $\Longrightarrow C(x) \neq \emptyset$
Since $\psi \in \Psi$ such that $\psi \setminus M = \pi$ $\Longrightarrow \psi \in \Sigma(x)$
Since $\psi\,(x) = p$ $\Longrightarrow p \in \Sigma(x)$ therefore $C(x) = \Sigma(x)$

Remark :
By above theorem we say the price p of x is consistent with (X,τ,K,M,π) if there exists $\psi \in \Psi$ such that $\psi \setminus M = \pi$ and $\psi\,(x) = p$.

Remark :
For $x \in X$, define :

1. $\overline{\pi}\,(x) = \inf\{\lim_\alpha \inf \pi(m_\alpha) : m_\alpha \geq x_\alpha \to x\}$
2. $\underline{\pi}\,(x) = \sup\{\lim_\alpha \sup \pi(m_\alpha) : m_\alpha \leq x_\alpha \to x\}$
3. If no such nets exists ,then $\underline{\pi}\,(x) = \infty$ *and* $\overline{\pi}\,(x) = \infty$

***Theorem* (2.12)**
If $(X,\tau,K,\mathrm{E},\Pi)$ is viable and $\overline{\pi}\,(x) > \underline{\pi}\,(x)$, then any price $p \in \left(\underline{\pi}\,(x), \overline{\pi}\,(x)\right)$ is a price of x consistent with (X,τ,K,M,π).

Proof :
Since $(X,\tau,K,\mathrm{E},\Pi)$ is viable , then there exists $\psi \in \Phi$ such that $\psi \setminus M = \pi$
$\Longrightarrow \underline{\pi} \leq \psi \leq \overline{\pi}$ $\Longrightarrow \psi\,(x) \in \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$ and $\psi\,(x)$ is finite .
$\Longrightarrow$ for any $p \in \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$, there exists $p' \in \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$ and $\lambda \in (0,1)$ such that :
$\lambda p' + (1-\lambda)\psi\,(x) = p$
Since $\overline{\pi}\,(x) > 0$ $\forall x \in K$, then $x \in \mathrm{E}$
Then there is $\psi\,(x) = p \in C(x)$, $\forall x \in X$
$\Longrightarrow (X,\tau,K,M,\pi)$ is viable
$\Longrightarrow \exists\, \varphi \in \Phi$ such that $\varphi \setminus M = \pi$ and $\varphi(x) = p'$
But $\psi' = \lambda\varphi + (1-\lambda)\psi$ where $\psi' \in \Psi$ and $\psi' \setminus M = \pi$
Thus $\psi'(x) = p$

<u>3.1 - *The Arbitrage*</u>

***Definition* (3.1.1)[5]**
We say that the price of x is determined by arbitrage from (X,τ,K,M,π) if there is a single p of x that is consistent with (X,τ,K,M,π) ,i.e [$C(x)$ is singleton] , and this unique of x is called the arbitrage value of x.

***Theorem* (3.1.2)**
If $(X,\tau,K,\mathrm{E},\Pi)$ is viable , then the price of x is determined by arbitrage from (X,τ,K,M,π) iff $\overline{\pi}\,(x) = \underline{\pi}\,(x)$.

Proof :
Since $(X, \tau, K, \mathrm{E}, \Pi)$ is viable ,
Then there exists $\psi \in \Psi$ such that $\psi \setminus M = \pi$ and $\psi(x) = p \quad \Rightarrow p \in C(x)$
Since x is determined by arbitrage from (X, τ, K, M, π)
$\Rightarrow C(x)$ is singleton
$\Rightarrow p$ is consistent with (X, τ, K, M, π)
$\Rightarrow x \in K$ where K is positive cone .
$\Rightarrow \overline{\pi}\,(x) > 0$ *and* $\underline{\pi}\,(x) > 0$ for all $x \in K$
Since p is arbitrage value and unique , also $p \in \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$
$\Rightarrow \underline{\pi}\,(x) = \overline{\pi}\,(x)$

