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## On Sensitivity and Expansivity of Linear Random Dynamical Systems

<b>Authors Names</b>	<b>ABSTRACT</b>
<p>a. Rafal Hamza Naif b. Ihsan Jabbar Kadhim</p> <p><b>Article History</b></p> <p>Received on:20 /4 /2021 Revised on:30 /5 /2021 Accepted on: 6/ 6/ 2021</p> <p><b>Keywords:</b></p> <p>linear random dynamical system, sensitive linear random dynamical and expansive linear random dynamical system</p> <p><b>DOI:</b> <a href="https://doi.org/10.29350/jops.2021.26.3.1305">https://doi.org/10.29350/jops.2021.26.3.1305</a></p>	<p>In this study we introduce the concepts of sensitivity and expansivity of linear random dynamical systems. We obtain various important properties and theorems about sensitivity random dynamical systems and expansivity random dynamical systems.</p>

### 1. Introduction

The main purpose of this Study is to present and popularize the notion of a linear random dynamical system(LRDS) and to give an impression of its space. The notion of (LRDS) cover the most important families of linear dynamical systems with randomness which are currently of interest. For instance, products of random maps-inparticular products of random matrices-are LRDS as well as (the solution flows of) stochastic and random ordinary and partial differential equations. I.M. Sobol (1993) [4], study Sensitivity estimates for nonlinear mathematical models. I.M. Sobol, S. Kucherenko (2009) [5], one of the most popular approaches for this strategy, Sobol indices, is a technique for performing variance-based sensitivity analysis. Ali Barzanouni and Ekta Shah(2010)[1], they show that if there exists a topologically expansive homeomorphism on a uniform space, then the space is always regular. during paradigm, we prove that in a general compound of topologically expansive homeomorphisms need not be topological expansive and also that couple of topological expansive homeomorphisms

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need not be topological expansive. Tullio Ceccherini-Silberstein and Michel Coornaert (2013)[11]. In this paper further consider the extent to which certain dynamical structures on metric spaces are also suitable for uniform spaces. Tarun Das, Keonhee Lee, David Richeson, and Jim Wiseman (2013)[10]. in particular, certain metric dynamical structures like expansivity, Anosov systems, Devaney's chaos and topological entropy have been extended to this uniform. R. Pulch, E.J.W. ter Maten, F. Augustin (2015) [9], investigate the sensitivity of the transfer function associated to random ODEs in the frequency domain. . these sensitivity measures were employed to investigate dynamical systems in different applications. I.J Kadhim, A.A ALI (2020) [3] introduce and study the expansivity of uniform dynamical systems, they obtain various important properties and theorems about sensitivity and expansivity of uniform continuous maps.

### Definition(1.1)[7](Random Dynamical System)

A measurable random dynamical system on the measurable space  $(X, B(X))$  over (or covering, or extending) an  $MDS(T, \Omega, F, P, \theta)$  with Measurability time is a mapping  $\varphi: T \times \Omega \times X \rightarrow X$ , with the following properties:

(i)  $\varphi$  is  $B(T) \otimes F \otimes B, B$ -measurable.

(ii) Cocycle property: The mapping  $\varphi(t, w) := \varphi(t, w, \cdot): X \rightarrow X$  from a cocycle over  $\theta(\cdot)$ , i.e. they satisfy  $\varphi(0, w) = id_X$  for all  $w \in \Omega$  (if  $0 \in T$ ),  $\varphi(t + s, w) = \varphi(t, \theta(s)w) \circ \varphi(s, w)$  for all  $s, t \in T, w \in \Omega$ . if there is no ambiguity the RDS is denoted by  $(\theta, \varphi)$  rather than  $(G, \Omega, X, \theta, \varphi)$ .

### Definition (1.2)[7] (Linear RDS):

A continuous RDS on a (for simplicity) finite-dimensional vector space is called a linear RDS, if  $\varphi(t, w) \in L(X)$  for each  $t \in T, w \in \Omega$ , where  $L(X)$  is the space of linear operators of  $X$ .

If we endow the vector space  $X$  with its natural manifold structure, then  $L(X) \subset C(X, X)$ . Hence a linear RDS is automatically  $C$ .

### Definition(1.3)[8]:

The metric dynamical system (MDS) is the 5-tuple  $(G, \Omega, F, P, \theta)$  where  $(\Omega, F, P)$  is a probability space and  $\theta: G \times \Omega \rightarrow \Omega$  is  $(\beta(G) \otimes F, F)$  –

$$\theta(e, w) = id,$$

$$\theta(g * h, w) = \theta(g, \theta(h, w)) \text{ and}$$

$$P(\theta_g F) = P(F), \forall F \in F \forall w \in G.$$

### Definition( 1.4)[7]:

Let  $(\Omega, F, P, \theta)$  be an MDS. A random variable  $\delta: \Omega \rightarrow R^+$  is said to be tempered, if there exists a full measurable subset  $\tilde{\Omega}$  of  $\Omega$  such that  $1/|n| \delta(\theta(n)\omega) = 0$ , for every  $\omega \in \tilde{\Omega}$ , where  $\log^+ = \{0, \log\}$ .

