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# The Performance of Some Restricted Estimators In Restricted Linear Regression Model

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**ABSTRACT**

In the linear regression model, the restricted biased estimation as one of important methods to address the high variance and multicollinearity problems. In this paper, we make the simulation study of some restricted biased estimators. The mean square error (MSE) criterion are used to make a comparison among them. According to the simulation study we observe that, the performance of the restricted modified unbiased ridge regression estimator (RMUR) was proposed by Bader and Mustafa Ismaeel Naif [16] is better than of these estimators. Numerical example has been considered to illustrate the performance of the estimators.

## 1. Introduction

Consider the standard linear regression model

$$Y = X\beta + e, \tag{1.1}$$

where $X$ is an $n \times p$ matrix of the explanatory variables, $Y$ is an $n \times 1$ vector of the response, $\beta$ is a $1 \times p$ vector of the unknown parameters and $e$ is an $n \times 1$ vector of the random errors with the mean $E(e) = 0$ and the variance $Var(e) = \sigma^2 I_n$, $I_n$ is identity matrix of order $n$ . sometimes, we have a linear restriction on $\beta$ as:

$$R\beta = r \tag{1.2}$$

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where $R$ is an $m \times p$ non zero matrix with rank$(R) = m < p$ and $r$ is an $m \times 1$ vector. The ordinary restricted least square estimator (RLS) is given by

$$\hat{\beta}_{RLS} = \hat{\beta} + S^{-1}R'(RS^{-1}R')^{-1}(r - R\hat{\beta}), \tag{1.3}$$

where $S^{-1} = (X'X)^{-1}$ and $\hat{\beta} = S^{-1}X'y$ is the least square estimator (OLSE) . The RLS estimator is widely used because it improves the variance value, but most researchers often encounter of the multicollinearity problem. That is, $X'X$ is always ill-conditioned due to the linear relationship among the regressors of $X$ matrix. Therefore, the unknown coefficient, estimated by RLS is usually instable and misleading( Najarian [15] ). In order to improve the performance of the estimators, the researchers proposed to use the bias estimation technique. Therefore, Sarkar [5], obtained the restricted ridge regression estimator (RRR) by combining in particular way the two approaches underling the RLS and the ORR estimator given as

$$\hat{\beta}_R = (S + kI)^{-1}X'Y \,. \tag{1.4}$$

The RRR estimator considered as a modify the RLS estimator by using ORR philosophy The RRR is denoted by $\beta^*(k)$ is given as:

$$\beta^*(k) = W\hat{\beta}_{RLS}, \tag{1.5}$$

where $W = (I + kS^{-1})^{-1}$.However the $\beta^*(k)$ dose not satisfy the linear restriction (1.2). Kacıranlar [17] proposed the restricted Liu estimator (RLE) by following the way that Sarkar [18] has done while obtaining the RRR estimator. By replacing, in the Liu estimator, the OLSE estimator by the RLS estimator they obtained the restricted Liu estimator as the follows:

$$\hat{\beta}_{rd} = (X'X + I)^{-1}(X'X + dI)\hat{\beta}_{RLS} \,. \tag{1.6}$$

Also the RLE estimator dose not satisfy the linear restriction (1.2). Alheety [2], obtained the restricted (k – d) class estimator to improve the restricted Liu estimator, by substituting the ORR for OLSE estimators. The restricted (k – d) estimator is general case and includes the RLS, RRR and RLE estimators. The restricted (k – d) class is denoted by $\hat{\beta}(k,d)$ as the follows

$$\hat{\beta}(k,d) = F_{k,d}\ \hat{\beta}_{RLS}, \tag{1.7}$$

where $F_{k,d} = (X'X + I)^{-1}(X'X + (k + d)I)(X'X + kI)^{-1}(X'X)$, the special cases of the restricted (k – d) class estimator are given by

$$\hat{\beta}(0,1) = \hat{\beta}_{RLS} \,. \tag{1.8}$$

$$\hat{\beta}(k, 1 - k) = \beta^*(k) \,. \tag{1.9}$$

$$\hat{\beta}(0,d) = \hat{\beta}(d), \tag{1.10}$$

Where $\hat{\beta}(d)$ is the liu estimator . Bader and Mustafa [4], introduced the restricted modified unbiased ridge regression (RMUR ) based on the modified unbiased ridge regression (MURR) was proposed by Batah and Gore [5],which is given as follows

$$\hat{\beta}_J(k) = \left[I - k\left(X'X + kI_p\right)^{-1}\right]\left(X'X + kI_p\right)^{-1}(X'Y + kJ)$$

The RMUR estimator is denoted by $\hat{\beta}_r^*(k)$ as the following

$$\hat{\beta}_r^*(k) = \hat{\beta}_J(k) + S_k^{-1}R'(RS_k^{-1}R')^{-1}\left(r - R\hat{\beta}_J(k)\right), \qquad (1.11)$$

Hence the RMUR estimator is hold the linear restriction (2). Therefore, the goal of this paper is to review and compare some restricted estimators .In section 2 ,we study the properties of the RLS, RRR, (k - d ) class and the RMUR estimators while in section 3, we make simulation study of these estimators by using the Matlab program . Section 4 contains numerical example, to illustrate the results Finally, the conclusions with some remarks are given in section 5 .

## 2.Restricted Estimators And Its Properties

In this section , we study the statistical properties of some restricted estimators and then, we show that, the performance of some the restricted estimators by using the mean square error matrix (MSE) criterion . The MSE of any estimator is given by:

$$MSE(\beta^*) = Var(\beta^*) + \left(bais(\beta^*)\right).\left(bias(\beta^*)\right)', \qquad (2.1)$$

where

$$Var(\beta^*) = E[(\beta^* - E(\beta^*)\left(\left(\beta^* - E(\beta^*)\right)'\right], \qquad (2.2)$$

and

$$Bias(\beta^*) = E(\beta^*) - \beta , \qquad (2.3)$$

where $E(\beta^*)$ the expected value of $\beta^*$

### 2.1 Restricted Ridge Regression (RRR)

Sarkar [18] proposed the restricted ridge regression (RRR) estimator . The variance, bias and mean square error of the RRR estimator are given

$$Var(\beta^*(k) = \sigma^2 WAW' \qquad (2.4)$$

$$bias\left(\beta^*(k)\right) = -kS_k^{-1}\beta \qquad (2.5)$$

$$MSE\left(\beta^*(k)\right) = \sigma^2 WAW' + k^2\beta' S_k^{-2}\beta . \qquad (2.6)$$

Thus the scalar men square error of the RRR estimator is given by

$$mse(\beta^*(k)) = \sigma^2 tr(WAW') + k^2 tr(\beta' S_k^{-2} \beta), \tag{2.7}$$

Where $tr$ denote the trace of matrix.