Conversely :
Let $x \in X \quad , x \notin M$
p is price of x consistent with (X, τ, K, M, π) , then there exists $\psi \in \Psi$ such that $\psi \setminus M = \pi$ and $\psi\,(x) = p$.
Since $\psi\,(x) \in \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$ and $\underline{\pi}\,(x) = \overline{\pi}\,(x)$
$\quad \Rightarrow \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right]$ is singleton
$\quad \Rightarrow \left[\underline{\pi}\,(x), \overline{\pi}\,(x)\right] = \{p\} = C(x)$
Therefore the price of x is determined by arbitrage .
Since $\psi\,(x) = p$ for all $x \in X \quad , x \notin M$
Then we can extended the market model as : $\mathrm{E} = [M \cup \{x\}] \quad , x \notin M$ and $\Pi : \mathrm{E} \to R$ where $\Pi(m + \lambda x) = \pi(m) + \lambda p$
Where $m \in M$ *and* $p = \psi\,(x) \quad , \forall x \notin M$
So $(X, \tau, K, \mathrm{E}, \Pi)$ is viable

***Definition* (3.1.3)** [5]
A positive element $x \in K$ is called an arbitrage opportunity if $\pi(x) = 0$.
The market model(X, τ, K, M, π) is free of arbitrage (or no arbitrage) if no such opportunity exists .
In other words , there are no arbitrage opportunity if :
1) π is strictly positive linear functional on M .
2) If $\in M \cap K$, then $\pi(x) > 0$.

***Theorem* (3.1.4)**
The market model(X, τ, K, M, π) satisfies the condition (NA) of no arbitrage iff $M_{\circ} \cap K = \emptyset$.

Proof :
Suppose The market model(X, τ, K, M, π) satisfies the (NA) condition
Then π is strictly positive linear functional on M and if $x \in M \cap K$
$\Rightarrow \ x \in M$ *and* $x \in K \quad \Rightarrow \ \pi(x) > 0$
Since $M_{\circ} = ker(\pi) = \{y \in X : \pi(y) = 0\}$
$\Rightarrow \ x \notin M_{\circ} \ \Rightarrow \ M_{\circ} \cap K = \emptyset$

Conversely :
Let $M_{\circ} \cap K \neq \emptyset \ \Rightarrow \ \exists x \in M_{\circ} \cap K$
$\Rightarrow \ x \in M_{\circ}$ *and* $x \in K$
$\Rightarrow \ \pi(x) = 0$ and x is positive
$\Rightarrow \ x$ is arbitrage opportunity
Therefore if $M_{\circ} \cap K = \emptyset$, then The market model(X, τ, K, M, π) satisfies (NA) conditions .

3.2 - *The Arbitrage with fixed costs*

***Definition* (3.2.1) [4]**

An arbitrage opportunity with fixed costs $[\mathcal{A}^1]$ is a strategy θ such that exist $i\,,j$ in $I=[0,T]$ where $0\le i\le j\le T$ and event $B\in\mathcal{F}_i$, for which $\bar{\theta}$ is null after date j , $\theta\in\mathcal{R}$, $V_i^\theta+C_i^\theta\le 0$ on B , $V_j^\theta>0$ and either $V_i^\theta+C_i^\theta$ or V_j^θ is deferent from 0 . Where $\mathcal{R}$ the set of all strategies with fixed costs

***Theorem* (3.2.2)**

There exists an $[\mathcal{A}^1]$ if and only if there exists a net gain arbitrage opportunity with fixed costs .

