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ABSTRACT

In this paper, we will give some results about the symmetric numerical semigroups such that $S_k = < 7, 7k + 4 >$ where $k \geq 1, k \in \mathbb{Z}$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

1. Introduction

Let $\mathbb{N} = \{0,1,2,...,n,...\}$ and $\mathbb{Z}$ be integer set. S is called a numerical semigroup if

(i) $a_1 + a_2 \in S$ for $\forall a_1, a_2 \in S$

(ii) $\gcd(S) = 1$

(iii) $0 \in S$

where $S \subseteq \mathbb{N}$ (Here, $\gcd(S)=$ greatest common divisor the elements of S).

A numerical semigroup S can be written that

$$S = < a_1, a_2, ..., a_n > = \left\{ \sum_{i=1}^{n} k_i a_i : k_i \in \mathbb{N} \right\}$$

(for details see [4]).

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$U \subseteq \mathbb{N}$ is minimal system of generators of S if $<U> = S$ and there isn't any subset $V \subseteq U$ such that $<V> = S$ .Also, $m(S) = \min\{x \in S : x > 0\}$ is called as multiplicity of S (see [3]). Let S be a numerical semigroup, then $F(S) = \max(\mathbb{N} \setminus S)$ is called as Frobenius number of S .

$$n(S) = Card(\{0,1,2,\ldots,F(S)\} \cap S)$$

is called as the determine number of S (see [5]).

If S is a numerical semigroup such that $S = <a_1, a_2, \ldots, a_n>$, then we observe that

$$S = <a_1, a_2, \ldots, a_n> = \{s_0 = 0, s_1, s_2, \ldots, s_{n-1}, s_n = F(S)+1, \rightarrow \ldots\},$$

where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S)+1$ belongs to S for $i = 1,2,\ldots,n = n(S)$ (see [6]).

If $b \in \mathbb{N}$ and $b \notin S$, then b is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S) = \mathbb{N} \setminus S$. The $G(S) = \#(H(S))$ is called the genus of S. It known that

$$G(S) + n(S) = F(S) + 1$$

(see [4]) .

S is called symmetric numerical semigroup if $F(S) - t$ belongs to S, for $t \in \mathbb{N} \setminus S$. It is know the numerical semigroup $S = <a_1, a_2>$ is symmetric and $F(S) = a_1 a_2 - a_1 - a_2$. In this case, we write

$$n(S) = \frac{F(S)+1}{2}$$

(see [1]).

A numerical semigroup S is called Arf if $a_1 + a_2 - a_3 \in S$, for all $a_1, a_2, a_3 \in S$ such that $a_1 \geq a_2 \geq a_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and it is denoted by $Arf(S)$ (for detail see [2, 3]). If S is a numerical semigroup such that $S = <a_1, a_2, \ldots, a_n>$, then $L(S) = \langle a_1, a_2 - a_1, a_3 - v_1, \ldots, a_n - v_1 \rangle$ is called Lipman numerical semigroup of S and it is known that

$$L_0(S) = S \subseteq L_1(S) = L(L_0(S)) \subseteq L_2 = L(L_1(S)) \subseteq \ldots \subseteq L_m = L(L_{m-1}(S)) \subseteq \ldots \subseteq \mathbb{N}$$

(see [7]).

In this paper, we will give some results about the symmetric numerical semigroups such that $S_k = <7, 7k+4>$ where $k \geq 1, k \in ¢$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

2. Main Results

Theorem 1. Let $S_k = <7, 7k+4>$ be numerical semigroups , where $k \geq 1, k \in ¢$. Then, we have

(a) $F(S_k) = 42k+17$

(b) $n(S_k) = 21k+9$

(c) $G(S_k) = 21k+9$.

Proof. Let $S_k = <7, 7k+4>$ be numerical semigroups , where $k \geq 1, k \in ¢$. Then, S_k is symmetric and we find that

(a) $F(S_k) = 7(7k+4) - 7 - 7k - 4 = 42k+17$.

(b) $n(S_k) = \frac{F(S_k)+1}{2} = \frac{42k+17+1}{2} = 21k+9$.

(c) $G(S_k) = 42k+17+1-21k-9 = 21k+9$ from $G(S_k) = F(S_k)+1-n(S_k)$.