## 2.2 Restricted (k – d) Class Estimator

Alheety [2] introduced the restricted (k – d) class estimator. The variance, bias and mean square error of the (k – d) class estimator are given

$$Var\left(\hat{\beta}(k,d)\right) = \sigma^2 F_{k,d}\, S^{-1} F_{k,d}{}' \tag{2.8}$$

$$bias\left(\hat{\beta}(k,d)\right) = (S+I)^{-1}(d(S+kI)^{-1}S - I)\beta, \tag{2.9}$$

$$\mathrm{MSE}\left(\hat{\beta}(k,d)\right) = \sigma^2 F_{k,d}\, S^{-1} F_{k,d}' + (S+I)^{-1}(d(S+kI)^{-1}S - I)\beta\beta'$$

$$(S+I)^{-1}(d(S+kI)^{-1}S - I). \tag{2.10}$$

So, the scalar mean square error is denoted by

$$mse(\beta^*) = tr\, Var(\beta^*) + \| E(\beta^*) - \beta \|^2$$

So that, the scalar mean square error of (k – d) class estimator is given by:

$$mse\left(\hat{\beta}(k,d)\right) = \sigma^2 tr(F_{k,d}\, S^{-1} F_{k,d}') + \beta'(d(S+kI)^{-1}S - I)$$

$$(S+I)^{-2}(d(S+kI)^{-1}S - I)\beta\,. \tag{2.11}$$

## 2.3 Restricted Modified Unbiased Ridge Regression (RMUR) Estimator

We want to show that the properties of the RMUR estimator, so that, the variance, bias and mean square error of the RMUR estimator are given

$$Var(\hat{\beta}_r^*(k) = \sigma^2 N_k S_k M S_k^{-1} S_K' M' N_k' \tag{2.12}$$

$$bias\left(\hat{\beta}_r^*(k)\right) = -kN_k\beta \tag{2.13}$$

$$MSE\left(\hat{\beta}_r^*(k)\right) = \sigma^2 N_k S_k M S_k^{-1} S_k' M' N_k' + k^2 N_k \beta\beta' N_k'\,, \tag{2.14}$$

where $N_k = (S_k^{-1} - S_k^{-1} R'(RS_k^{-1}R')^{-1} RS_k^{-1}), M = (I - kS_k^{-1}) = SS_k^{-1}$. So that the scalar mean square error is given by

$$mse(\hat{\beta}_r^*(k)) = \sigma^2 tr(N_k SSS_k^{-1} N_k) + k^2 tr(N_k \beta\beta' N_k) \tag{2.15}$$

## 2.4.Estimated ridge parameter k

To study the performance of the MSE , we using the different values of ridge parameter k to evaluate mean square error. Hoerl and Kennard [8], introduced $k$ is denoted by $k_{HK}$ as the follows:

$$k_{HK} = \frac{\hat{\sigma}^2}{\hat{\varphi}_{max}^2 OLSE}, \tag{2.16}$$

Where $\hat{\varphi}_{max}^2$ is the maximum element of $\hat{\varphi}_{OLSE}$. Hoerl [8], suggested $k$ is denoted by

$$k_{HKB} = \frac{p\hat{\sigma}^2}{\hat{\varphi}_{OLSE}'\hat{\varphi}_{OLSE}}. \tag{2.17}$$

Lawless and Wang [12], suggested $k$ denoted by $k_{LW}$ as the follows:

$$k_{LW} = \frac{p\hat{\sigma}^2}{\hat{\varphi}_{OLSE}'X'X\hat{\varphi}_{OLSE}}. \tag{2.18}$$

Hocking [7], proposed $k$ denoted by $k_{HSL}$ as the follows:

$$k_{HSL} = \hat{\sigma}^2 \frac{\left(\sum_{i=1}^{p}(\lambda_i \hat{\varphi}_{OLSE})^2\right)}{\sum_{i=1}^{p}\left(\lambda_i \hat{\varphi}_{OLSE}^2\right)^2} . \tag{2.19}$$

Nomura [16], suggested $k$ is denoted by $k_{HMO}$ as the follows:

$$k_{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left(\frac{\hat{\varphi}_{iOLSE}}{1+\left(1+\lambda_i\left(\frac{\hat{\varphi}_{iOLSE}^2}{\hat{\sigma}^2}\right)^{\frac{1}{2}}\right)}\right)}. \tag{2.20}$$

Kibria [11], introduced the estimators for $k$ based on arithmetic mean (AM), geometric mean (GM), and median of $\frac{\hat{\sigma}^2}{\hat{\varphi}_i^2}$. These are defined as follows:

The estimator based on AM is denoted by $k_{AM}$ as the follows:

$$k_{AM} = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\varphi}_{iOLSE}}, \tag{2.21}$$

based on (GM), the estimator $k_{GM}$ as the follows:

$$k_{GM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p}\hat{\varphi}^2_{OLSE}\right)^{1/p}}, \tag{2.22}$$

based on median, the ridge parameter $k_{MED}$ is the follows:

$$k_{MED} = midian\left\{\frac{\hat{\sigma}^2}{\hat{\varphi}^2_{OLSE}}\right\}. \tag{2.23}$$

Khalaf and Shukur [9] suggested based on $k_{HK}$ denoted by $k_{KS}$ as

$$k_{KS} = \frac{\lambda_{max}\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_{max}\hat{\varphi}^2_{max}OLSE}, \tag{2.24}$$

Where $\lambda_{max}$ the maximum eigenvalues of $X'X$.

Alkhamisi et al. [3] suggested the following estimators of $k$ based on Kibria [11] , Khalaf and Shukur [9] , denoted by $k_{s_{airth}}$ as:

$$k_{s_{airth}} = \frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_i\hat{\varphi}^2_iOLSE}. \tag{2.25}$$

$$k_{s_{md}} = midian\left(\frac{\lambda_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+\lambda_i\hat{\varphi}^2_iOLSE}\right). \tag{2.26}$$

M .N. Lattef . et al [14] proposed the following estimators of $k$

$$k_{MU1} = \frac{\lambda_{med}\sum_{i=1}^{p}\hat{\varphi}^2_iOLSE}{\lambda_{max}}. \tag{2.27}$$

$$k_{MU2} = \left|\frac{p\hat{\sigma}^2}{\hat{\varphi}_{OLSE}{}'\hat{\varphi}_{OLSE}} - \frac{p\hat{\sigma}^2}{\hat{\varphi}'_{OLSE}X'X\hat{\varphi}_{OLSE}}\right|. \tag{2.28}$$

$$k_{MU3} = min\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\hat{\varphi}^2_iOLSE}{\hat{\sigma}^2}}\right). \tag{2.29}$$

$$k_{MU4} = max\left(\sqrt{\frac{\lambda_{min}\sum_{i=1}^{p}\hat{\varphi}^2_iOLSE}{\hat{\sigma}^2}}\right). \tag{2.30}$$

## 3. Simulation study

In this section , In this section, we make simulation study of the some restricted estimators by using the Matlab program. The purpose of this study to find the performance of this estimators. This simulation has been designed depends on specific factors that are expected to influence the properties of the estimators which may be subjected to a statistical investigation . Since the degree of the collinearity among several explanatory variables ($Xs$) is very essential, Kibria [10], was followed to generate $Xs$ using the following equation :

$$X_{ij} = (1-\emptyset^2)^{1/2} Z_{ij} + \emptyset Z_{ip} \, , i = 1,2, \ldots n, j = 1,2, \ldots p \tag{3.1}$$

where the $Z_{ij}$ independent standard normal pseudo-random numbers and $\emptyset$ represents the correlation between any two variables. These variables are standardized so that $X'X$ is being in correlation form. The response variable y is considered by:

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_{i2} X_{i2} \ldots \ldots \ldots .. + \beta_p X_{ip} + e_i, \quad i = 1,2, \ldots n \tag{3.2}$$

where $e_i$ is independent and identically distributed random variables (i.i.d.) $N(0, \sigma^2)$. Therefore, zero intercept for (3.2) will be assumed. Also the number of explanatory variables $p = 4$ while the values of $\sigma$ are chosen as (0.1,1, 5, 15). The correlation $\emptyset$ will choose as (0.85, 0.95, 0.99) and sample size n as (50,100,150,).The coefficients $\beta_1$, $\beta_2$, …, $\beta_p$ are selected as the eigenvectors corresponding to the largest eigenvalue of the matrix $X'X$ subject to constraint $\beta'\beta$ = 1. Thus, for all $n$ , $\sigma$ , $\emptyset$ , $p$ ,sets of $Xs$ are created. The experiment was replicated 2000 times by creating new error terms. Estimated mean square error (EMSE) is calculated as follows:

$$EMSE(\beta^*) = \frac{1}{2000} \sum_{i=1}^{2000} (\beta^* - \beta)'(\beta^* - \beta),$$

where $\beta^*$ would be any estimators (RLS,RRR, (k – d) class or RMUR).

Table1: : Estimated MSE when $n = 50$ $,\emptyset = .85$ $,p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 1.5206 | 0.8447 | 0.8358 | 0.3429 | 1 | $k_{HK}$ | 1.0477 | 0.9119 | 0.9104 | 0.4130 |
| | $k_{HKB}$ | 1.5206 | 0.8446 | 0.8358 | 0.3429 | | $k_{HKB}$ | 1.0477 | 0.9099 | 0.9104 | 0.4301 |
| | $k_{LW}$ | 1.5206 | 0.8447 | 0.8358 | 0.3429 | | $k_{LW}$ | 1.0477 | 0.9119 | 0.9104 | 0.4128 |
| | $k_{HSL}$ | 1.5206 | 0.8386 | 0.8353 | 0.3537 | | $k_{HSL}$ | 1.0477 | 0.9019 | 0.8929 | 0.9975 |
| | $k_{HMO}$ | 1.5206 | 0.8432 | 0.8357 | 0.3453 | | $k_{HMO}$ | 1.0477 | 0.8692 | 0.9092 | 0.7367 |
| | $k_{AM}$ | 1.5206 | 0.8394 | 0.8354 | 0.3522 | | $k_{AM}$ | 1.0477 | 0.8753 | 0.9094 | 0.7031 |
| | $k_{GM}$ | 1.5206 | 0.8447 | 0.8358 | 0.3429 | | $k_{GM}$ | 1.0477 | 0.9122 | 0.9104 | 0.4108 |
| | $k_{MED}$ | 1.5206 | 0.8443 | 0.8357 | 0.3435 | | $k_{MED}$ | 1.0477 | 0.9125 | 0.9104 | 0.4088 |
| | $k_{KS}$ | 1.5206 | 0.8440 | 0.8357 | 0.3440 | | $k_{KS}$ | 1.0477 | 0.9122 | 0.9104 | 0.4108 |
| | $k_{s_{arith}}$ | 1.5206 | 0.8443 | 0.8357 | 0.3435 | | $k_{s_{arith}}$ | 1.0477 | 0.9125 | 0.9104 | 0.4083 |
| | $k_{SMD}$ | 1.5206 | 0.8445 | 0.8357 | 0.3432 | | $k_{SMD}$ | 1.0477 | 0.9125 | 0.9104 | 0.4085 |
| | $k_{MU1}$ | 1.5206 | 0.8105 | 0.8332 | 0.4126 | | $k_{MU1}$ | 1.0477 | 0.7932 | 0.8998 | 0.9791 |
| | $k_{MU2}$ | 1.5206 | 0.8147 | 0.8335 | 0.4032 | | $k_{MU2}$ | 1.0477 | 0.8732 | 0.9094 | 0.7150 |
| | $k_{MU3}$ | 1.5206 | 0.8440 | 0.8357 | 0.3440 | | $k_{MU3}$ | 1.0477 | 0.9122 | 0.9104 | 0.4108 |
| | $k_{MU4}$ | 1.5206 | 0.8445 | 0.8357 | 0.3432 | | $k_{MU4}$ | 1.0477 | 0.9125 | 0.9104 | 0.4085 |

Table 2: : Estimated MSE when $n = 50$ $,\emptyset = .85$ $,p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 1.0559 | 0.4920 | 0.4921 | 0.7350 | 15 | $k_{HK}$ | 0.8984 | 0.8879 | 0.8892 | 0.8085 |
| | $k_{HKB}$ | 1.0559 | 0.4923 | 0.4922 | 0.7233 | | $k_{HKB}$ | 0.8984 | 0.8852 | 0.8889 | 0.8293 |
| | $k_{LW}$ | 1.0559 | 0.4921 | 0.4921 | 0.7291 | | $k_{LW}$ | 0.8984 | 0.8865 | 0.8890 | 0.8195 |
| | $k_{HSL}$ | 1.0559 | 0.4926 | 0.4922 | 0.7179 | | $k_{HSL}$ | 0.8984 | 0.8769 | 0.8879 | 0.8804 |
| | $k_{HMO}$ | 1.0559 | 0.4925 | 0.4922 | 0.7205 | | $k_{HMO}$ | 0.8984 | 0.8819 | 0.8885 | 0.8521 |
| | $k_{AM}$ | 1.0559 | 0.4924 | 0.4922 | 0.7206 | | $k_{AM}$ | 0.8984 | 0.8789 | 0.8881 | 0.8696 |
| | $k_{GM}$ | 1.0559 | 0.4918 | 0.4921 | 0.7475 | | $k_{GM}$ | 0.8984 | 0.8901 | 0.8894 | 0.7938 |
| | $k_{MED}$ | 1.0559 | 0.4917 | 0.4921 | 0.7545 | | $k_{MED}$ | 0.8984 | 0.8905 | 0.8895 | 0.7940 |
| | $k_{KS}$ | 1.0559 | 0.4919 | 0.4921 | 0.7400 | | $k_{KS}$ | 0.8984 | 0.8898 | 0.8894 | 0.7945 |
| | $k_{s_{arith}}$ | 1.0559 | 0.4917 | 0.4921 | 0.7603 | | $k_{s_{arith}}$ | 0.8984 | 0.8907 | 0.8895 | 0.7954 |
| | $k_{SMD}$ | 1.0559 | 0.4917 | 0.4921 | 0.7584 | | $k_{SMD}$ | 0.8984 | 0.8907 | 0.8895 | 0.7949 |
| | $k_{MU1}$ | 1.0559 | 0.4952 | 0.4925 | 0.7195 | | $k_{MU1}$ | 0.8984 | 0.8728 | 0.8873 | 0.8996 |
| | $k_{MU2}$ | 1.0559 | 0.4946 | 0.4925 | 0.7165 | | $k_{MU2}$ | 0.8984 | 0.8770 | 0.8879 | 0.8801 |
| | $k_{MU3}$ | 1.0559 | 0.4919 | 0.4921 | 0.7400 | | $k_{MU3}$ | 0.8984 | 0.8898 | 0.8894 | 0.7945 |
| | $k_{MU4}$ | 1.0559 | 0.4917 | 0.4921 | 0.7584 | | $k_{MU4}$ | 0.8984 | 0.8907 | 0.8895 | 0.7949 |