Proof :

Suppose that exists $[\mathcal{A}^1]$

Then there is $\theta\in\mathcal{R}$ and dates $i\,,j\in\ [0,T]$ such that $0\le i\le j\le T$ and event $B\ in\ \mathcal{F}_i$ and $\bar{\theta}$ is null after date j where $V_i^\theta+C_i^\theta\le 0$ on , $V_j^\theta>0$

Set $\theta\in\mathcal{R}$ and $=\{\theta_t^n : t\in I\}\,, n=1,2,\dots..$

Define the function $\mu:(\mathrm{N},\ge)\to X$, where N is the set of natural number , $\ge$ is relation on N and X is non empty set .

To prove that μ is a net we should prove that $(\mathrm{N},\ge)$ is directed set .

- $\forall\ a\in N\ \Longrightarrow a\ \ge a\ \Longrightarrow\ \ge$ is reflexive
- $\forall\ a,b,c\in N\ \ and\ a\ge b\ \wedge b\ \ge c\Longrightarrow a\ge c\ \Longrightarrow\ \ge$ is transitive
- $\forall\ a,b\in N\ \Longrightarrow\ \exists\ d\in N\ such\ that\ d\ge a\ \wedge d\ \ge b$

Therefore $(\mathrm{N},\ge)$ is directed set .

$\Longrightarrow \forall n\in N\ \exists!\,\theta^n\ \in\mathcal{R}\ \ such\ that\ \mu(n)=\ \theta^n$

Since $V_i^\theta+C_i^\theta\le 0\ \Longrightarrow\ V_i^{\theta^n}+C_i^{\theta^n}\le 0$, and $V_j^\theta>0\ \Longrightarrow\ V_j^\theta>0$, $\forall n$

$\Longrightarrow \theta^n\in\mathcal{R}$, then there is $B\in\ \mathcal{F}_i$ and sequence $(\epsilon_i^n)_{n\in N}$ of random variable convergence to $\epsilon_i>0$ on B i.e $(\epsilon_i^n\to\epsilon_i\ as\ n\to\infty)$

So there is a sequence of trading strategies $(\theta^n)_{n\in N}$ with fixed costs

Then μ is a net of arbitrage opportunity with fixed costs

Conversely :

Suppose that there is a net μ of arbitrage opportunity with fixed costs ϵ_i^n on B (is $\mathcal{F}_i$ measurable)

$\Longrightarrow \forall\ n\in N\ \exists!\,\theta^n\ \in\mathcal{R}\ \ such\ that\ \mu(n)=\ \theta^n$

$\Longrightarrow \mu^{-1}(\theta^n)=n \qquad ,n=1,2,3\dots\dots..$

Then there is a sequence of trading strategies $(\theta^n)_{n\in N}$ in $\mathcal{R}$

Let there is a date $j\in I=[0,T]$ such that $0\le i\le j\le T$ $\overline{\theta^n}$ is null after date j Since $(\epsilon_i^n)_{n\in N}$ be a sequences of random variables in $L^1(\Omega,\mathcal{F}_i,P)$ and converging to $\epsilon_i>0$ on B belong to $L^1(\Omega,\mathcal{F},P)$

$\Longrightarrow \epsilon_i^n\ \le\epsilon_i\quad \forall n\in N\ \ and\ \ \epsilon_i>0$, and $V_i^{\theta^n}\le 0\ \Longrightarrow\ V_i^{\theta^n}+C_i^{\theta^n}\le 0$

Since $=\{\theta_t^n : t\in I\}$, and $\bar{\theta}$ is null after date $j\quad\Longrightarrow\quad V_i^\theta\ +\ C_i^\theta\ \le 0$

Now to prove that $V_j^\theta>0 \qquad , \ 0\le i\le j\le T$

Since $\Delta V_t^\theta=\ V_t^\theta-\ V_{t-1}^\theta$ (since $i\le j$ and $V_t^\theta=\theta_t\cdot Z_t$)

$$\begin{aligned}\Longrightarrow \Delta V_j^\theta=\ V_j^\theta-\ V_i^\theta &=\ \theta_j\cdot Z_j-\theta_i\cdot Z_i\\ &=\ \theta_j\cdot Z_j-\theta_j\cdot Z_i-\theta_j\cdot Z_i+\theta_i\cdot Z_i\\ &=\ \theta_j\cdot\left(Z_j-Z_i\right)+\left(\theta_j-\theta_i\right)\cdot Z_i\\ &=\ \theta_j\cdot\Delta Z_j+\Delta\theta_j\cdot Z_i\end{aligned}$$

