Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 2 Article 5

4-7-2021

Coefficient Bounds and Fekete-Szegö Problem for a New Subclasses of Holomorphic Bi-Univalent Functions Defined by Horadam polynomials

Najah Ali Jiben Al-Zaidi
Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniyah, Iraq,, najah.ali@qu.edu.iq

Abbas Kareem Wanas
Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq, abbas.kareem.w@qu.edu.iq

Recommended Citation

Al-Zaidi, Najah Ali Jiben and Wanas, Abbas Kareem (2021) "Coefficient Bounds and Fekete-Szegö Problem for a New Subclasses of Holomorphic Bi-Univalent Functions Defined by Horadam polynomials," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 2, Article 5.
DOI: 10.29350/qjps.2021.26.2.1276
Available at: https://qjps.researchcommons.org/home/vol26/iss2/5

Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (2) (2021) pp. Math. 52–65

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

Coefficient Bounds and Fekete-Szegö Problem for a New Subclasses of Holomorphic Bi-Univalent Functions Defined by Horadam polynomials

Authors Names

a. Najah Ali Jiben Al-Ziadi

b. Abbas Kareem Wanas

Article History

Received on: 7/2/2021

Revised on: 5/3/2021

Accepted on: 9/3/2021

Keywords:

Holomorphic functions

Bi-univalent functions

Coefficient bounds

Horadam polynomials

Fekete-Szegö problem

DOI: https://doi.org/10.29350/jops.2021.26. 2.1276

ABSTRACT

In the present paper, by making use the Horadam polynomials, we introduce and investigate two new subclasses $\mathcal{H}_\Sigma(\alpha,\mu,r)$ and $\mathcal{R}_\Sigma(\Upsilon,\delta,\lambda,r)$ of the function class Σ of holomorphic bi-univalent functions in the open unit disk Δ. For functions belonging to these subclasses, we obtain upper bounds for the second and third coefficients and discuss Fekete-Szegö problem. Furthermore, we point out several new special cases of our results.

1. Introduction

Let $\mathcal{A}$ symbolize the class of function $f(z)$ which are holomorphic in the open unit disk

$$\Delta = \{ z : z \in \mathbb{C} \; and \; |z| < 1\}$$

and normalized under the conditions $f(0) = 0$ and $f'(0) = 1$ and having the following shape:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ symbolize the subclass of functions in $\mathcal{A}$ which are univalent in Δ. Since univalent functions are one–to–one, they are invertible and inverse functions need not be defined on the

[a] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniyah, Iraq, E-Mail: najah.ali@qu.edu.iq

[b] Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq, E-Mail: abbas.kareem.w@qu.edu.iq

entire unit disk Δ. However, the famous Koebe one–quarter theorem [5] ensure that the image of the unit disk Δ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, each function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

Let D be the class of functions F which is holomorphic in Δ with

$$F(0) = 0 \text{ and } |F(z)| < 1 \quad (z \in \Delta).$$

Let $\mathcal{N}(z)$ and $\mathcal{M}(z)$ be holomorphic in Δ then the function $\mathcal{N}(z)$ is said to subordinate to $\mathcal{M}(z)$ in Δ written by

$$\mathcal{N}(z) \prec \mathcal{M}(z) \qquad (z \in \Delta), \tag{1.3}$$

such that $\mathcal{N}(z) = \mathcal{M}(F(z))$ $(z \in \Delta)$. From the definition of the subordination, it is easy to show that the subordination (1.3) implies that

$$\mathcal{N}(0) = \mathcal{M}(0) \text{ and } \mathcal{N}(\Delta) \subset \mathcal{M}(\Delta). \tag{1.4}$$

In particular, if $\mathcal{M}(z)$ is univalent in Δ, then the subordination (1.3) is equivalent to the condition (1.4).

The function $f \in \mathcal{A}$ is considered bi-univalent in Δ if together f^{-1} and f are univalent in Δ. Indicated by the Taylor-Maclaurin series expansion (1.1), the class of all bi-univalent functions in Δ can be symbolized by Σ . In the year 2010, Srivastava et al. [10] refreshed the study of various classes of bi-univalent functions. Moreover, many penmans explored bounds for different subclasses of bi-univalent functions (see, for example [3,4,8,11,12]). The coefficient estimate problem involving the bound of $|a_n|$ ($n \in \mathbb{N} - \{1,2\}, \mathbb{N} = \{1,2,3, \ldots\}$) is still an open problem.