### Lemma (1.5)[7]:

Let  $(\Omega, F, P, \theta)$  be an MDS, then every constant function  $\delta: \Omega \rightarrow R^+$  is tempered random variable.

**Lemma (1.6)[7]**

The set of real-valued tempered random variable with respect to an MDS  $(G, \Omega, F, P, \theta)$

$$R = \{\delta: \Omega \rightarrow R^+ : \delta \text{ is a tempered random variable}\}$$

is a semigroup with  $+: R \times R \rightarrow R$  defined by  $+(\delta_1, \delta_2) = \delta_1(\omega) + \delta_2(\omega)$ .

**Lemma( 1.7)[7]:**

The set of real-valued tempered random variable with respect to an MDS  $(\Omega, F, P, \theta)$

$$R := \{\delta: \Omega \rightarrow R^+ : \delta \text{ is a tempered random variable}\}$$

is an abelian group with  $\cdot: R \times R \rightarrow R$  defined by  $(\delta_1, \delta_2) = \delta_1(\omega) \cdot \delta_2(\omega)$ .

**Definition(1.8) [7]:Equivalence of RDS**

Let  $(\theta, \varphi_1)$  and  $(\theta, \varphi_2)$  be two RDSs. The order paire

$$(\xi, \Phi): (\theta, \varphi_1) \rightarrow (\theta, \varphi_2)$$

is said to be isomorphism between the two RDS's  $(\theta, \varphi_1)$  and  $(\theta, \varphi_2)$  if

- (i)  $\xi(\theta(t)\omega) = \theta(t)\xi(\omega)$  for every  $\omega \in \Omega$ .
- (ii)  $\Phi: X_1 \rightarrow X_2$  is homeomorphism map and
- (iii)  $\Phi(\varphi_1(t, \omega)x) = \varphi_2(t, \xi(\omega))\Phi(x)$ .

If there exists such order paire we say that  $(\theta, \varphi_1)$  and  $(\theta, \varphi_2)$  are **equivalent** via  $(\xi, \Phi)$  and write  $(\theta, \varphi_1) \cong_{(\xi, \Phi)} (\theta, \varphi_2)$ .

**Lemma( 1.9):**

Let  $(N_1, \Omega_1, F_1, P_1, \theta_1) \cong_{(\alpha, \xi)} (N, \Omega_2, F_2, P_2, \theta_2)$ . If  $\delta_1: \Omega_1 \rightarrow R^+$  is tempered random variable, then  $\delta_2: \Omega_2 \rightarrow R^+$  which defined by  $\delta_2 = \delta_1 \circ \xi^{-1}$  is tempered random variable

**Proof:** The function  $\delta_2: \Omega_2 \rightarrow R^+$  is measurable being a composition of two measurable functions  $\delta$  and  $\xi^{-1}$ . Since  $\delta_1: \Omega_1 \rightarrow R^+$  is tempered random variable, then there exists a full measurable subset  $\tilde{\Omega}_1$  of  $\Omega_1$  such that  $\frac{1}{|n|} \lim_{n \rightarrow \infty} \log \delta_1(\theta_1(n)\omega_1) = 0$ , for every  $\omega_1 \in \tilde{\Omega}_1$ .

Since  $\xi$  is measurable, then  $\tilde{\Omega}_2 = \xi(\tilde{\Omega}_1)$  be a full measure subset of  $\Omega_2$ , let  $\omega_2 \in \tilde{\Omega}_2$ , then there exists  $\omega_1 \in \tilde{\Omega}_1$  such that  $\omega_2 = \xi(\omega_1)$ . Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_2(\theta_2(n)\omega_2) &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_2 (\theta_2 (\alpha \circ n)\xi(\omega_1)) \\ &= \lim_{n \rightarrow \infty} \log \delta_1 \circ \xi^{-1} (\xi(\theta_1(n)\omega_1)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_1 (\theta_1(n)\omega_1) = 0. \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_2 (\theta_2(n)\omega_2) = 0$ , for every  $\omega_2 \in \tilde{\Omega}_2$ .