Table 3: Estimated MSE when $n = 50$ , $\emptyset = .95$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 1.8721 | 1.6500 | 1.5992 | 0.3562 | 1 | $k_{HK}$ | 2.0332 | 0.8923 | 0.8641 | 0.4159 |
| | $k_{HKB}$ | 1.8721 | 1.6497 | 1.5992 | 0.3563 | | $k_{HKB}$ | 2.0332 | 0.8903 | 0.8637 | 0.4158 |
| | $k_{LW}$ | 1.8721 | 1.6500 | 1.5992 | 0.3562 | | $k_{LW}$ | 2.0332 | 0.8937 | 0.8643 | 0.4161 |
| | $k_{HSL}$ | 1.8721 | 1.6111 | 1.5919 | 0.3669 | | $k_{HSL}$ | 2.0332 | 0.8883 | 0.8634 | 0.4158 |
| | $k_{HMO}$ | 1.8721 | 1.6421 | 1.5977 | 0.3585 | | $k_{HMO}$ | 2.0332 | 0.8895 | 0.8636 | 0.4158 |
| | $k_{AM}$ | 1.8721 | 1.6316 | 1.5957 | 0.3614 | | $k_{AM}$ | 2.0332 | 0.8922 | 0.8641 | 0.4159 |
| | $k_{GM}$ | 1.8721 | 1.6500 | 1.5992 | 0.3562 | | $k_{GM}$ | 2.0332 | 0.8948 | 0.8645 | 0.4163 |
| | $k_{MED}$ | 1.8721 | 1.6493 | 1.5991 | 0.3564 | | $k_{MED}$ | 2.0332 | 0.9005 | 0.8656 | 0.4182 |
| | $k_{KS}$ | 1.8721 | 1.6489 | 1.5990 | 0.3566 | | $k_{KS}$ | 2.0332 | 0.8948 | 0.8645 | 0.4163 |
| | $k_{S_{arith}}$ | 1.8721 | 1.6491 | 1.5990 | 0.3565 | | $k_{S_{arith}}$ | 2.0332 | 0.9024 | 0.8659 | 0.4193 |
| | $k_{SMD}$ | 1.8721 | 1.6495 | 1.5991 | 0.3564 | | $k_{SMD}$ | 2.0332 | 0.9018 | 0.8658 | 0.4189 |
| | $k_{MU1}$ | 1.8721 | 1.4121 | 1.5522 | 0.4279 | | $k_{MU1}$ | 2.0332 | 0.7999 | 0.8467 | 0.4455 |
| | $k_{MU2}$ | 1.8721 | 1.4438 | 1.5588 | 0.4168 | | $k_{MU2}$ | 2.0332 | 0.7551 | 0.8379 | 0.4652 |
| | $k_{MU3}$ | 1.8721 | 1.6489 | 1.5990 | 0.3566 | | $k_{MU3}$ | 2.0332 | 0.8948 | 0.8645 | 0.4163 |
| | $k_{MU4}$ | 1.8721 | 1.6495 | 1.5991 | 0.3564 | | $k_{MU4}$ | 2.0332 | 0.9018 | 0.8658 | 0.4189 |

Table 4: Estimated MSE when $n = 50$ , $\emptyset = .95$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 0.9034 | 0.8340 | 0.8343 | 0.7563 | 15 | $k_{HK}$ | 0.9490 | 0.9131 | 0.9066 | 0.7322 |
| | $k_{HKB}$ | 0.9034 | 0.8309 | 0.8338 | 0.7738 | | $k_{HKB}$ | 0.9490 | 0.9082 | 0.9055 | 0.7264 |
| | $k_{LW}$ | 0.9034 | 0.8244 | 0.8326 | 0.8128 | | $k_{LW}$ | 0.9490 | 0.8910 | 0.9016 | 0.7451 |
| | $k_{HSL}$ | 0.9034 | 0.8251 | 0.8327 | 0.8082 | | $k_{HSL}$ | 0.9490 | 0.8896 | 0.9013 | 0.7468 |
| | $k_{HMO}$ | 0.9034 | 0.8283 | 0.8333 | 0.7892 | | $k_{HMO}$ | 0.9490 | 0.9019 | 0.9041 | 0.7319 |
| | $k_{AM}$ | 0.9034 | 0.8255 | 0.8328 | 0.8056 | | $k_{AM}$ | 0.9490 | 0.9017 | 0.9040 | 0.7321 |
| | $k_{GM}$ | 0.9034 | 0.8355 | 0.8345 | 0.7504 | | $k_{GM}$ | 0.9490 | 0.9139 | 0.9068 | 0.7361 |
| | $k_{MED}$ | 0.9034 | 0.8362 | 0.8346 | 0.7560 | | $k_{MED}$ | 0.9490 | 0.9139 | 0.9068 | 0.7365 |
| | $k_{KS}$ | 0.9034 | 0.8349 | 0.8344 | 0.7520 | | $k_{KS}$ | 0.9490 | 0.9076 | 0.9054 | 0.7267 |
| | $k_{S_{arith}}$ | 0.9034 | 0.8367 | 0.8347 | 0.7817 | | $k_{S_{arith}}$ | 0.9490 | 0.9161 | 0.9072 | 0.7630 |
| | $k_{SMD}$ | 0.9034 | 0.8366 | 0.8347 | 0.7764 | | $k_{SMD}$ | 0.9490 | 0.9158 | 0.9072 | 0.7583 |
| | $k_{MU1}$ | 0.9034 | 0.8197 | 0.8316 | 0.8431 | | $k_{MU1}$ | 0.9490 | 0.8440 | 0.8901 | 0.8043 |
| | $k_{MU2}$ | 0.9034 | 0.8256 | 0.8328 | 0.8052 | | $k_{MU2}$ | 0.9490 | 0.8655 | 0.8955 | 0.7770 |
| | $k_{MU3}$ | 0.9034 | 0.8349 | 0.8344 | 0.7520 | | $k_{MU3}$ | 0.9490 | 0.9076 | 0.9054 | 0.7267 |
| | $k_{MU4}$ | 0.9034 | 0.8366 | 0.8347 | 0.7764 | | $k_{MU4}$ | 0.9490 | 0.9158 | 0.9072 | 0.7583 |

