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ABSTRACT

Let R be an associative ring with center Z(R) , I be a nonzero ideal of R and θ be an automorphism of R . An 3-additive mapping M:RxRxR →R is called a symmetric left θ-3-centralizer if $M(u_1y,u_2,u_3)=M(u_1,u_2,u_3)\theta(y)$ holds for all y, u_1, u_2, u_3 ∈R . In this paper , we shall investigate the commutativity of prime rings admitting symmetric left θ-3-centralizer satisfying any one of the following conditions :

(i)$M([u ,y], u_2, u_3) \pm [\theta(u), \theta(y)] = 0$, (ii)$M((u \circ y), u_2, u_3) \pm (\theta(u) \circ \theta(y)) = 0$

(iii)$M(u^2, u_2, u_3) \pm \theta(u^2) = 0$, (iv) $M(u y, u_2, u_3) \pm \theta(uy) = 0$

(v) $M(u y, u_2, u_3) \pm \theta(uy) \in Z(R)$

For all u_2,u_3 ∈R and u ,y ∈ I

1.Introduction

Let R be an associative ring with center Z(R) . In [3], Ashraf , and Ali, investigate the commutativity of prime rings satisfying certain identities involving left multiplier on a nonzero ideal . In [1] , Ali and Huang , get many sufficient conditions of commutativity on left α-multipliers when R is semiprime ring on a nonzero ideal . In [2] , Abduljaleel , and Majeed, discussed the commutativity of prime ring admitting a right α- centralizer where α is an endomorphism of R on a lie ideal . For more information see [5-9] . In[10] , Saed , study the commutativity of prime Γ-ring by using the notion of symmetric left Γ-n- centralizers .

This paper is organized as follows . In section two , we recall some well-known definitions and examples that will be used in this paper . In section three , we present the notion of symmetric left θ-3-centralizer of R , where θ is a mapping on R , and study the commutativity of prime rings

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admitting a symmetric left θ-3-centralizer , where θ is an automorphism of R satisfying many conditions on a nonzero ideal .

2. Basic Concept

Definition 2.1:[4] A nonempty set R is said to be associative ring if in R there are defined two operations , denoted " + " and " . " respectively, such that for all a,b,c in R:

1. a+b is in R

2. a +b= b +a

3. (a +b) +c = a +(b +c)

4. There is an element 0 in R such a +0 = a (for every a in R)

5. There exists an element -a in R such that a +(-a) = 0

6. a .b is in R

7. a.(b.c) = (a.b).c

8. a. (b +c) = a.b + a.c and (b +c).a = b.c + c.a

Definition 2.2:[4] A subring I of a ring R is called (two-sided) ideal of R if for every r∈R and every a∈ I , ra ∈ I and ar ∈ I .

Examples 2.3:[4] (1)In a ring R the trivial subrings : R and {0} are both ideals .

(2)The set of even integers is an ideal in the ring of integers .

Definition 2.4:[4] Let R be a ring , the center of R denoted by Z(R) and is defined by :
Z(R) = { x∈R : xr = rx , for all r∈R } .

Definition 2.5:[3] A ring R is called a prime ring if for any a,b ∈R , aRb = {0} implies that either a= 0 or b= 0 .

Example 2.6:[3] The ring of real numbers with the usual operation of addition and multiplication is prime ring .

Definition 2.7:[3] Let R be a ring . An additive mapping H : R →R is called a left(resp. right) centralizer of R if H(xy) = H(x)y (resp. H(xy) = xH(y)), holds for all x,y ∈R .A centralizer of a ring R is both left and right centralizer .

Definition 2.8:[1] Let R be a ring . An additive mapping H : R →R is called a left(resp. right)α-centralizer of R if H(xy) = H(x) α(y) (resp. H(xy) = α(x) H(y)) holds for all x,y ∈R , where α is an endomorphism of R . A α- centralizer of a ring R is both left and right α- centralizer .

Definition 2.9:[2] Let R be a ring . For any x,y ∈R , the symbol [x,y] will denote the commutator xy – yx and the symbol x ∘ y will denote the anticommutator xy + yx .