Hence  $\delta_2$  is a tempered random variable. ■

**Lemma( 1.10):**

Let  $(N_i, \Omega_i, F_i, P_i, \theta_i), i = 1,2$ , be an MDS. If  $\delta_i: \Omega_i \rightarrow R^+$  is tempered random variable,  $i = 1,2$ , then the function  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by  $\delta(\omega_1, \omega_2) = \{\delta_1 (\omega_1), \delta_2 (\omega_2)\}$  is tempered random variable

**Proof:** Suppose that  $\delta_i: \Omega_i \rightarrow R^+$  is tempered random variable,  $i = 1,2$ , then  $\delta_i: \Omega_i \rightarrow R^+$  is measurable,  $i = 1,2$  and there exists a full measurable subset  $\tilde{\Omega}_i$  of  $\Omega_i$  such that  $\lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_i (\theta_i (n, \omega_i)) = 0$ ,

for every  $\omega_i \in \tilde{\Omega}_i, i = 1,2$ . Clearly that  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by

$$\delta(\omega_1, \omega_2) = \max\{\delta_1 (\omega_1), \delta_2 (\omega_2)\}$$

is measurable. Set  $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2$ , then  $\tilde{\Omega}$  is a full measure subset of  $\Omega_1 \times \Omega_2$ . Let  $\omega \in \tilde{\Omega}$ , then  $\omega = (\omega_1, \omega_2)$  for some  $\omega_1 \in \tilde{\Omega}_1$  and some  $\omega_2 \in \tilde{\Omega}_2$ .

Now

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta(\theta(n, \omega)) &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta(\theta_1 \times \theta_2 ((n, n), (\omega_1, \omega_2))) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta(\theta_1 (n, \omega_1), \theta_2 (n, \omega_2)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \max\{\delta_1 (\theta_1 (n, \omega_1)), \delta_2 (\theta_2 (n, \omega_2))\} = 0. \end{aligned}$$

This means that  $\delta(\omega_1, \omega_2) = \max\{\delta_1 (\omega_1), \delta_2 (\omega_2)\}$  is tempered random variable. ■

**Lemma( 1.11):**

Let  $(N_i, \Omega_i, F_i, P_i, \theta_i), i = 1,2$ , be an MDS. If  $\delta_i: \Omega_i \rightarrow R^+$  is tempered random variable,  $i = 1,2$ , then the function  $\delta^i: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by  $\delta^i(\omega_1, \omega_2) = \delta_i (\omega_i), i = 1,2$ , is tempered random variables

**Proof:** Suppose that  $\delta_i: \Omega_i \rightarrow R^+$  is tempered random variable,  $i = 1,2$ , then  $\delta_i: \Omega_i \rightarrow R^+$  is measurable,  $i = 1,2$  and there exists a full measurable subset  $\tilde{\Omega}_i$  of  $\Omega_i$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_i (\theta_i(n)\omega_i) = 0$$

for every  $\omega_i \in \tilde{\Omega}_i, i = 1,2$ . Clearly that  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by

$\delta^1 (\omega_1, \omega_2) = \delta_1 (\omega_1)$  is measurable .Set  $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2$ , then  $\tilde{\Omega}$  is a full measure subset of  $\Omega_1 \times \Omega_2$ . Let  $\omega \in \tilde{\Omega}$ , then  $\omega = (\omega_1, \omega_2)$  for some  $\omega_1 \in \tilde{\Omega}_1$  and some  $\omega_2 \in \tilde{\Omega}_2$ . Now

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta^1(\theta(n)\omega) &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta^1 (\theta_1 \times \theta_2 ((n, n))(\omega_1, \omega_2)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta^1 (\theta_1(n)\omega_1, \theta_2(n)\omega_2) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|n|} \log \delta_1 (\theta_1(n)\omega_1) = 0. \end{aligned}$$

This means that  $\delta^1 (\omega_1, \omega_2) = \delta_1 (\omega_1)$  is tempered random variable. Similarly  $\delta^2 (\omega_1, \omega_2) = \delta_2 (\omega_2)$  is tempered random variable. ■

**Definition (1.12)[7]**

The ball

$$U(\omega) = B_{\delta(\omega)} (x_0) = \{x \in X: \|(x - x_0)\| < \delta(\omega)\}$$

is called tempered if  $\delta$  is tempered random variable.

**2- Sensitivity of Random Dynamical Systems**

In this section the concept of sensitive of RDS's is introduced and some new properties of such systems are proved.

**Definition( 2.1):**

An LRDS  $(\theta, \varphi)$  is said to be sensitive if there exists a tempered random variable  $\delta$  such that for any  $x \in X$  and any (deterministic) open neighborhood  $U$  of  $x$ , there exist  $y \in U$ , and full measure subset  $\tilde{\Omega}$  of  $\Omega$  such that every  $\omega \in \tilde{\Omega}$  we have

$$\| (\varphi(n, \omega)x - \varphi(n, \omega)y) \| > \delta(\theta(n)\omega).$$

**Remark(2.2):**

If  $\Omega$  is a singleton, and  $\delta$  is a positive constant, then we get the definition of sensitivity in case of the deterministic dynamical system.