Table 5: Estimated MSE when $n = 50$ , $\emptyset = .99$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 2.0985 | 1.8248 | 1.6943 | 0.3617 | 1 | $k_{HK}$ | 2.3074 | 1.3764 | 1.2504 | 0.4218 |
| | $k_{HKB}$ | 2.0985 | 1.8229 | 1.6936 | 0.3619 | | $k_{HKB}$ | 2.3074 | 1.3624 | 1.2442 | 0.3979 |
| | $k_{LW}$ | 2.0985 | 1.8248 | 1.6943 | 0.3617 | | $k_{LW}$ | 2.3074 | 1.3553 | 1.2411 | 0.3933 |
| | $k_{HSL}$ | 2.0985 | 0.6762 | 1.0057 | 0.7582 | | $k_{HSL}$ | 2.3074 | 1.3478 | 1.2378 | 0.3905 |
| | $k_{HMO}$ | 2.0985 | 1.7152 | 1.6498 | 0.3685 | | $k_{HMO}$ | 2.3074 | 1.3548 | 1.2409 | 0.3931 |
| | $k_{AM}$ | 2.0985 | 1.7264 | 1.6544 | 0.3678 | | $k_{AM}$ | 2.3074 | 1.3469 | 1.2374 | 0.3903 |
| | $k_{GM}$ | 2.0985 | 1.8248 | 1.6943 | 0.3617 | | $k_{GM}$ | 2.3074 | 1.3769 | 1.2506 | 0.4232 |
| | $k_{MED}$ | 2.0985 | 1.8242 | 1.6941 | 0.3617 | | $k_{MED}$ | 2.3074 | 1.3789 | 1.2515 | 0.4308 |
| | $k_{KS}$ | 2.0985 | 1.8237 | 1.6939 | 0.3618 | | $k_{KS}$ | 2.3074 | 1.3582 | 1.2424 | 0.3949 |
| | $k_{S_{arith}}$ | 2.0985 | 1.8242 | 1.6941 | 0.3618 | | $k_{S_{arith}}$ | 2.3074 | 1.3859 | 1.2546 | 0.4745 |
| | $k_{SMD}$ | 2.0985 | 1.8243 | 1.6941 | 0.3617 | | $k_{SMD}$ | 2.3074 | 1.3848 | 1.2541 | 0.4652 |
| | $k_{MU1}$ | 2.0985 | 0.6760 | 1.0045 | 0.7602 | | $k_{MU1}$ | 2.3074 | 1.0464 | 1.0981 | 0.4079 |
| | $k_{MU2}$ | 2.0985 | 1.2025 | 1.4196 | 0.4233 | | $k_{MU2}$ | 2.3074 | 1.0051 | 1.0777 | 0.4125 |
| | $k_{MU3}$ | 2.0985 | 1.8237 | 1.6939 | 0.3618 | | $k_{MU3}$ | 2.3074 | 1.3582 | 1.2424 | 0.3949 |
| | $k_{MU4}$ | 2.0985 | 1.8243 | 1.6941 | 0.3617 | | $k_{MU4}$ | 2.3074 | 1.3848 | 1.2541 | 0.4652 |

Table 6: Estimated MSE when $n = 50$ , $\emptyset = .99$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 2.5383 | 0.6358 | 0.5513 | 0.7725 | 15 | $k_{HK}$ | 3.7283 | 1.2421 | 1.0998 | 0.9160 |
| | $k_{HKB}$ | 2.5383 | 0.6217 | 0.5450 | 0.6684 | | $k_{HKB}$ | 3.7283 | 1.2077 | 1.0841 | 0.8531 |
| | $k_{LW}$ | 2.5383 | 0.4826 | 0.4794 | 0.6105 | | $k_{LW}$ | 3.7283 | 0.5409 | 0.7289 | 0.8303 |
| | $k_{HSL}$ | 2.5383 | 0.5265 | 0.5009 | 0.6066 | | $k_{HSL}$ | 3.7283 | 0.4409 | 0.6528 | 0.8346 |
| | $k_{HMO}$ | 2.5383 | 0.5980 | 0.5342 | 0.6196 | | $k_{HMO}$ | 3.7283 | 1.0910 | 1.0296 | 0.8244 |
| | $k_{AM}$ | 2.5383 | 0.6075 | 0.5385 | 0.6310 | | $k_{AM}$ | 3.7283 | 1.2138 | 1.0869 | 0.8592 |
| | $k_{GM}$ | 2.5383 | 0.6361 | 0.5514 | 0.7770 | | $k_{GM}$ | 3.7283 | 1.2486 | 1.1027 | 0.9421 |
| | $k_{MED}$ | 2.5383 | 0.6116 | 0.5404 | 0.6384 | | $k_{MED}$ | 3.7283 | 1.2135 | 1.0867 | 0.8588 |
| | $k_{KS}$ | 2.5383 | 0.5309 | 0.5030 | 0.6064 | | $k_{KS}$ | 3.7283 | 1.0186 | 0.9948 | 0.8229 |
| | $k_{S_{arith}}$ | 2.5383 | 0.6430 | 0.5545 | 0.9008 | | $k_{S_{arith}}$ | 3.7283 | 1.2909 | 1.1218 | 1.8783 |
| | $k_{SMD}$ | 2.5383 | 0.6408 | 0.5535 | 0.8523 | | $k_{SMD}$ | 3.7283 | 1.2873 | 1.1202 | 1.6509 |
| | $k_{MU1}$ | 2.5383 | 0.2684 | 0.3409 | 0.6678 | | $k_{MU1}$ | 3.7283 | 0.3518 | 0.4635 | 0.8571 |
| | $k_{MU2}$ | 2.5383 | 0.3355 | 0.3969 | 0.6372 | | $k_{MU2}$ | 3.7283 | 0.6712 | 0.8112 | 0.8268 |
| | $k_{MU3}$ | 2.5383 | 0.5309 | 0.5030 | 0.6064 | | $k_{MU3}$ | 3.7283 | 1.0186 | 0.9948 | 0.8229 |
| | $k_{MU4}$ | 2.5383 | 0.6408 | 0.5535 | 0.8523 | | $k_{MU4}$ | 3.7283 | 1.2873 | 1.1202 | 1.6509 |