The Horadam polynomials $h_k(r)$ are defined by the following recurrence relation (see [7])

$$h_k(r) = prh_{k-1}(r) + qh_{k-2}(r), \ \ (r \in \mathbb{R},\ \ k \in \mathbb{N} - \{1,2\}, \mathbb{N} = \{1,2,3, \ldots\}), \tag{1.5}$$

with $h_1(r) = e, h_2(r) = br$, where e, b, p, and q are some real constants. It is very clear from (1.5) that $h_3(r) = pbr^2 + eq$. The generating function of the Horadam polynomials $h_k(r)$ is given by (see [6])

$$\Psi(r,z) = \sum_{k=1}^{\infty} h_k(r)z^{k-1} = \frac{e + (b - ep)rz}{1 - prz - qz^2} \ .$$

2. Coefficient bounds and Fekete–Szegö inequality for the class $\mathcal{H}_\Sigma(\alpha,\mu,r)$

Definition (2.1): A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\alpha,\mu,r)$ for $0 \le \alpha \le 1$, $0 \le \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$(1-\alpha)\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)+\alpha\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right) \prec \Psi(r,z)+1-e \tag{2.1}$$

and

$$(1-\alpha)\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)+\alpha\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right) \prec \Psi(r,w)+1-e, \tag{2.2}$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\alpha = 0$ in Definition (2.1), we have the following definition:

Definition (2.2): A function $f \in \Sigma$ is said to be in the class $T_\Sigma(\mu,r)$ for $0 \le \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) \prec \Psi(r,z)+1-e \tag{2.3}$$

and

$$\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right) \prec \Psi(r,w)+1-e, \tag{2.4}$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\alpha = 1$ in Definition (2.1), we have the following definition:

Definition (2.3): A function $f \in \Sigma$ is said to be in the class $C_\Sigma(\mu,r)$ for $0 \le \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right) \prec \Psi(r,z)+1-e \tag{2.5}$$

and

$$\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right) \prec \Psi(r,w)+1-e, \tag{2.6}$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

Remark (2.1)

1) For $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha,\mu,r)$ shortens to the function class $\mathcal{M}_\sigma(\alpha,r)$ presented and investigated by Abirami et al. [1].
2) For $\alpha = 0$ and $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha,\mu,r)$ shortens to the function class $\mathcal{W}_\Sigma(r)$ presented and investigated by Srivastava et al. [9].
3) For $\alpha = 1$ and $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha,\mu,r)$ shortens to the function class $\mathcal{K}_\Sigma(r)$ presented and investigated by Abirami et al. [1].

Theorem (2.1): For $0 \le \alpha \le 1$, $0 \le \mu < 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\alpha,\mu,r)$. Then

$$|a_2| \le \frac{|br|\sqrt{|br|}}{\sqrt{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2p\right]br^2-(1+\alpha)^2(1-\mu)^2eq\right|}} \tag{2.7}$$

and

$$|a_3| \le \frac{b^2r^2}{(1+\alpha)^2(1-\mu)^2} + \frac{|br|}{2(1+2\alpha)(1-\mu)}, \tag{2.8}$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|br|}{2(1+2\alpha)(1-\mu)} \quad if \\ \left(|\nu-1| \le \dfrac{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2p\right]br^2-(1+\alpha)^2(1-\mu)^2eq\right|}{2(1+2\alpha)(1-\mu)b^2r^2}\right) \\ \dfrac{|br|^3|\nu-1|}{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2p\right]br^2-(1+\alpha)^2(1-\mu)^2eq\right|} \quad if \\ \left(|\nu-1| \ge \dfrac{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2p\right]br^2-(1+\alpha)^2(1-\mu)^2eq\right|}{2(1+2\alpha)(1-\mu)b^2r^2}\right). \end{cases} \tag{2.9}$$