Since θ is constant for all t where $t_{n-1}<t\le t_n\ \Longrightarrow \Delta\theta_j=0$

$\Longrightarrow \Delta V_j^\theta = \theta_j \cdot \Delta Z_j > 0$, then we have $V_j^\theta > 0$
Since V^θ is portfolio value $\Longrightarrow V_i^\theta + C_i^\theta$ or V_j^θ is deferent from 0
Hence there exists an $[\mathcal{A}^1]$.

***Definition* (3.2.3) [4]**
A frictionless strong arbitrage opportunity $[\mathcal{A}^2]$ is a strategy θ such that exist i , j in $I = [0, T]$ where $0 \le i \le j \le T$ and event $B \in \mathcal{F}_i$, for which $\bar{\theta}$ is null after date j , $\theta \in \mathcal{B}$, $V_i^\theta < 0$ on B , $V_j^\theta \ge 0$.
Where $\mathcal{B}$ the set of all strategies without fixed costs .

***Theorem* (3.2.4)**
There exists an $[\mathcal{A}^2]$ if and only if there exists a frictionless net gain arbitrage opportunity.

Proof :
Suppose that exists $[\mathcal{A}^2]$
Then there is $\theta \in \mathcal{B}$ and dates i , $j \in [0, T]$ such that $0 \le i \le j \le T$ and event B *in* $\mathcal{F}_i$ and $\bar{\theta}$ is null after date j where $V_i^\theta < 0$ on , $V_j^\theta \ge 0$
Set $\theta \in \mathcal{B}$ and $= \{\theta_t^n : t \in I\}$, $n = 1,2, \dots ..$
Define the function $\mu : (\mathrm{N}, \ge) \to X$, where N is the set of natural number , $\ge$ is relation on N and X is non empty set .
To prove that μ is a net we should prove that $(\mathrm{N}, \ge)$ is directed set .

- $\forall\, a \in N \Longrightarrow a \ge a \Longrightarrow \ge$ is reflexive
- $\forall\, a, b, c \in N$ *and* $a \ge b \wedge b \ge c \Longrightarrow a \ge c \Longrightarrow \ge$ is transitive
- $\forall\, a, b \in N \Longrightarrow \exists\, d \in N$ *such that* $d \ge a \wedge d \ge b$

Therefore $(\mathrm{N}, \ge)$ is directed set .
$\Longrightarrow \forall n \in N\ \exists!\, \theta^n \in \mathcal{B}$ *such that* $\mu(n) = \theta^n$
Since $V_i^\theta < 0 \Longrightarrow V_i^{\theta^n} < 0$ and $V_j^\theta \ge 0 \Longrightarrow V_j^\theta \ge 0$, $\forall n$
So there is a sequence of trading strategies $(\theta^n)_{n \in N}$
Then μ is a net of frictionless arbitrage opportunity .

Conversely :
Suppose that there is a net μ of frictionless arbitrage opportunity
$\Longrightarrow \forall\, n \in N\ \exists!\, \theta^n \in \mathcal{B}$ *such that* $\mu(n) = \theta^n$
$\Longrightarrow \mu^{-1}(\theta^n) = n \qquad , n = 1,2,3 \dots \dots ..$
Then there is a sequence of trading strategies $(\theta^n)_{n \in N}$ in $\mathcal{B}$
Let there is a date $j \in I = [0, T]$ such that $0 \le i \le j \le T$ $\overline{\theta^n}$ is null after date j And $V_i^{\theta^n} < 0$, since $= \{\theta_t^n : t \in I\}$, and $\bar{\theta}$ is null after date j
$\Longrightarrow \quad V_i^\theta < 0$
Now to prove that $V_j^\theta \ge 0 \qquad , 0 \le i \le j \le T$
Since $\Delta V_t^\theta = V_t^\theta - V_{t-1}^\theta$ (since $i \le j$ and $V_t^\theta = \theta_t \cdot Z_t$)