Table 7: Estimated MSE when $n = 100$ , $\emptyset = .85$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 1.1833 | 0.8280 | 0.8260 | 0.3415 | 1 | $k_{HK}$ | 1.2376 | 1.2326 | 1.2298 | 0.3826 |
| | $k_{HKB}$ | 1.1833 | 0.8280 | 0.8260 | 0.3416 | | $k_{HKB}$ | 1.2376 | 1.2319 | 1.2298 | 0.3866 |
| | $k_{LW}$ | 1.1833 | 0.8280 | 0.8260 | 0.3415 | | $k_{LW}$ | 1.2376 | 1.2326 | 1.2298 | 0.3826 |
| | $k_{HSL}$ | 1.1833 | 0.8195 | 0.8258 | 0.4220 | | $k_{HSL}$ | 1.2376 | 1.2048 | 1.2292 | 0.5474 |
| | $k_{HMO}$ | 1.1833 | 0.8277 | 0.8260 | 0.3445 | | $k_{HMO}$ | 1.2376 | 1.2243 | 1.2296 | 0.4324 |
| | $k_{AM}$ | 1.1833 | 0.8275 | 0.8260 | 0.3463 | | $k_{AM}$ | 1.2376 | 1.2038 | 1.2292 | 0.5530 |
| | $k_{GM}$ | 1.1833 | 0.8280 | 0.8260 | 0.3415 | | $k_{GM}$ | 1.2376 | 1.2327 | 1.2298 | 0.3823 |
| | $k_{MED}$ | 1.1833 | 0.8280 | 0.8260 | 0.3419 | | $k_{MED}$ | 1.2376 | 1.2328 | 1.2298 | 0.3818 |
| | $k_{KS}$ | 1.1833 | 0.8280 | 0.8260 | 0.3420 | | $k_{KS}$ | 1.2376 | 1.2327 | 1.2298 | 0.3823 |
| | $k_{S_{arith}}$ | 1.1833 | 0.8280 | 0.8260 | 0.3420 | | $k_{S_{arith}}$ | 1.2376 | 1.2328 | 1.2298 | 0.3817 |
| | $k_{SMD}$ | 1.1833 | 0.8280 | 0.8260 | 0.3417 | | $k_{SMD}$ | 1.2376 | 1.2328 | 1.2298 | 0.3817 |
| | $k_{MU1}$ | 1.1833 | 0.8034 | 0.8253 | 0.5725 | | $k_{MU1}$ | 1.2376 | 1.1937 | 1.2289 | 0.6035 |
| | $k_{MU2}$ | 1.1833 | 0.8193 | 0.8258 | 0.4242 | | $k_{MU2}$ | 1.2376 | 1.2022 | 1.2291 | 0.5616 |
| | $k_{MU3}$ | 1.1833 | 0.8280 | 0.8260 | 0.3420 | | $k_{MU3}$ | 1.2376 | 1.2327 | 1.2298 | 0.3823 |
| | $k_{MU4}$ | 1.1833 | 0.8280 | 0.8260 | 0.3417 | | $k_{MU4}$ | 1.2376 | 1.2328 | 1.2298 | 0.3817 |

Table 8: Estimated MSE when $n = 100$ , $\emptyset = .85$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 0.9245 | 1.0811 | 1.0809 | 0.6809 | 15 | $k_{HK}$ | 1.1521 | 1.4723 | 1.4744 | 1.0914 |
| | $k_{HKB}$ | 0.9245 | 1.0805 | 1.0809 | 0.6717 | | $k_{HKB}$ | 1.1521 | 1.4654 | 1.4741 | 1.0605 |
| | $k_{LW}$ | 0.9245 | 1.0812 | 1.0809 | 0.6840 | | $k_{LW}$ | 1.1521 | 1.4688 | 1.4743 | 1.0738 |
| | $k_{HSL}$ | 0.9245 | 1.0762 | 1.0807 | 0.6855 | | $k_{HSL}$ | 1.1521 | 1.4418 | 1.4734 | 1.0144 |
| | $k_{HMO}$ | 0.9245 | 1.0796 | 1.0809 | 0.6683 | | $k_{HMO}$ | 1.1521 | 1.4556 | 1.4738 | 1.0344 |
| | $k_{AM}$ | 0.9245 | 1.0807 | 1.0809 | 0.6736 | | $k_{AM}$ | 1.1521 | 1.4431 | 1.4734 | 1.0158 |
| | $k_{GM}$ | 0.9245 | 1.0819 | 1.0809 | 0.7060 | | $k_{GM}$ | 1.1521 | 1.4758 | 1.4745 | 1.1150 |
| | $k_{MED}$ | 0.9245 | 1.0820 | 1.0810 | 0.7157 | | $k_{MED}$ | 1.1521 | 1.4761 | 1.4745 | 1.1179 |
| | $k_{KS}$ | 0.9245 | 1.0818 | 1.0809 | 0.7048 | | $k_{KS}$ | 1.1521 | 1.4757 | 1.4745 | 1.1144 |
| | $k_{S_{arith}}$ | 0.9245 | 1.0821 | 1.0810 | 0.7196 | | $k_{S_{arith}}$ | 1.1521 | 1.4762 | 1.4745 | 1.1191 |
| | $k_{SMD}$ | 0.9245 | 1.0821 | 1.0810 | 0.7183 | | $k_{SMD}$ | 1.1521 | 1.4762 | 1.4745 | 1.1187 |
| | $k_{MU1}$ | 0.9245 | 1.0674 | 1.0803 | 0.7491 | | $k_{MU1}$ | 1.1521 | 1.4422 | 1.4734 | 1.0149 |
| | $k_{MU2}$ | 0.9245 | 1.0713 | 1.0805 | 0.7226 | | $k_{MU2}$ | 1.1521 | 1.4538 | 1.4738 | 1.0310 |
| | $k_{MU3}$ | 0.9245 | 1.0818 | 1.0809 | 0.7048 | | $k_{MU3}$ | 1.1521 | 1.4757 | 1.4745 | 1.1144 |
| | $k_{MU4}$ | 0.9245 | 1.0821 | 1.0810 | 0.7183 | | $k_{MU4}$ | 1.1521 | 1.4762 | 1.4745 | 1.1187 |

Table 9: Estimated MSE when $n = 100$ ,$\emptyset = .95$ ,$p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $k_{HK}$ | 1.8047 | 1.4752 | 1.4575 | 0.3544 | | $k_{HK}$ | 0.9323 | 0.9207 | 0.9209 | 0.8966 |
| | $k_{HKB}$ | 1.8047 | 1.4752 | 1.4575 | 0.3544 | | $k_{HKB}$ | 0.9323 | 0.9209 | 0.9209 | 0.9013 |
| | $k_{LW}$ | 1.8047 | 1.4752 | 1.4575 | 0.3544 | | $k_{LW}$ | 0.9323 | 0.9212 | 0.9209 | 0.9081 |
| | $k_{HSL}$ | 1.8047 | 1.4420 | 1.4550 | 0.3839 | | $k_{HSL}$ | 0.9323 | 0.9237 | 0.9212 | 0.9369 |
| | $k_{HMO}$ | 1.8047 | 1.4694 | 1.4571 | 0.3597 | | $k_{HMO}$ | 0.9323 | 0.9213 | 0.9209 | 0.9096 |
| | $k_{AM}$ | 1.8047 | 1.4436 | 1.4551 | 0.3825 | | $k_{AM}$ | 0.9323 | 0.9210 | 0.9209 | 0.9044 |
| | $k_{GM}$ | 1.8047 | 1.4752 | 1.4575 | 0.3544 | 1 | $k_{GM}$ | 0.9323 | 0.9207 | 0.9209 | 0.8928 |
| 0.1 | $k_{MED}$ | 1.8047 | 1.4751 | 1.4575 | 0.3545 | | $k_{MED}$ | 0.9323 | 0.9206 | 0.9209 | 0.8907 |
| | $k_{KS}$ | 1.8047 | 1.4750 | 1.4575 | 0.3546 | | $k_{KS}$ | 0.9323 | 0.9207 | 0.9209 | 0.8954 |
| | $k_{S_{arith}}$ | 1.8047 | 1.4750 | 1.4575 | 0.3545 | | $k_{S_{arith}}$ | 0.9323 | 0.9206 | 0.9209 | 0.8886 |
| | $k_{SMD}$ | 1.8047 | 1.4751 | 1.4575 | 0.3545 | | $k_{SMD}$ | 0.9323 | 0.9206 | 0.9209 | 0.8889 |
| | $k_{MU1}$ | 1.8047 | 1.3675 | 1.4493 | 0.4568 | | $k_{MU1}$ | 0.9323 | 0.9260 | 0.9214 | 0.9525 |
| | $k_{MU2}$ | 1.8047 | 1.3617 | 1.4488 | 0.4628 | | $k_{MU2}$ | 0.9323 | 0.9222 | 0.9210 | 0.9228 |
| | $k_{MU3}$ | 1.8047 | 1.4750 | 1.4575 | 0.3546 | | $k_{MU3}$ | 0.9323 | 0.9207 | 0.9209 | 0.8954 |
| | $k_{MU4}$ | 1.8047 | 1.4751 | 1.4575 | 0.3545 | | $k_{MU4}$ | 0.9323 | 0.9206 | 0.9209 | 0.8889 |