**Theorem (2.3):**

Let  $(\theta_1, \varphi_1) \cong_{(\xi, \Phi)} (\theta_2, \varphi_2)$  with  $\Phi$  is isometric. If  $(\theta_1, \varphi_1)$  is sensitive, then  $(\theta_2, \varphi_2)$  is sensitive

**Proof:** Suppose that  $(\theta_1, \varphi_1)$  is sensitive RDS. To show that the  $(\theta_2, \varphi_2)$  is sensitive RDS. Let  $x_1^2 \in X_2$  and let  $U_2$  be any open neighborhood of  $x_1^2 = \Phi^{-1}(x_1^2)$  in  $U_1$ . By hypothesis, there exist tempered random variable  $\delta_1, x_1^1 \in U_1$ , and full measure subset  $\tilde{\Omega}_1$  of  $\Omega_1$  such that every  $\omega_1 \in \tilde{\Omega}_1$  we have

$$\|(\varphi_1(n, \omega_1)x_1^1 - \varphi_1(n, \omega_1)x_2^1)\| > \delta_1(\theta_1(n)\omega_1).$$

Since  $x_1^2 \in U_1$ , then  $x_2^2 = \Phi(x_1^2) \in U_2$  and since  $\xi$  is measure-preserved, then  $\tilde{\Omega}_2 = \xi(\tilde{\Omega}_1)$  is of full measure.  $\delta_2 = \delta_1 \circ \xi^{-1}$  is a tempered random variable, Let  $\omega_2 \in \tilde{\Omega}_2$ , then there exists  $\omega_1 \in \tilde{\Omega}_1$  such that  $\omega_2 = \xi(\omega_1)$ . Now,

$$\begin{aligned} & \|(\varphi_2(n, \omega_2)x_1^2 - \varphi_2(n, \omega_2)x_2^2)\| = \|(\varphi_2(n, \xi(\omega_1))\Phi(x_1^1) - \varphi_2(n, \xi(\omega_1))\Phi(x_2^1))\| \\ & = \|(\Phi\varphi_1(n, \omega_1)x_1^1 - \Phi\varphi_1(n, \omega_1)x_2^1)\| \\ & = \|(\varphi_1(n, \omega_1)x_1^1 - \varphi_1(n, \omega_1)x_2^1)\| > \delta_1(\theta_1(n)\omega_1). \end{aligned}$$

Since  $\delta_1(\theta_1(n)\omega_1) = \delta_1(\theta_1(n)\xi^{-1}(\omega_2))$

$$\begin{aligned} & = \delta_1(\theta_1(n \times \xi^{-1}(n, \omega_2))) \\ & = \delta_1(\xi^{-1}(\theta_2(n)\omega_2)) = \delta_1 \circ \xi^{-1}(\theta_2(n)\omega_2), \end{aligned}$$

then  $\|(\varphi_2(n, \omega_2)x_1^2 - \varphi_2(n, \omega_2)x_2^2)\| > \delta_2(\theta_2(n)\omega_2)$ . This means that  $(\theta_2, \varphi_2)$  is sensitive. ■

The following theorem shows that the product of two sensitive RDS's is also sensitive. We shall consider the max metric on  $X_1 \times X_2$ .

**Theorem (2.4):**

If  $(\theta_1, \varphi_1)$  is insensitive and  $(\theta_2, \varphi_2)$  is sensitive RDS's, then  $(\theta_1 \times \theta_2, \varphi_1 \times \varphi_2)$  is sensitive.

**Proof:** Let  $z_1 = (x_1^1, x_1^2) \in X_1 \times X_2$  and  $W$  be any open neighborhood of  $z_1$  in  $X_1 \times X_2$ . Then there exist open neighborhood  $U_1$  of  $x_1^1$  in  $X_1$  and an open neighborhood  $U_2$  of  $x_1^2$  in  $X_2$  such that  $U_1 \times U_2 \subseteq W$ . Since  $(\theta_1, \varphi_1)$  is insensitive, then there exist tempered random variable  $\delta_i, x_2^i \in U_i$  and a full measure subset  $\tilde{\Omega}_i$  of  $\Omega_i$ ,  $i = 1, 2$ , such that every  $\omega_i \in \tilde{\Omega}_i$  we have

$$\| (\varphi_i(n, \omega_i)x_1^i - \varphi_i(n, \omega_i)x_2^i) \| > \delta_i(\theta_i(n)\omega_i).$$