Theorem 2. Let $S_k = <7, 7k+4>$ be numerical semigroups , where $k \geq 1, k \in ¢$. Then, $Arf(S_k) = \{0, 7, 14, 21, ..., 7k, 7k+4, 7k+7, \rightarrow ...\}$.

Proof. It is trivial $m_0 = 7$ since $L_0(S_k) = S_k = <7, 7k+4>$. Thus, we write $L_1(S_k) = <7, 7k-3>$. In this case,

(1) If $7k-3 < 7$ (if $k=1$) then $S_1 = \langle 7, 11 \rangle$ and $L_1(S_1) = <7, 4> = <4, 7>$, $m_1 = 4$.
$L_2(S_1) = <4, 3> = <3, 4>$, $m_2 = 3$ and $L_3(S_1) = <3, 1> = <1> = ¥$, $m_3 = 1$.

Thus, we obtain that $Arf(S_1) = \{0, 7, 11, 14, \rightarrow ...\}$.

(2) If $7k-3 > 7$ (if $k \geq 2$) then $L_1(S_k) = <7, 7k-3>$ and $m_1 = 7$. In this case, we write $L_2(S_k) = <7, 7k-10>$.

(a) If $k = 2$ then $L_2(S_2) = <7,4> = <4,7>$, $m_2 = 4$. $L_3(S_2) = <4,3> = <3,4>$, $m_3 = 3$ and $L_4(S_2) = <3,1> = <1> = \mathbb{N}$, $m_4 = 1$. So, we have $Arf(S_2) = \{0,7,14,18,21,\rightarrow \ldots\}$.

(b) If $k > 2$ then $L_2(S_k) = <7,7k-10>$, $m_2 = 7$ and $L_3(S_k) = <7,7k-17>$. In this case,

(i) if $k = 3$ then $L_3(S_3) = <7,4> = <4,7>$, $m_3 = 4$. $L_4(S_3) = <4,3> = <3,4>$, $m_4 = 3$ and $L_5(S_3) = <1> = \mathbb{N}$, $m_5 = 1$. Thus we obtain that $Arf(S_3) = \{0,7,14,21,25,28,\rightarrow \ldots\}$.

(ii) If $k > 3$ then $L_3(S_k) = <7,7k-17>$, $m_3 = 7$ and $L_4(S_k) = <7,7k-24>$. In this case,

(1) if $k = 4$ then $L_4(S_4) = <7,4> = <4,7>$, $m_4 = 4$. $L_5(S_4) = <4,3> = <3,4>$, $m_5 = 3$

and

$$L_6(S_4) = <1> = \mathbb{N},\ m_6 = 1.$$

Thus we find $Arf(S_k) = \{0,7,14,21,28,32,35,\rightarrow \ldots\}$.

(2) If $k > 4$ then $L_4(S_k) = <7,7k-24>$, $m_4 = 7$ and we write $L_5(S_k) = <7,7k-31>$. If we continue the operations then we obtain Arf closure of $Arf(S_k)$ as follows

$$Arf(S_k) = \{0,7,14,21,\ldots,7k,7k+4,7k+7,\rightarrow \ldots\}.$$

Thus, the proof is completed.

Proposition 3 . Let $S_k = <7,7k+4>$ be numerical semigroups , where $k \geq 1$, $k \in \mathbb{N}$. Then, we have

(a) $F(Arf(S_k)) = 7k+6$

(b) $n(Arf(S_k)) = k+2$

(c) $G(Arf(S_k)) = 6k+5$.

Proof. Let $S_k = <7,7k+4>$ be numerical semigroups , where $k \geq 1$, $k \in \mathbb{N}$. Then,

we write that $F(Arf(S_k)) = 7k+6$ from Theorem 2. On the other hand, we find that

$n(Arf(S_k)) = \#(\{0,1,2,\ldots,7k+6\} \cap Arf(S)) = \#(\{0,7,14,\ldots,7k,7k+4\}) = k+2$ and we obtain $G(Arf(S_k)) = 7k+6+1-k-2 = 6k+5$ since $G(Arf(S_k)) = F(Arf(S_k)) + 1 - n(Arf(S_k))$.