Table 10: Estimated MSE when $n = 100$ ,$\emptyset = .95$ ,$p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | 0.7553 | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $k_{HK}$ | 0.8897 | 0.8054 | 0.8058 | 0.7060 | | $k_{HK}$ | 0.9323 | 0.9207 | 0.9209 | 0.8966 |
| | $k_{HKB}$ | 0.8897 | 0.8058 | 0.8058 | 0.7029 | | $k_{HKB}$ | 0.9323 | 0.9209 | 0.9209 | 0.9013 |
| | $k_{LW}$ | 0.8897 | 0.8059 | 0.8058 | 0.7430 | | $k_{LW}$ | 0.9323 | 0.9212 | 0.9209 | 0.9081 |
| | $k_{HSL}$ | 0.8897 | 0.8086 | 0.8061 | 0.7089 | | $k_{HSL}$ | 0.9323 | 0.9237 | 0.9212 | 0.9369 |
| | $k_{HMO}$ | 0.8897 | 0.8066 | 0.8059 | 0.7101 | | $k_{HMO}$ | 0.9323 | 0.9213 | 0.9209 | 0.9096 |
| | $k_{AM}$ | 0.8897 | 0.8067 | 0.8059 | 0.7878 | | $k_{AM}$ | 0.9323 | 0.9210 | 0.9209 | 0.9044 |
| | $k_{GM}$ | 0.8897 | 0.8053 | 0.8057 | 0.8159 | | $k_{GM}$ | 0.9323 | 0.9207 | 0.9209 | 0.8928 |
| | $k_{MED}$ | 0.8897 | 0.8052 | 0.8057 | 0.7506 | | $k_{MED}$ | 0.9323 | 0.9206 | 0.9209 | 0.8907 |
| 5 | $k_{KS}$ | 0.8897 | 0.8054 | 0.8058 | 0.8541 | 15 | $k_{KS}$ | 0.9323 | 0.9207 | 0.9209 | 0.8954 |
| | $k_{S_{arith}}$ | 0.8897 | 0.8052 | 0.8057 | 0.8464 | | $k_{S_{arith}}$ | 0.9323 | 0.9206 | 0.9209 | 0.8886 |
| | $k_{SMD}$ | 0.8897 | 0.8052 | 0.8057 | 0.8073 | | $k_{SMD}$ | 0.9323 | 0.9206 | 0.9209 | 0.8889 |
| | $k_{MU1}$ | 0.8897 | 0.8129 | 0.8066 | 0.7595 | | $k_{MU1}$ | 0.9323 | 0.9260 | 0.9214 | 0.9525 |
| | $k_{MU2}$ | 0.8897 | 0.8095 | 0.8063 | 0.7506 | | $k_{MU2}$ | 0.9323 | 0.9222 | 0.9210 | 0.9228 |
| | $k_{MU3}$ | 0.8897 | 0.8054 | 0.8058 | 0.8464 | | $k_{MU3}$ | 0.9323 | 0.9207 | 0.9209 | 0.8954 |
| | $k_{MU4}$ | 0.8897 | 0.8052 | 0.8057 | 0.7553 | | $k_{MU4}$ | 0.9323 | 0.9206 | 0.9209 | 0.8889 |

Table 11: Estimated MSE when $n = 100$ , $\emptyset = .99$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 2.3266 | 2.0074 | 1.9012 | 0.3625 | 1 | $k_{HK}$ | 1.2975 | 0.3170 | 0.3174 | 0.7353 |
| | $k_{HKB}$ | 2.3266 | 2.0071 | 1.9011 | 0.3624 | | $k_{HKB}$ | 1.2975 | 0.3170 | 0.3175 | 0.7173 |
| | $k_{LW}$ | 2.3266 | 2.0074 | 1.9012 | 0.3625 | | $k_{LW}$ | 1.2975 | 0.3205 | 0.3184 | 0.7102 |
| | $k_{HSL}$ | 2.3266 | 1.9751 | 1.8920 | 0.3648 | | $k_{HSL}$ | 1.2975 | 0.3189 | 0.3181 | 0.7097 |
| | $k_{HMO}$ | 2.3266 | 1.9997 | 1.8990 | 0.3620 | | $k_{HMO}$ | 1.2975 | 0.3175 | 0.3177 | 0.7102 |
| | $k_{AM}$ | 2.3266 | 1.9774 | 1.8926 | 0.3646 | | $k_{AM}$ | 1.2975 | 0.3186 | 0.3181 | 0.7097 |
| | $k_{GM}$ | 2.3266 | 2.0074 | 1.9012 | 0.3625 | | $k_{GM}$ | 1.2975 | 0.3170 | 0.3174 | 0.7377 |
| | $k_{MED}$ | 2.3266 | 2.0073 | 1.9011 | 0.3625 | | $k_{MED}$ | 1.2975 | 0.3170 | 0.3174 | 0.7294 |
| | $k_{KS}$ | 2.3266 | 2.0072 | 1.9011 | 0.3625 | | $k_{KS}$ | 1.2975 | 0.3171 | 0.3175 | 0.7133 |
| | $k_{S_{arith}}$ | 2.3266 | 2.0073 | 1.9011 | 0.3625 | | $k_{S_{arith}}$ | 1.2975 | 0.3170 | 0.3174 | 0.7648 |
| | $k_{SMD}$ | 2.3266 | 2.0073 | 1.9011 | 0.3625 | | $k_{SMD}$ | 1.2975 | 0.3170 | 0.3174 | 0.7606 |
| | $k_{MU1}$ | 2.3266 | 1.7199 | 1.8160 | 0.3944 | | $k_{MU1}$ | 1.2975 | 0.3388 | 0.3210 | 0.7160 |
| | $k_{MU2}$ | 2.3266 | 1.7236 | 1.8171 | 0.3939 | | $k_{MU2}$ | 1.2975 | 0.3216 | 0.3186 | 0.7105 |
| | $k_{MU3}$ | 2.3266 | 2.0072 | 1.9011 | 0.3625 | | $k_{MU3}$ | 1.2975 | 0.3171 | 0.3175 | 0.7133 |
| | $k_{MU4}$ | 2.3266 | 2.0073 | 1.9011 | 0.3625 | | $k_{MU4}$ | 1.2975 | 0.3170 | 0.3174 | 0.7606 |