$$\begin{aligned}
\Longrightarrow \Delta V_j^\theta = V_j^\theta - V_i^\theta &= \theta_j \cdot Z_j - \theta_i \cdot Z_i \\
&= \theta_j \cdot Z_j - \theta_j \cdot Z_i - \theta_j \cdot Z_i + \theta_i \cdot Z_i \\
&= \theta_j \cdot (Z_j - Z_i) + (\theta_j - \theta_i) \cdot Z_i \\
&= \theta_j \cdot \Delta Z_j + \Delta \theta_j \cdot Z_i
\end{aligned}$$

Since θ is constant for all t where $t_{n-1} < t \le t_n \Longrightarrow \Delta \theta_j = 0$
$\Longrightarrow \Delta V_j^\theta = \theta_j \cdot \Delta Z_j \ge 0$, then we have $V_j^\theta \ge 0$, Hence there exists an $[\mathcal{A}^2]$.

Proposition **(3.2.5)**
There exists an $[\mathcal{A}^1]$ if and only if there exists an $[\mathcal{A}^2]$.

Proof :
Suppose there exists an $[\mathcal{A}^1]$,
Then there is a strategy $\theta \in \mathcal{R}$ and event $in\ \mathcal{F}_i$, also dates $i , j \in [0, T]$ such that $0 \leq i \leq j \leq T$ and $\bar{\theta}$ is null after date j where $V_i^\theta + C_i^\theta \leq 0$ on B , $V_j^\theta > 0$
If $C_i^\theta = 0$ for all $i \in I = [0, T] \Longrightarrow \theta$ is frictionless
$\Longrightarrow V_i^\theta + C_i^\theta = V_i^\theta + 0 = V_i^\theta \leq 0$
Since either $V_i^\theta + C_i^\theta$ or V_j^θ is deferent from 0 $\Longrightarrow V_j^\theta > 0$
Hence the strategy θ is frictionless arbitrage opportunity $[\mathcal{A}^2]$.

Conversely :
Let there is exists $[\mathcal{A}^2]$
Then there is $\theta \in \mathcal{B}$ and dates $i , j \in [0, T]$ such that $0 \leq i \leq j \leq T$ and event $B\ in\ \mathcal{F}_i$ and $\bar{\theta}$ is null after date j where $V_i^\theta < 0$ on , $V_j^\theta \geq 0$
Let there is a sequences $(\epsilon_i^n)_{n \in N}$ of random variables in $L^1(\Omega, \mathcal{F}_i, P)$ and converging to $\epsilon_i > 0$ on B belong to $L^1(\Omega, \mathcal{F}, P)$
Since $V_i^\theta < 0$ and $\epsilon_i^n \to \epsilon_i > 0$, $\epsilon_i^n \leq \epsilon_i$
$\Longrightarrow V_i^{\theta^n} + \epsilon_i^{\theta^n} \leq \epsilon_i \Longrightarrow V_i^{\theta^n} + \epsilon_i^{\theta^n} \leq 0$
Since $\theta = \{\theta_t^n : t \in I\} \Longrightarrow V_i^\theta + \epsilon_i^\theta \leq 0$ …….(1)
Since $V_j^\theta > 0$ …………(2)
Since V^θ is portfolio value $\Longrightarrow V_i^\theta + C_i^\theta$ or V_j^θ is deferent from 0
Then from (1) , (2) we have $\theta \in \mathcal{R}$ and θ is $[\mathcal{A}^1]$

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