Remark 2.10 :[2]

Let R be a ring and x ,y, z ∈R , then

(i) $[xy, z] = x[y, z] + [x, z]y$

(ii) $[x, yz] = y[x, z] + [x, y]z$

(iii) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$

(iv) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$

(v) $(xy) \circ z = x(y \circ z)$

Definition 2.11:[10] Let M be a Γ-ring and n be a fixed positive integer . An n-additive mapping T: MxMxMx….xM → M is said to be left Γ-n- centralizer if the following equations hold for all y , $r_1, r_2, \ldots, r_n \in M$ and $\gamma \in \Gamma$.

$T_1(r_1 \gamma y, r_2, \ldots, r_n) = T_1(r_1, r_2, \ldots, r_n)\gamma y$

$T_2(r_1, r_2 \gamma y, \ldots, r_n) = T_2(r_1, r_2, \ldots, r_n)\gamma y$

⋮

$T_n(r_1, r_2, \ldots, r_n \gamma y) = T_n(r_1, r_2, \ldots, r_n)\gamma y$

T is said to be a symmetric left Γ-n- centralizers if all the above equations are equivalent to all other .That is ,

$T(r_1 \gamma y, r_2, \ldots, r_n) = T(r_1, r_2, \ldots, r_n)\gamma y$

For all $y, r_1, r_2, \ldots, r_n \in M$ and $\gamma \in \Gamma$.

3. On Symmetric Left θ-3-Centralizers and Commutativity of Prime Rings

Now, we introduce the concept of symmetric left θ-3-centralizer

Definition 3.1 :

Let R be a ring and θ is a mapping on R . An 3-additive mapping M : RxRxR →R is said to be left θ-3-centralizer if the following equations hold for all $y, u_1, u_2, u_3 \in R$

$M_1(u_1 y, u_2, u_3) = M_1(u_1, u_2, u_3)\theta(y)$

$M_2(u_1, u_2 y, u_3) = M_2(u_1, u_2, u_3)\theta(y)$

$M_3(u_1, u_2, u_3 y) = M_3(u_1, u_2, u_3)\theta(y)$

M is said to be a symmetric left θ-3-centralizer if all the above equations are equivalent to all other . That is ,

M ($u_1 y$, u_2 , u_3) = M (u_1 , u_2 , u_3) θ(y)

For all y , u_1 , u_2 , u_3 ∈ R .

Example 3.2: Consider R = $\left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}\right\}$ be a ring and $\mathbb{Z}$ is a ring of integer numbers . Let M : RxRxR →R be a mapping defined by

M $\left(\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1 a_2 a_3 \\ 0 & 0 \end{pmatrix}$, for all

$\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & 0 \end{pmatrix} \in$ R

And θ : R →R is defined by

$\theta\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, for $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in$ R

Then M is a symmetric left θ-3-centralizer .

Theorem 3.3 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ-3-centralizer M such that M (u , u_2 , u_3) $\neq$ θ(u), for all u∈I . Further , if M([u ,y], u_2, u_3) - [θ(u), θ(y)] = 0 , for all u_2,u_3 ∈R and u ,y ∈ I , then R is commutative .

Proof :

By the given hypothesis we have

M([u ,y], u_2, u_3) - [θ(u), θ(y)] = 0 , for all u_2,u_3 ∈R and u ,y ∈ I (3.1)

This can be rewritten as

(M (u , u_2 , u_3) - θ(u)) θ(y) – (M (y , u_2 , u_3) - θ(y)) θ(u) = 0 (3.2)

Replacing u by ur in (3.2) , where r∈R , we find that

(M (u , u_2, u_3) - θ(u)) θ(r) θ(y) – (M (y, u_2, u_3) - θ(y)) θ(u) θ(r) = 0 (3.3)

Multiply equation (3.2) from the right side by θ(r)

(M (u, u_2, u_3) - θ(u)) θ(y) θ(r) – (M (y, u_2, u_3) - θ(y)) θ(u) θ(r) = 0 (3.4)

Comparing equation (3.3) and (3.4) , we get

(M (u , u_2, u_3) - θ(u))(θ(r) θ(y) - θ(y) θ(r)) =

(M (u , u_2, u_3) - θ(u)) [θ(r), θ(y)] = 0 (3.5)

Again , replace r by rs in (3.5) , where s∈R , and using it , to get

$(M(u, u_2, u_3) - \theta(u))\, \theta(r)\, [\theta(s), \theta(y)] = 0$

i.e., $(M(u, u_2, u_3) - \theta(u))\, R[\theta(s), \theta(y)] = \{0\}$

Since R is prime ring , then either $(M(u, u_2, u_3) - \theta(u)) = 0$ or $[\theta(s), \theta(y)] = 0$, for all $s, u_2, u_3 \in R$ and $u, y \in I$.