Proof: Let $f \in \mathcal{H}_\Sigma(\alpha,\mu,r)$. Then there are two holomorphic functions $k, y: \Delta \to \Delta$ given by

$$|k(z)| = k_1z + k_2z^2 + k_3z^3 + \cdots \qquad (z \in \Delta) \tag{2.10}$$

and

$$|y(w)| = y_1w + y_2w^2 + y_3w^3 + \cdots \qquad (w \in \Delta), \tag{2.11}$$

with $k(0) = y(0) = 0$, $|k(z)| < 1$, $|y(w)| < 1$ and $z, w \in \Delta$ such that

$$(1-\alpha)\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) + \alpha\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right) \prec \Psi(r,k(z)) + 1 - e$$

and

$$(1-\alpha)\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)+\alpha\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right)\prec \Psi(r,y(w))+1-e.$$

Or, equivalently,

$$(1-\alpha)\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)+\alpha\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right)$$

$$=1+h_1(r)-e+h_2(r)k(z)+\ h_3(r)[k(z)]^2+\cdots \tag{2.12}$$

and

$$(1-\alpha)\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)+\alpha\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right)$$

$$=1+h_1(r)-e+h_2(r)y(w)+h_3(r)[y(w)]^2+\cdots. \tag{2.13}$$

Combining (2.10), (2.11), (2.12) and (2.13) yields

$$(1-\alpha)\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)+\alpha\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right)$$

$$=1+h_2(r)k_1z+[h_2(r)k_2+h_3(r)k_1^2]z^2+\cdots \tag{2.14}$$

and

$$(1-\alpha)\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)+\alpha\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right)$$

$$=1+h_2(r)y_1w+[h_2(r)y_2+h_3(r)y_1^2]w^2+\cdots. \tag{2.15}$$

It is clear that if $|k(z)|<1$ and $|y(w)|<1$, $z,w\in\Delta$, then

$$|k_i|\le 1 \ and\ |y_i|\le 1 \quad (\, i\in\mathbb{N}\,). \tag{2.16}$$

From (2.14) and (2.15), it follows that

$$(1+\alpha)(1-\mu)a_2=h_2(r)k_1, \tag{2.17}$$

$$(1+3\alpha)(\mu^2-1)a_2^2+2(1+2\alpha)(1-\mu)a_3=h_2(r)k_2+h_3(r)k_1^2, \tag{2.18}$$

$$-(1+\alpha)(1-\mu)a_2=h_2(r)y_1 \tag{2.19}$$

and

$$[(\mu^2-4\mu+3)+\alpha(3\mu^2-8\mu+5)]a_2^2-2(1+2\alpha)(1-\mu)a_3=h_2(r)y_2+h_3(r)y_1^2. \tag{2.20}$$

From (2.17) and (2.19), we get

$$k_1=-y_1 \tag{2.21}$$

and

$$2(1+\alpha)^2(1-\mu)^2a_2^2 = [h_2(r)]^2(k_1^2+y_1^2). \tag{2.22}$$

If we add (2.18) to (2.20), we attain

$$[2(\mu^2-2\mu+1)+2\alpha(3\mu^2-4\mu+1)]a_2^2 = h_2(r)(k_2+y_2)+h_3(r)(k_1^2+y_1^2). \tag{2.23}$$

By using (2.22) in (2.23), we deduce that

$$\left[2(\mu^2-2\mu+1)+2\alpha(3\mu^2-4\mu+1)-\frac{2(1+\alpha)^2(1-\mu)^2h_3(r)}{[h_2(r)]^2}\right]a_2^2 = h_2(r)(k_2+y_2), \tag{2.24}$$

which yields

$$|a_2| \le \frac{|br|\sqrt{|br|}}{\sqrt{|[((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1))b-(1+\alpha)^2(1-\mu)^2p]br^2-(1+\alpha)^2(1-\mu)^2eq|}}.$$

Next, by subtracting (2.20) from (2.18), we have

$$4(1+2\alpha)(1-\mu)(a_3-a_2^2) = h_2(r)(k_2-y_2)+h_3(r)(k_1^2-y_1^2). \tag{2.25}$$

In view of (2.21) and (2.22), we obtain from (2.25)

$$a_3 = \frac{[h_2(r)]^2(k_1^2+y_1^2)}{2(1+\alpha)^2(1-\mu)^2}+\frac{h_2(r)(k_2-y_2)}{4(1+2\alpha)(1-\mu)}.$$