The function  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by

$$\delta((\omega_1, \omega_2)) = \max\{\delta_1(\omega_1), \delta_2(\omega_2)\}$$

is tempered random variable Set  $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2$ , then it is a full measure subset of  $\Omega = \Omega_1 \times \Omega_2$ . Set  $z_2 = (x_2^1, x_2^2) \in U_1 \times U_2 \subseteq W$ . Let  $\omega \in \tilde{\Omega}$ , then  $\omega = (\omega_1, \omega_2)$  for some  $\omega_1 \in \tilde{\Omega}_1$  and  $\omega_2 \in \tilde{\Omega}_2$ . Now

$$\begin{aligned} & \| (\varphi(n, \omega)z_1 - \varphi(n, \omega)z_2) \| \\ &= \max\{ \| ((\varphi(n, \omega_1)x_1^1 - \varphi(n, \omega_1)x_2^1)) \|, \| ((\varphi(n, \omega_2)x_1^2 - \varphi(n, \omega_2)x_2^2)) \| \} \\ &\geq \max\{\delta_1(\theta_1(n)\omega_1), \delta_2(\theta_2(n)\omega_2)\} = \delta(\theta(n)\omega). \end{aligned}$$

This means that  $(\theta_1 \times \theta_2, \varphi_1 \times \varphi_2)$  is sensitive. ■

### 3- Expansive Random Dynamical Systems

#### **Definition (3.1):**

The LRDS  $(\theta, \varphi)$  is said to be expansive if for every  $x, y \in X$  with  $x \neq y$ , there exist tempered random variable  $\delta$  (called an expansivity characteristic variable) and a full measure subset  $\tilde{\Omega}$  of  $\Omega$  such that for every  $\omega \in \tilde{\Omega}$  we have

$$\| (\varphi(n, \omega)x - \varphi(n, \omega)y) \| > \delta(\theta(n)\omega).$$

**Notation (3.2):**  $F_\delta(x) = \{y \in X : \| \varphi(n, \omega)x - \varphi(n, \omega)y \| \leq \delta(\theta(n)\omega)\}$ .

**Theorem (3.3) :** The set  $F_\delta(x)$  is convex closed subset of  $X$ .

**Proof :**  $F_\delta(x) = \{y \in X : \| \varphi(n, \omega)x - \varphi(n, \omega)y \| \leq \delta(\theta(n)\omega), \omega \in N\}$

let  $y_1, y_2 \in F_\delta(x)$   $\alpha \in [0,1]$ . To prove that  $\alpha y_1 + (1 - \alpha)y_2 \in F_\delta(x)$ .

Since  $\alpha y_1 + (1 - \alpha)y_2 \in X$ , we have

$$\begin{aligned} & \| \varphi(n, \omega)x - \varphi(n, \omega)(\alpha y_1 + (1 - \alpha)y_2) \| \\ &= \| \varphi(n, \omega)x - \alpha \varphi(n, \omega)y_1 + (1 - \alpha) \varphi(n, \omega)y_2 \| \\ &= \| \varphi(n, \omega)x + \alpha \varphi(n, \omega)x - \alpha \varphi(n, \omega)x - \alpha \varphi(n, \omega)y_1 - (1 - \alpha) \varphi(n, \omega)y_2 \| \end{aligned}$$



$$\begin{aligned}
&= \| [\alpha \varphi(n, \omega)x - \alpha \varphi(n, \omega)y_1] + [(1 - \alpha)\varphi(n, \omega)x - (1 - \alpha)\varphi(n, \omega)y_2] \| \\
&\leq |\alpha| \| [\varphi(n, \omega)x - \varphi(n, \omega)y_1] \| + |(1 - \alpha)| \| \varphi(n, \omega)x - \varphi(n, \omega)y_2 \| \\
&\leq |\alpha| \delta_1(\theta((n)\omega)) + |(1 - \alpha)| \delta_2(\theta((n)\omega)) = \delta_3\theta((n)\omega).
\end{aligned}$$

Hence  $F_\delta(x)$  is convex set. Now we need to show that  $F_\delta(x)$  is closed.

Let  $y \in \overline{F_\delta(x)} \Rightarrow \exists$  sequence  $\{y_k\} \in F_\delta(x)$  with  $y_k \rightarrow y$

Since  $y_k \in F_\delta(x), \forall k$ , then

$$\| \varphi(n, \omega)x - \varphi(n, \omega)y_k \| \leq \delta(\theta((n)\omega))$$