Corollary 4 . Let $S_k = < 7, 7k + 4 >$ be numerical semigroups , where $k \geq 1, k \in \mathbb{N}$. Then, we have

(a) $F(S_k) = F(Arf(S_k)) + 35k + 11$

(b) $n(S_k) = n(Arf(S_k)) + 20k + 7$

(c) $G(S_k) = G(Arf(S_k)) + 15k + 4$.

Proof. Let $S_k = < 7, 7k + 4 >$ be numerical semigroups , where $k \geq 1, k \in \mathbb{N}$. We write that

(a) $F(Arf(S_k)) + 35k + 11 = 7k + 6 + 35k + 11 = 42k + 17 = F(S_k)$. However, we find that

(b) $n(Arf(S_k)) + 20k + 7 = k + 2 + 20k + 7 = 21k + 9 = n(S_k)$,

(c) $G(Arf(S_k)) + 15k + 4 = 6k + 5 + 15k + 4 = 21k + 9 = G(S_k)$.

Corollary 5. Let $S_k = < 7, 7k + 4 >$ be numerical semigroups , where $k \geq 1, k \in \mathbb{N}$. Then, it satisfies following conditions:

(a) $F(S_{k+1}) = F(S_k) + 42$

(b) $n(S_{k+1}) = n(S_k) + 21$

(c) $G(S_{k+1}) = G(S_k) + 21$.

Corollary 6. Let $S_k = < 7, 7k + 4 >$ be numerical semigroups , where $k \geq 1, k \in \mathbb{N}$. Then, it satisfies following conditions:

(a) $F(Arf(S_{k+1})) = F(Arf(S_k)) + 7$

(b) $n(Arf(S_{k+1})) = n(Arf(S_k)) + 1$

(c) $G(Arf(S_{k+1})) = G(Arf(S_k)) + 6$.

Example 7. We put $k = 1$ **in** $S_k = < 7, 7k + 4 >$ symmetric numerical semigroup. Then we have $S_1 = < 7, 11 > = \{0, 7, 11, 14, 18, 21, 22, 25, 28, 29, 32, 33, 35, 36, 39, 40, 42, 43, 44, 46, 47, 49, 50, 51, 53, 54, 55, 56, 57, 58, 60, \rightarrow \ldots\}$. In this case, we obtain $F(S_1) = 59$, $n(S_1) = 30$,

$H(S_1) = \{1,2,3,4,5,6,8,9,10,12,13,15,16,17,19,20,23,24,26,27,30,31,34,37,38,41,45,48,52,59\}$
$G(S_1) = 30$, $Arf(S_1) = \{0,7,11,14,\rightarrow \ldots\}$, $F(Arf(S_1)) = 13$, $n(Arf(S_1)) = 3$ $G(Arf(S_1)) = 11$. Thus, we find that

$F(Arf(S_1)) + 46 = 13 + 46 = 59 = F(S_1)$, $n(Arf(S_1)) + 27 = 3 + 27 = 30 = n(S_1)$

and $G(Arf(S_1)) + 19 = 11 + 19 = 30 = G(S_1)$.

If $k = 2$ then we write $S_2 = <7,18> = \{0,7,14,18,21,25,28,\ldots,100,102,\rightarrow \ldots\}$.

Thus, we have $F(S_2) = 101$, $n(S_2) = 51$, $G(S_2) = 51$, $Arf(S_2) = \{0,7,14,18,21,\rightarrow \ldots\}$, $F(Arf(S_2)) = 20$, $n(Arf(S_2)) = 4$ and $G(Arf(S_2)) = 17$.

So, we write that

$$F(S_1) + 42 = 59 + 42 = 101 = F(S_2),$$

$n(S_1) + 21 = 30 + 21 = 51 = n(S_2)$ and $G(S_1) + 21 = 30 + 21 = 51 = G(S_2)$. Also, we obtain that $F(Arf(S_1)) + 7 = 13 + 7 = 20 = F(Arf(S_2))$, $n(Arf(S_1)) + 1 = 3 + 1 = 4 = n(Arf(S_2))$ and $G(Arf(S_1)) + 6 = 11 + 6 = 17 = G(Arf(S_2))$.

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