Hence using (1.5), we deduce that

$$|a_3| \le \frac{b^2r^2}{(1+\alpha)^2(1-\mu)^2}+\frac{|br|}{2(1+2\alpha)(1-\mu)}.$$

Finally, by using (2.24) and (2.25) for some $\nu \in \mathbb{R}$, we obtain

$$a_3-\nu a_2^2 = \frac{h_2(r)(k_2-y_2)}{4(1+2\alpha)(1-\mu)} + \frac{[h_2(r)]^3(1-\nu)(k_2+y_2)}{[2(\mu^2-2\mu+1)+2\alpha(3\mu^2-4\mu+1)][h_2(r)]^2-2(1+\alpha)^2(1-\mu)^2h_3(r)}$$

$$= \frac{h_2(r)}{2}\left[\left(\Omega(\nu,r)+\frac{1}{2(1+2\alpha)(1-\mu)}\right)k_2+\left(\Omega(\nu,r)-\frac{1}{2(1+2\alpha)(1-\mu)}\right)y_2\right],$$

where

$$\Omega(\nu,r) = \frac{[h_2(r)]^2(1-\nu)}{[(\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)][h_2(r)]^2-(1+\alpha)^2(1-\mu)^2h_3(r)}.$$

According to (1.5), we deduce that

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|br|}{2(1+2\alpha)(1-\mu)} & if \ \left(0 \leq \ |\Omega(\nu,r)| \ \leq \ \dfrac{1}{2(1+2\alpha)(1-\mu)}\right) \\ |br||\Omega(\nu,r)| & if \ \left(|\Omega(\nu,r)| \geq \dfrac{1}{2(1+2\alpha)(1-\mu)}\right). \end{cases}$$

After some computations, we get

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|br|}{2(1+2\alpha)(1-\mu)} \quad if \\ \left(|\nu - 1| \leq \dfrac{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2 p\right]br^2-(1+\alpha)^2(1-\mu)^2 eq\right|}{2(1+2\alpha)(1-\mu)b^2r^2}\right) \\ \dfrac{|br|^3|\nu-1|}{\left|\left[((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1))b-(1+\alpha)^2(1-\mu)^2 p\right]br^2-(1+\alpha)^2(1-\mu)^2 eq\right|} \quad if \\ \left(|\nu - 1| \geq \dfrac{\left|\left[\left((\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1)\right)b-(1+\alpha)^2(1-\mu)^2 p\right]br^2-(1+\alpha)^2(1-\mu)^2 eq\right|}{2(1+2\alpha)(1-\mu)b^2r^2}\right). \end{cases}$$

By putting $\alpha = 0$ in Theorem (2.1), we attain the corollary in the below:

Corollary (2.1): Let the function $f(z)$ indicated by (1.1) be in the class $T_\Sigma(\mu, r)(\ 0 \leq \mu < 1 \ , r \in \mathbb{R} \ )$. Then

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|[(\mu^2 - 2\mu + 1)b - (1-\mu)^2 p]br^2 - (1-\mu)^2 eq|}}$$

and

$$|a_3| \leq \frac{b^2r^2}{(1-\mu)^2} + \frac{|br|}{2(1-\mu)},$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|br|}{2(1-\mu)} \quad if \\ \left(|\nu - 1| \leq \dfrac{|[(\mu^2 - 2\mu + 1)b - (1-\mu)^2 p]br^2 - (1-\mu)^2 eq|}{2(1-\mu)b^2r^2}\right) \\ \dfrac{|br|^3|\nu - 1|}{|[(\mu^2 - 2\mu + 1)b - (1-\mu)^2 p]br^2 - (1-\mu)^2 eq|} \quad if \\ \left(|\nu - 1| \geq \dfrac{|[(\mu^2 - 2\mu + 1)b - (1-\mu)^2 p]br^2 - (1-\mu)^2 eq|}{2(1-\mu)b^2r^2}\right). \end{cases}$$

By putting $\alpha = 1$ in Theorem (2.1), we attain the corollary in the below:

Corollary (2.2): Let $f(z)$ indicated by (1.1) be in the class $\mathcal{C}_\Sigma(\mu, r)(\ 0 \leq \mu < 1 \ , r \in \mathbb{R} \ )$. Then

$$|a_2| \le \frac{|br|\sqrt{|br|}}{\sqrt{|[(4\mu^2-6\mu+2)b-4(1-\mu)^2p]br^2-4(1-\mu)^2eq|}}$$

and

$$|a_3| \le \frac{b^2r^2}{4(1-\mu)^2} + \frac{|br|}{6(1-\mu)},$$

and for some $\nu \in \mathbb{R}$,

$$|a_3-\nu a_2^2| \le \begin{cases} \dfrac{|br|}{6(1-\mu)} \quad if \\ \left(|\nu-1| \le \dfrac{|[(2\mu^2-3\mu+1)b-2(1-\mu)^2p]br^2-2(1-\mu)^2eq|}{3(1-\mu)b^2r^2}\right) \\ \dfrac{|br|^3|\nu-1|}{|[(4\mu^2-6\mu+2)b-4(1-\mu)^2p]br^2-4(1-\mu)^2eq|} \quad if \\ \left(|\nu-1| \ge \dfrac{|[(2\mu^2-3\mu+1)b-2(1-\mu)^2p]br^2-2(1-\mu)^2eq|}{3(1-\mu)b^2r^2}\right). \end{cases}$$

Remark (2.2): In Theorem (2.1), if we choose

1) $\mu = 0$, then we get the outcomes which was proven by Abirami et al. [1].
2) $\alpha = 0$ and $\mu = 0$, then we get the outcomes which was proven by Srivastava et al. [9].
3) $\alpha = 1$ and $\mu = 0$, then we attain the outcomes which was proven by Abirami et al. [1].

3. Coefficient bounds and Fekete–Szegö inequality for the class $\mathcal{R}_\Sigma(\Upsilon,\delta,\lambda,r)$

Definition (3.1): A function $f \in \Sigma$ is said to be in the class $\mathcal{R}_\Sigma(\Upsilon,\delta,\lambda,r)$ for $\Upsilon \ge 0, 0 \le \delta \le 1, \lambda \in \mathbb{C}\backslash\{0\}$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$1+\frac{1}{\lambda}\left[\Upsilon\delta(zf''(z)-2)+(\delta(\Upsilon+1)+\Upsilon)f'(z)+(1-\Upsilon)(1-\delta)\frac{f(z)}{z}-1\right] \prec \Psi(r,z)+1-e \tag{3.1}$$

and

$$1+\frac{1}{\lambda}\left[\Upsilon\delta(wg''(w)-2)+(\delta(\Upsilon+1)+\Upsilon)g'(w)+(1-\Upsilon)(1-\delta)\frac{g(w)}{w}-1\right] \prec \Psi(r,w)+1-e, \tag{3.2}$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\delta = 0$ in Definition (3.1), we have the following definition:

Definition (3.2): A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}(\Upsilon, \lambda, r)$ for $\Upsilon \geq 0, \lambda \in \mathbb{C}\backslash\{0\}$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$1 + \frac{1}{\lambda}\left[\Upsilon f'(z) + (1-\Upsilon)\frac{f(z)}{z} - 1\right] \prec \Psi(r, z) + 1 - e \tag{3.3}$$

and

$$1 + \frac{1}{\lambda}\left[\Upsilon g'(w) + (1-\Upsilon)\frac{g(w)}{w} - 1\right] \prec \Psi(r, w) + 1 - e, \tag{3.4}$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

Remark (3.1)

For $\Upsilon = \lambda = 1$ and $\delta = 0$, the function class $\mathcal{R}_{\Sigma}(\Upsilon, \delta, \lambda, r)$ shortens to the function class Σ' presented and investigated by Alamoush [2].