Thus application of our hypotheses implies that $[\theta(s), \theta(y)] = 0$, for all $y \in I$ and $s \in R$ and therefore $I \subseteq Z(R)$.

Hence , R is commutative .

Using similar arguments as used in proof of the Theorem 3.3 , we can prove the following :

Theorem 3.4 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ-3-centralizer M such that $M(u, u_2, u_3) \neq -\theta(u)$, for all u∈I . Further , if $M([u, y], u_2, u_3) + [\theta(u), \theta(y)] = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative .

Theorem 3.5 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ-3-centralizer M such that $M(u, u_2, u_3) \neq \theta(u)$, for all u∈I . Further , if $M((u \circ y), u_2, u_3) - (\theta(u) \circ \theta(y)) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative

Proof :

By the hypotheses , we have

$M((u \circ y), u_2, u_3) - (\theta(u) \circ \theta(y)) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$ (3.6)

This implies that

$(M(u, u_2, u_3) - \theta(u))\, \theta(y) + (M(y, u_2, u_3) - \theta(y))\, \theta(u) = 0$ (3.7)

Replacing u by ur in (3.7) , where r∈R , we obtain

$(M(u, u_2, u_3) - \theta(u))\, \theta(r)\, \theta(y) + (M(y, u_2, u_3) - \theta(y))\, \theta(u)\, \theta(r) = 0$ (3.8)

Multiply equation (3.7) from the right side by $\theta(r)$

$(M(u, u_2, u_3) - \theta(u))\, \theta(y)\, \theta(r) + (M(y, u_2, u_3) - \theta(y))\, \theta(u)\, \theta(r) = 0$ (3.9)

Comparing equation (3.8) and (3.9) , we get

$(M(u, u_2, u_3) - \theta(u))(\theta(r)\, \theta(y) - \theta(y)\, \theta(r)) =$

$(M(u, u_2, u_3) - \theta(u))\, [\theta(r), \theta(y)] = 0$ (3.10)

Again , replace r by rs in (3.5) , where s∈R , and using it , to get

$(M(u, u_2, u_3) - \theta(u))\, \theta(r)\, [\theta(s), \theta(y)] = 0$

i.e., (M (u , u_2, u_3) - θ(u)) $R[\theta(s), \theta(y)]$ ={0}

Since R is prime ring , then either (M (u , u_2, u_3) - θ(u)) = 0 or [θ(s), θ(y)] = 0 , for all s , u_2,u_3 ∈R and u , y ∈ I .

Thus, application of our hypotheses implies that [θ(s), θ(y)] = 0 , for all y ∈ I and s ∈R and therefore I ⊆ Z(R) .

Hence , R is commutative .

Using similar arguments as used in proof of the Theorem 3.5 , we can prove the following :

Theorem 3.6 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ-3-centralizer M such that M (u , u_2 , u_3) $\neq -\theta$(u), for all u∈I . Further , if M((u ∘ y), u_2, u_3) + (θ(u) ∘ θ(y)) = 0 **,** for all u_2,u_3 ∈R and u , y ∈ I , then R is commutative .

Theorem 3.7 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ-3-centralizer . If M (u^2 , u_2 , u_3) = θ(u^2), for all u_2,u_3 ∈R and u ∈ I , then R is commutative .

Proof :

We are given that

M (u^2 , u_2 , u_3) = θ(u^2), for all u_2,u_3 ∈R and u ∈ I (3.11)

For all u , y ∈ I , we have

M ($(u+y)^2$, u_2 , u_3) = θ($(u+y)^2$) , which implies that

M((u ∘ y), u_2, u_3) - (θ(u) ∘ θ(y)) = 0 **,** for all u_2,u_3 ∈R and u ,y ∈ I

By using Theorem 3.5 , we get R is commutative .

Similarly , we can prove the following :

Theorem 3.8 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ-3-centralizer . If M (u^2 , u_2 , u_3) = - θ(u^2), for all u_2,u_3 ∈R and u ∈ I , then R is commutative .