Since  $y_k \rightarrow y$  as  $k \rightarrow \infty$ , then by continuity of  $\varphi(n, \omega): X \rightarrow X$  we have

$$\varphi(n, \omega)y_k \rightarrow \varphi(n, \omega)y.$$

Now

$$\begin{aligned}
&\| \varphi(n, \omega)x - \varphi(n, \omega)y \| \\
&= \| \varphi(n, \omega)x - \varphi(n, \omega)y_k + \varphi(n, \omega)y_k - \varphi(n, \omega)y \| \\
&\leq \| \varphi(n, \omega)x - \varphi(n, \omega)y_k \| + \| \varphi(n, \omega)y_k - \varphi(n, \omega)y \| \\
&\leq \delta(\theta((n)\omega)) + \frac{1}{n} \\
&= \delta_1(\theta((n)\omega)), \text{ where } \delta_1 = \delta + \frac{1}{n}. \text{ Since } \delta_1 \text{ is tempered random variable then } y \in F_\delta(x), \text{ so } \\
&F_\delta(x) \text{ is closed.}
\end{aligned}$$

### Theorem(3.4):

The LRDS  $(\theta, \varphi)$  is expansive if and only if there exists tempered random variable  $\delta$  and a full measure subset  $\tilde{\Omega}$  of  $\Omega$  such that  $\Gamma_\delta(x) = \{x\}$  where

$$\Gamma_\delta(x) = \{y \in X: \| (\varphi(n, \omega)x - \varphi(n, \omega)y) \| \leq \delta(\theta(n)\omega), \omega \in \tilde{\Omega}\}.$$

**Proof:** Suppose that  $(\theta, \varphi)$  is expansive. Assume contrary that there exists  $y \in \Gamma_\delta(x)$  with  $x \neq y$ . Then there exist tempered random variable  $\delta$  and a full measure subset  $\tilde{\Omega}$  of  $\Omega$

$$\|(\varphi(n, \omega)x - \varphi(n, \omega)y)\| \leq \delta(\theta(n)\omega), \text{ for every } \omega \in \tilde{\Omega}.$$

But this contradicts the fact that  $(\theta, \varphi)$  is expansive.

Conversely, suppose that  $\Gamma_\delta(x) = \{x\}$ . Let  $x, y \in X$  with  $x \neq y$ . If  $(\theta, \varphi)$  is not expansive, then for every tempered random variable  $\delta$  and a full measure subset  $\tilde{\Omega}$  of  $\Omega$  such that

$$\|(\varphi(n, \omega)x - \varphi(n, \omega)y)\| \leq \delta(\theta(n)\omega), \text{ for every } \omega \in \tilde{\Omega}.$$

Thus  $y \in \Gamma_\delta(x)$ . But  $\Gamma_\delta(x) = \{x\}$ , this implies that  $x = y$  and this contradict our hypothesis. Therefore  $(\theta, \varphi)$  is expansive .■

The following theorem shows that the sensitivity is weaker than expansivity.

**Proposition(3.5):**

Every expansive LRDS is sensitive.

**Proof:** Suppose that  $(\theta, \varphi)$  is expansive. Let  $x \in X$  and  $U$  open neighborhood of  $x$ . We can find  $y \in U$  such that  $x \neq y$ . By hypothesis there exists a tempered random variable  $\delta$  and a full measure subset  $\tilde{\Omega}$  of  $\Omega$  such that every  $\omega \in \tilde{\Omega}$  we have  $\|(\varphi(n, \omega)x - \varphi(n, \omega)y)\| > \delta(\theta(n)\omega)$ . This means that  $(\theta, \varphi)$  is sensitive.■

The following theorem shows that the expansivity is invariant by conjugations.

**Theorem (3.6):**

Let  $(\theta_1, \varphi_1) \cong_{(\xi, \Phi)} (\theta_2, \varphi_2)$  with  $\Phi$  is isometric. If  $(\theta_1, \varphi_1)$  is expansive LRDS, then  $(\theta_2, \varphi_2)$  is expansive LRDS

**Proof:** Suppose that  $(\theta_1, \varphi_1)$  is expansive. Let  $x_1^2, x_2^2 \in X_2$  with  $x_1^2 \neq x_2^2$ , then  $x_1^1 = \Phi^{-1}(x_1^2), x_2^1 = \Phi^{-1}(x_2^2) \in X$  with  $x_1^1 \neq x_2^1$ . By hypothesis, there exist tempered random variable  $\delta_1$  and a full measure subset  $\tilde{\Omega}_1$  of  $\Omega_1$  such that every  $\omega_1 \in \tilde{\Omega}_1$  we have

$$\|(\varphi_1(n, \omega_1)x_1^1 - \varphi_1(n, \omega_1)x_2^1)\| > \delta_1(\theta_1(n)\omega_1).$$

the function  $\delta_2: \Omega_2 \rightarrow R^+$  is tempered random variable, Since  $\xi$  is measure-preserved, then  $\tilde{\Omega}_2 = \xi(\tilde{\Omega}_1)$  is of full measure. Let  $\omega_2 \in \tilde{\Omega}_2$ , then there exist