Theorem (3.1): For $\Upsilon \geq 0, 0 \leq \delta \leq 1, \lambda \in \mathbb{C}\backslash\{0\}$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{R}_{\Sigma}(\Upsilon, \delta, \lambda, r)$. Then

$$|a_2| \leq \frac{|\lambda br|\sqrt{|br|}}{\sqrt{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b - (\delta(5\Upsilon+1)+\Upsilon+1)^2 p]br^2 - (\delta(5\Upsilon+1)+\Upsilon+1)^2 eq|}} \tag{3.5}$$

and

$$|a_3| \leq \frac{|\lambda|^2 b^2 r^2}{(\delta(5\Upsilon+1)+\Upsilon+1)^2} + \frac{|\lambda br|}{(2\delta(5\Upsilon+1)+2\Upsilon+1)}, \tag{3.6}$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\lambda br|}{2\delta(5\Upsilon+1)+2\Upsilon+1} \quad if \\ \left(|\nu-1| \leq \frac{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2 p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2 eq|}{|\lambda|(br)^2(2\delta(5\Upsilon+1)+2\Upsilon+1)}\right) \\ \frac{|\lambda|^2|br|^3|\nu-1|}{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2 p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2 eq|} \quad if \\ \left(|\nu-1| \geq \frac{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2 p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2 eq|}{|\lambda|(br)^2(2\delta(5\Upsilon+1)+2\Upsilon+1)}\right). \end{cases} \tag{3.7}$$

Proof: Let $f \in \mathcal{R}_{\Sigma}(\Upsilon, \delta, \lambda, r)$. Then there are two holomorphic functions $k, y: \Delta \to \Delta$ given by

$$|k(z)| = k_1 z + k_2 z^2 + k_3 z^3 + \cdots \qquad (z \in \Delta) \tag{3.8}$$

and

$$|y(w)| = y_1 w + y_2 w^2 + y_3 w^3 + \cdots \qquad (w \in \Delta), \tag{3.9}$$

with $k(0) = y(0) = 0, |k(z)| < 1, |y(w)| < 1$ and $z, w \in \Delta$ such that

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(zf''(z) - 2) + (\delta(\Upsilon + 1) + \Upsilon)f'(z) + (1 - \Upsilon)(1 - \delta)\frac{f(z)}{z} - 1\right] \prec \Psi(r, k(z)) + 1 - e$$

and

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(wg''(w) - 2) + (\delta(\Upsilon + 1) + \Upsilon)g'(w) + (1 - \Upsilon)(1 - \delta)\frac{g(w)}{w} - 1\right] \prec \Psi(r, y(w)) + 1 - e.$$

Or, equivalently,

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(zf''(z) - 2) + (\delta(\Upsilon + 1) + \Upsilon)f'(z) + (1 - \Upsilon)(1 - \delta)\frac{f(z)}{z} - 1\right]$$

$$= 1 + h_1(r) - e + h_2(r)k(z) + h_3(r)[k(z)]^2 + \cdots \tag{3.10}$$

and

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(wg''(w) - 2) + (\delta(\Upsilon + 1) + \Upsilon)g'(w) + (1 - \Upsilon)(1 - \delta)\frac{g(w)}{w} - 1\right]$$

$$= 1 + h_1(r) - e + h_2(r)y(w) + h_3(r)[y(w)]^2 + \cdots. \tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11) yields

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(zf''(z) - 2) + (\delta(\Upsilon + 1) + \Upsilon)f'(z) + (1 - \Upsilon)(1 - \delta)\frac{f(z)}{z} - 1\right]$$

$$= 1 + h_2(r)k_1 z + [h_2(r)k_2 + h_3(r)k_1^2]z^2 + \cdots \tag{3.12}$$

and

$$1 + \frac{1}{\lambda}\left[\Upsilon\delta(wg''(w) - 2) + (\delta(\Upsilon + 1) + \Upsilon)g'(w) + (1 - \Upsilon)(1 - \delta)\frac{g(w)}{w} - 1\right]$$

$$= 1 + h_2(r)y_1 w + [h_2(r)y_2 + h_3(r)y_1^2]w^2 + \cdots. \tag{3.13}$$

It is clear that if $|k(z)| < 1$ and $|y(w)| < 1, z, w \in \Delta$, then

$$|k_i| \le 1 \; and \; |y_i| \le 1 \quad (i \in \mathbb{N}). \tag{3.14}$$

Equating the coefficients in (3.12) and (3.13), we find that

$$\frac{\delta(5\Upsilon + 1) + \Upsilon + 1}{\lambda} a_2 = h_2(r)k_1, \tag{3.15}$$

$$\frac{2\delta(5\Upsilon + 1) + 2\Upsilon + 1}{\lambda} a_3 = h_2(r)k_2 + h_3(r)k_1^2, \tag{3.16}$$

$$-\frac{\delta(5\Upsilon+1)+\Upsilon+1}{\lambda}a_2=h_2(r)y_1 \tag{3.17}$$

and

$$\frac{2\delta(5\Upsilon+1)+2\Upsilon+1}{\lambda}(2a_2^2-a_3)=h_2(r)y_2+h_3(r)y_1^2. \tag{3.18}$$