Theorem 3.9 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ-3-centralizer such that M (u , u_2 , u_3) $\neq \pm\theta$(u) , for all u∈I . Further , if M(u y, u_2, u_3) $\pm$ θ(u y) = 0 **,** for all u_2,u_3 ∈R and u , y ∈ I , then R is commutative .

Proof :

For any u_2 , u_3 ∈R and u , y ∈ I

We have M(u y, u_2, u_3) = θ(u y)

Then we have $M(u y - y u, u_2, u_3) = \theta(u y - y u)$

This implies that $M([u, y], u_2, u_3) - [\theta(u), \theta(y)] = 0$,for all $u_2, u_3 \in R$ and $u, y \in I$, and hence by Theorem 3.3 , R is commutative .

On the other hand if R satisfy the condition $M(u y, u_2, u_3) + \theta(u y) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$

Then we have $M(u y + y u, u_2, u_3) = -\theta(u y + y u)$

This implies that $M((u \circ y), u_2, u_3) + (\theta(u) \circ \theta(y)) = 0$ **,** for all $u_2, u_3 \in R$ and $u, y \in I$. Thus, by Theorem 3.6 , R is commutative .

Theorem 3.10 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ-3-centralizer such that $M(u, u_2, u_3) \neq \theta(u)$, for all $u \in I$. Further , if $M(u y, u_2, u_3) - \theta(u y) \in Z(R)$ **,** for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative .

Proof :

For all $u_2, u_3 \in R$ and $u, y \in I$ **,** we have

$M(u y, u_2, u_3) - \theta(u y) \in Z(R)$

This can be rewritten as

$$(M(u, u_2, u_3) - \theta(u))\theta(y) \in Z(R) \qquad (3.12)$$

Replacing u by ur in (3.12) , where $r \in R$, we obtain

$$(M(u, u_2, u_3) - \theta(u))\theta(r)\,\theta(y) \in Z(R) \qquad (3.13)$$

Multiply equation (3.12) from the right side by $\theta(r)$

$$(M(u, u_2, u_3) - \theta(u))\theta(y)\theta(r) \in Z(R) \qquad (3.14)$$

$[M(u, u_2, u_3) - \theta(u))\theta(y), \ \theta(r)] = 0$,

For all $u_2, u_3 \in R$ and $u, y \in I$ (3.15)

This implies that ,

$(M(u, u_2, u_3) - \theta(u))[\theta(y), \theta(r)] + [M(u, u_2, u_3) - \theta(u), \theta(r)]\theta(y) = 0$

For all $u_2, u_3 \in R$ and $u, y \in I$ (3.16)

Again , replace u by us in (3.16) , where $s \in R$, and using it , to get

$$(M(u, u_2, u_3) - \theta(u))\,\theta(s)\,[\theta(y), \theta(r)] + [(M(u, u_2, u_3) - \theta(u))\,\theta(s), \theta(r)]\theta(y) = 0 \qquad (3.17)$$

Comparing equation (3.16) and (3.17) , we get

$(M(u, u_2, u_3) - \theta(u))\,\theta(s)\,[\theta(y), \theta(r)] = 0$

For all r, s , u_2 , u_3 ∈R and u ,y ∈ I

This yields that

$(M(u, u_2, u_3) - \theta(u)) R [\theta(y), \theta(r)] = \{0\}$. For all r, u_2 , u_3 ∈R and u , y ∈ I

The primeness of R implies that either $[\theta(y), \theta(r)] = 0$ or

$M(u, u_2, u_3) - \theta(u) = 0$, for all r , u_2 , u_3 ∈R and u , y ∈ I .

Since $M(u, u_2, u_3) \neq \theta(u)$, implies that $[\theta(y), \theta(r)] = 0$, for all y∈ I and r∈R and therefore I ⊆ Z(R) .

Hence , R is commutative .

Using similar arguments as used in proof of the Theorem 3.10 , we can prove the following :

Theorem 3.11 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ-3- centralizer such that $M(u, u_2, u_3) \neq -\theta(u)$, for all u∈I . Further , if $M(u y, u_2, u_3) + \theta(u y) \in Z(R)$ **,** for all u_2,u_3 ∈R and u , y ∈ I , then R is commutative .

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