$$\begin{aligned} &= \| (\varphi_2(n, \xi(\omega_1))\Phi(x_1^1) - \varphi_1(n, \xi(\omega_1))\Phi(x_2^1)) \| \\ &= \| (\Phi\varphi_1(n, \omega_1)x_1^1 - \Phi\varphi_1(n, \omega_1)x_2^1) \| \\ &= \| (\varphi_1(n, \omega_1)x_1^1 - \varphi_1(n, \omega_1)x_2^1) \| > \delta_1(\theta_1(n)\omega_1). \end{aligned}$$

$$\text{Since } \delta_1(\theta_1(n)\omega_1) = \delta_1(\theta_1(n)\xi^{-1}(\omega_2))$$

$$= \delta_1(\theta_1(n \times \xi^{-1}(n, \omega_2)))$$

$$= \delta_1(\xi^{-1}(\theta_2(n)\omega_2)) = \delta_1 \circ \xi^{-1}(\theta_2(n)\omega_2),$$

then  $\| (\varphi_2(n, \omega_2)x_1^2 - \varphi_2(n, \omega_2)x_2^2) \| > \delta_2(\theta_2(n)\omega_2)$ . This means that  $(\theta_2, \varphi_2)$  is expansive RDS. ■

The following theorem shows that the product of two expansive LRDS's is also expansive.

**Theorem (3.7):**

If  $(\theta_1, \varphi_1)$  is expansive or  $(\theta_2, \varphi_2)$  is expansive, then  $(\theta_1 \times \theta_2, \varphi_1 \times \varphi_2)$  is expansive

**Proof:** Let  $z_1 = (x_1^1, x_2^1), z_2 = (x_1^2, x_2^2) \in X_1 \times X_2$  with  $z_1 \neq z_2$ . Then either  $x_1^1 \neq x_1^2$  or  $x_2^1 \neq x_2^2$ .

Thus we have three cases:

**Case(I):** If  $x_1^1 \neq x_1^2$ , but  $x_2^1 = x_2^2$  then by hypothesis there exist tempered random variable  $\delta_1$  and a full measure subset  $\tilde{\Omega}_1$  of  $\Omega_1$  such that for every  $\omega_1 \in \tilde{\Omega}_1$  we have

$$\| (\varphi_1(n, \omega_1)x_1^1 - \varphi_1(n, \omega_1)x_1^2) \| > \delta_1(\theta_1(n)\omega_1).$$

the function  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by  $\delta((\omega_1, \omega_2)) = \delta_1(\omega_1)$  is random variable. Set  $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2$  a full measure subset of  $\Omega = \Omega_1 \times \Omega_2$ . Let  $\omega \in \tilde{\Omega}$ , then  $\omega = (\omega_1, \omega_2)$  for some  $\omega_1 \in \tilde{\Omega}_1$  and  $\omega_2 \in \tilde{\Omega}_2$ . Now

$$\begin{aligned} &\| (\varphi(n, \omega)z_1 - \varphi(n, \omega)z_2) \| \\ &= \max \| ((\varphi(n, \omega_1)x_1^1 - \varphi(n, \omega_1)x_1^2)) \|, \| ((\varphi(n, \omega_2)x_2^1 - \varphi(n, \omega_2)x_2^2)) \| \\ &\geq \| ((\varphi(n, \omega_1)x_1^1 - \varphi(n, \omega_1)x_1^2)) \| \\ &> \delta_1(\theta_1(n)\omega_1) = \delta(\theta(n)\omega). \end{aligned}$$

**Case(II):** If  $x_1^1 = x_1^2$ , but  $x_2^1 \neq x_2^2$ , then we can use the same argument that given in Case(I) to get the same result.

**Case(III):** If  $x_1^1 \neq x_1^2$  and  $x_2^1 \neq x_2^2$ . By hypothesis there exist tempered random variable  $\delta_i$  and a full measure subset  $\tilde{\Omega}_i$  of  $\Omega_i$  such that for every  $\omega_i \in \tilde{\Omega}_i$  we have  $\|(\varphi_i(n, \omega_i)x_i^1 - \varphi_i(n, \omega_i)x_i^2)\| > \delta_i(\theta_i(n)\omega_i)$ . The function  $\delta: \Omega_1 \times \Omega_2 \rightarrow R^+$  defined by  $\delta((\omega_1, \omega_2)) = \max\{\delta_1(\omega_1), \delta_2(\omega_2)\}$  is random variable. Set  $\tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2$  a full measure subset of  $\Omega = \Omega_1 \times \Omega_2$ . Let  $\omega \in \tilde{\Omega}$ , then  $\omega = (\omega_1, \omega_2)$  for some  $\omega_1 \in \tilde{\Omega}_1$  and  $\omega_2 \in \tilde{\Omega}_2$ . Now  $\|(\varphi(n, \omega)z_1 - \varphi(n, \omega)z_2)\|$   
 $= \max \|((\varphi(n, \omega_1)x_1^1 - \varphi(n, \omega_1)x_1^2))\|, \|((\varphi(n, \omega_2)x_2^1 - \varphi(n, \omega_2)x_2^2))\|$   
 $\geq \max\{\delta_1(\omega_1), \delta_2(\omega_2)\} = \delta(\theta(n)\omega)$ .