From (3.15) and (3.17), we get

$$k_1=-y_1 \tag{3.19}$$

and

$$\frac{2(\delta(5\Upsilon+1)+\Upsilon+1)^2}{\lambda^2}a_2^2=[h_2(r)]^2(k_1^2+y_1^2). \tag{3.20}$$

If we add (3.16) to (3.18), we attain

$$\frac{2(2\delta(5\Upsilon+1)+2\Upsilon+1)}{\lambda}a_2^2=h_2(r)(k_2+y_2)+h_3(r)(k_1^2+y_1^2). \tag{3.21}$$

By using (3.20) in (3.21), we deduce that

$$\left[\frac{2(2\delta(5\Upsilon+1)+2\Upsilon+1)}{\lambda}-\frac{2(\delta(5\Upsilon+1)+\Upsilon+1)^2h_3(r)}{\lambda^2[h_2(r)]^2}\right]a_2^2=h_2(r)(k_2+y_2), \tag{3.22}$$

which yields

$$|a_2|\le\frac{|\lambda br|\sqrt{|br|}}{\sqrt{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2eq|}}.$$

Next, by subtracting (3.18) from (3.16), we have

$$\left(\frac{2(2\delta(5\Upsilon+1)+2\Upsilon+1)}{\lambda}\right)(a_3-a_2^2)=h_2(r)(k_2-y_2)+h_3(r)(k_1^2-y_1^2). \tag{3.23}$$

In view of (3.19) and (3.20), we obtain from (3.23)

$$a_3=\frac{\lambda^2[h_2(r)]^2(k_1^2+y_1^2)}{2(\delta(5\Upsilon+1)+\Upsilon+1)^2}+\frac{\lambda h_2(r)(k_2-y_2)}{2(2\delta(5\Upsilon+1)+2\Upsilon+1)}.$$

Hence using (1.5), we deduce that

$$|a_3|\le\frac{|\lambda|^2b^2r^2}{(\delta(5\Upsilon+1)+\Upsilon+1)^2}+\frac{|\lambda br|}{(2\delta(5\Upsilon+1)+2\Upsilon+1)}.$$

Finally, by using (3.22) and (3.23) for some $\nu\in\mathbb{R}$, we obtain

$$a_3 - \nu a_2^2 = \frac{\lambda h_2(r)(k_2 - y_2)}{2(2\delta(5\Upsilon+1)+2\Upsilon+1)} + \frac{\lambda^2[h_2(r)]^3(1-\nu)(k_2+y_2)}{2\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)[h_2(r)]^2 - 2(\delta(5\Upsilon+1)+\Upsilon+1)^2 h_3(r)}$$

$$= \frac{\lambda h_2(r)}{2}\left[\left(\Omega(\nu,r) + \frac{1}{2\delta(5\Upsilon+1)+2\Upsilon+1}\right)k_2 + \left(\Omega(\nu,r) - \frac{1}{2\delta(5\Upsilon+1)+2\Upsilon+1}\right)y_2\right],$$

where

$$\Omega(\nu,r) = \frac{\lambda[h_2(r)]^2(1-\nu)}{\lambda[2\delta(5\Upsilon+1)+2\Upsilon+1][h_2(r)]^2 - (\delta(5\Upsilon+1)+\Upsilon+1)^2 h_3(r)}.$$

According to (1.5), we deduce that

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|\lambda br|}{2\delta(5\Upsilon+1)+2\Upsilon+1} & if \left(0 \le |\Omega(\nu,r)| \le \dfrac{1}{2\delta(5\Upsilon+1)+2\Upsilon+1}\right) \\ |\lambda br||\Omega(\nu,r)| & if \left(|\Omega(\nu,r)| \ge \dfrac{1}{2\delta(5\Upsilon+1)+2\Upsilon+1}\right). \end{cases}$$