This means that  $(\theta_1 \times \theta_2, \varphi_1 \times \varphi_2)$  is expansive. ■

In the following we shall characterize the concept of expansive RDS in a manner analogous that in the deterministic

**Definition (3.8):**

Let  $(\theta, \varphi)$  be an LRDS. Let  $U$  be a finite random open cover of  $X$ . Then  $U$  is said to be generator for  $(\theta, \varphi)$  if there exists a divergent sequence  $n$  in  $N$  such that for every  $\omega \in \Omega$  and each choice of  $A_n \in U$ ,  $(n \in Z)$  we have  $\bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)A_n(\theta(n)\omega)$  is either empty or a single point.

**Remark (3.9):**

If  $\Omega$  is a singleton, then we get the definition of generator cover in case of the deterministic dynamical system.

**Theorem(3.10):**

The LRDS  $(\theta, \varphi)$  has a generator if and only if it is expansive.

**Proof:** Suppose  $(\theta, \varphi)$  is expansive with expansive variable  $\delta$ . Put  $\varepsilon(\omega) = \frac{\delta(\omega)}{2}$ . Since  $X$  is compact, there is a finite random cover consisting of sets of the form  $U_{\varepsilon(\omega)}(x) = \{y: \|x - y\| < \varepsilon(\omega)\}$ . Suppose that there exists a choice of sets  $A_n \in U$   $(n \in Z)$  such that  $\bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)A_n(\theta(n)\omega)$  contains two distinct points  $x$  and  $y$ . If  $A_n(\omega) = U_{\varepsilon(\omega)}(x_n)$ , then  $x, y \in \bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)U_{\varepsilon(\theta(n)\omega)}(x_n)$ , then  $x, y \in \varphi(n, \omega)U_{\varepsilon(\theta(n)\omega)}(x_n)$  for each  $n \in Z$  and consequently  $\varphi(n, \omega)x, \varphi(n, \omega)y \in U_{\varepsilon(\theta(n)\omega)}(x_n)$  for each  $n \in Z$ , then for the divergent sequence  $n$  in  $G$  we have  $\|(\varphi(n, \omega)x - x_n)\| < \varepsilon(\theta(n)\omega)$  and  $\|(\varphi(n, \omega)y - x_n)\| < \varepsilon(\theta(n)\omega)$ . Hence

$$\|(\varphi(n, \omega)x - \varphi(n, \omega)y)\| < \varepsilon(\theta(n)\omega) + \varepsilon(\theta(n)\omega) = \delta(\theta(n)\omega)$$

for the divergent sequence  $n$ . But this contradicts the assumption that  $\delta$  is expansive variable. Thus

$$\bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)A_n(\theta(n)\omega)$$

contains at most one point for each choice of  $A_n \in U$  ( $n \in Z$ ). In general,  $U$  will not be a generator, but we shall make use of it to construct one. For each  $x \in X$ , let  $x \in A_x \in U$ . Choose a random open ball  $V_x$  of  $x$  with  $V_x \subseteq A_x$ . The sets  $V_x$  cover  $X$  and we can choose a finite random subcover  $W$ . For each choice  $B_n \in W$  ( $n \in Z$ ),  $B_n \subset A_n \in U$  and so  $\bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)B_n(\theta(n)\omega)$  contains at most one point. Thus  $W$  is a generator.

Conversely, suppose that  $U$  is a generator for  $(\theta, \varphi)$  for some divergent sequence  $n$ . Then there exists a tempered random variable  $\delta$  such that for each  $x \in X$  there is some  $U \in U$  that contains  $B_{\delta(\omega)}(x)$ .

Suppose that  $\|(\varphi(n, \omega)x - \varphi(n, \omega)y)\| \leq \delta(\theta(n)\omega)$ . Choose  $A_n \in U$  such that  $B_{\delta(\theta(n)\omega)}(\varphi(n, \omega)x) \subseteq A_n$ . Then  $\varphi(n, \omega)y \in A_n$  and, clearly,  $\varphi(n, \omega)x \in A_n$ . It follows that  $x, y \in \bigcap_{n=-\infty}^{\infty} \varphi(n, \omega)A_n(\theta(n)\omega)$ . Since  $U$  is a generator  $x = y$ . Thus  $(\theta, \varphi)$  is expansive.

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