After some computations, we get

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|\lambda br|}{2\delta(5\Upsilon+1)+2\Upsilon+1} \quad if \\ \left(|\nu-1| \le \dfrac{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2eq|}{|\lambda|(br)^2(2\delta(5\Upsilon+1)+2\Upsilon+1)}\right) \\ \dfrac{|\lambda|^2|br|^3|\nu-1|}{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2eq|} \quad if \\ \left(|\nu-1| \ge \dfrac{|[\lambda(2\delta(5\Upsilon+1)+2\Upsilon+1)b-(\delta(5\Upsilon+1)+\Upsilon+1)^2p]br^2-(\delta(5\Upsilon+1)+\Upsilon+1)^2eq|}{|\lambda|(br)^2(2\delta(5\Upsilon+1)+2\Upsilon+1)}\right). \end{cases}$$

By putting $\delta = 0$ in Theorem (3.1), we attain the corollary in the below:

Corollary (3.1): Let the function $f(z)$ indicated by (1.1) be in the class $\mathcal{B}_\Sigma(\Upsilon,\lambda,r)(\Upsilon \ge 0,\ \lambda \in \mathbb{C}\backslash\{0\}, r \in \mathbb{R})$. Then

$$|a_2| \le \frac{|\lambda br|\sqrt{|br|}}{\sqrt{|[\lambda(2\Upsilon+1)b - (\Upsilon+1)^2p]br^2 - (\Upsilon+1)^2eq|}}$$

and

$$|a_3| \le \frac{|\lambda|^2 b^2 r^2}{(\Upsilon+1)^2} + \frac{|\lambda br|}{(2\Upsilon+1)},$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|\lambda br|}{(2\Upsilon+1)} \quad if \\ \left(|\nu - 1| \leq \dfrac{|[\lambda(2\Upsilon+1)b - (\Upsilon+1)^2 p]br^2 - (\Upsilon+1)^2 eq|}{|\lambda|(br)^2(2\Upsilon+1)}\right) \\ \dfrac{|\lambda|^2 |br|^3 |\nu - 1|}{|[\lambda(2\Upsilon+1)b - (\Upsilon+1)^2 p]br^2 - (\Upsilon+1)^2 eq|} \quad if \\ \left(|\nu - 1| \geq \dfrac{|[\lambda(2\Upsilon+1)b - (\Upsilon+1)^2 p]br^2 - (\Upsilon+1)^2 eq|}{|\lambda|(br)^2(2\Upsilon+1)}\right). \end{cases}$$

Remark (3.2)

If we put $\Upsilon = \lambda = 1$ and $\delta = 0$ in Theorem (3.1), we get the results which were given by Alamoush [2].

References

[1] C. Abirami, N. Magesh and J. Yamini, Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials, Abstract and Applied Analysis, 2020, Art. ID 7391058, 8 pp.

[2] A. G. Alamoush, On a subclass of bi-univalent functions associated to Horadam polynomials, Int. J. Open Problems Complex Analysis, 12(1)(2020), 58-65.

[3] I. T. Awolere and A. T. Oladipo, Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials, Khayyam J. Math., 5(1)(2019), 140-149.

[4] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C. R. Acad. Paris. Ser. I., 352(6)(2014), 479-484.

[5] P. L. Duren, Univalent Functions, Vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 1983.

[6] A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, Fibonacci Quart., 23(1985), 7-20.

[7] T. Horzum and E. G. Kocer, On some properties of Horadam polynomials, Int. Math. Forum, 4(2009), 1243-1252.

[8] B. Şeker, On a new subclass of bi-univalent functions defined by using Salagean operator, Turk. J. Math., 42(2018), 2891-2896.

[9] H. M. Srivastava, Ş. Altınkaya and S. Yalçın, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, Iran. J. Sci. Technol. Trans. A Sci.,43(2019), 1873-1879.

[10] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23(2010), 1188–1192.

[11] A. K. Wanas, Coefficient estimates for Bazilevič functions of bi-prestarlike functions, Miskolc Mathematical Notes, 21(2)(2020), 1031-1040.

[12] A. Zireh and S. Hajiparvaneh, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Bol. Soc. Paran. Mat., 39(4)(2021), 153-164.