Table 12: Estimated MSE when $n = 100$ , $\emptyset = .99$ , $p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 0.9664 | 0.5237 | 0.5288 | 0.7385 | 15 | $k_{HK}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 |
| | $k_{HKB}$ | 0.9664 | 0.5253 | 0.5295 | 0.7424 | | $k_{HKB}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 |
| | $k_{LW}$ | 0.9664 | 0.5448 | 0.5360 | 0.7295 | | $k_{LW}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 |
| | $k_{HSL}$ | 0.9664 | 0.5371 | 0.5336 | 0.7323 | | $k_{HSL}$ | 1.3026 | 0.7533 | 0.7969 | 0.7540 |
| | $k_{HMO}$ | 0.9664 | 0.5296 | 0.5310 | 0.7375 | | $k_{HMO}$ | 1.3026 | 0.7986 | 0.7977 | 0.3505 |
| | $k_{AM}$ | 0.9664 | 0.5326 | 0.5321 | 0.7349 | | $k_{AM}$ | 1.3026 | 0.7982 | 0.7977 | 0.3555 |
| | $k_{GM}$ | 0.9664 | 0.5236 | 0.5288 | 0.7379 | | $k_{GM}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 |
| | $k_{MED}$ | 0.9664 | 0.5238 | 0.5289 | 0.7392 | | $k_{MED}$ | 1.3026 | 0.7994 | 0.7978 | 0.3430 |
| | $k_{KS}$ | 0.9664 | 0.5262 | 0.5298 | 0.7417 | | $k_{KS}$ | 1.3026 | 0.7994 | 0.7978 | 0.3431 |
| | $k_{S_{arith}}$ | 0.9664 | 0.5230 | 0.5286 | 0.7412 | | $k_{S_{arith}}$ | 1.3026 | 0.7994 | 0.7978 | 0.3430 |
| | $k_{SMD}$ | 0.9664 | 0.5231 | 0.5286 | 0.7398 | | $k_{SMD}$ | 1.3026 | 0.7994 | 0.7978 | 0.3429 |
| | $k_{MU1}$ | 0.9664 | 0.5785 | 0.5455 | 0.7249 | | $k_{MU1}$ | 1.3026 | 0.7494 | 0.7968 | 0.7730 |
| | $k_{MU2}$ | 0.9664 | 0.5446 | 0.5360 | 0.7296 | | $k_{MU2}$ | 1.3026 | 0.7765 | 0.7974 | 0.5944 |
| | $k_{MU3}$ | 0.9664 | 0.5262 | 0.5298 | 0.7417 | | $k_{MU3}$ | 1.3026 | 0.7994 | 0.7978 | 0.3431 |
| | $k_{MU4}$ | 0.9664 | 0.5231 | 0.5286 | 0.7398 | | $k_{MU4}$ | 1.3026 | 0.7994 | 0.7978 | 0.3429 |

Table 13: Estimated MSE when $n = 150$ $, \emptyset = .85$ $, p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 | 1 | $k_{HK}$ | 1.2113 | 1.1169 | 1.1143 | 0.3820 |
| | $k_{HKB}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 | | $k_{HKB}$ | 1.2113 | 1.1164 | 1.1143 | 0.3860 |
| | $k_{LW}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 | | $k_{LW}$ | 1.2113 | 1.1169 | 1.1143 | 0.3820 |
| | $k_{HSL}$ | 1.3026 | 0.7533 | 0.7969 | 0.7540 | | $k_{HSL}$ | 1.2113 | 1.0974 | 1.1138 | 0.5270 |
| | $k_{HMO}$ | 1.3026 | 0.7986 | 0.7977 | 0.3505 | | $k_{HMO}$ | 1.2113 | 1.1105 | 1.1141 | 0.4304 |
| | $k_{AM}$ | 1.3026 | 0.7982 | 0.7977 | 0.3555 | | $k_{AM}$ | 1.2113 | 1.0968 | 1.1138 | 0.5308 |
| | $k_{GM}$ | 1.3026 | 0.7994 | 0.7978 | 0.3428 | | $k_{GM}$ | 1.2113 | 1.1170 | 1.1143 | 0.3817 |
| | $k_{MED}$ | 1.3026 | 0.7994 | 0.7978 | 0.3430 | | $k_{MED}$ | 1.2113 | 1.1170 | 1.1143 | 0.3812 |
| | $k_{KS}$ | 1.3026 | 0.7994 | 0.7978 | 0.3431 | | $k_{KS}$ | 1.2113 | 1.1170 | 1.1143 | 0.3817 |
| | $k_{S_{arith}}$ | 1.3026 | 0.7994 | 0.7978 | 0.3430 | | $k_{S_{arith}}$ | 1.2113 | 1.1171 | 1.1143 | 0.3811 |
| | $k_{SMD}$ | 1.3026 | 0.7994 | 0.7978 | 0.3429 | | $k_{SMD}$ | 1.2113 | 1.1171 | 1.1143 | 0.3812 |
| | $k_{MU1}$ | 1.3026 | 0.7494 | 0.7968 | 0.7730 | | $k_{MU1}$ | 1.2113 | 1.0850 | 1.1135 | 0.6045 |
| | $k_{MU2}$ | 1.3026 | 0.7765 | 0.7974 | 0.5944 | | $k_{MU2}$ | 1.2113 | 1.0881 | 1.1136 | 0.5866 |
| | $k_{MU3}$ | 1.3026 | 0.7994 | 0.7978 | 0.3431 | | $k_{MU3}$ | 1.2113 | 1.1170 | 1.1143 | 0.3817 |
| | $k_{MU4}$ | 1.3026 | 0.7994 | 0.7978 | 0.3429 | | $k_{MU4}$ | 1.2113 | 1.1171 | 1.1143 | 0.3812 |

Table 14: Estimated MSE when $n = 150$ $, \emptyset = .85$ $, p =$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 0.9113 | 0.9529 | 0.9528 | 0.6386 | 15 | $k_{HK}$ | 0.9610 | 1.2306 | 1.2308 | 0.8371 |
| | $k_{HKB}$ | 0.9113 | 0.9524 | 0.9528 | 0.6559 | | $k_{HKB}$ | 0.9610 | 1.2292 | 1.2307 | 0.8413 |
| | $k_{LW}$ | 0.9113 | 0.9530 | 0.9528 | 0.6368 | | $k_{LW}$ | 0.9610 | 1.2294 | 1.2307 | 0.8395 |
| | $k_{HSL}$ | 0.9113 | 0.9507 | 0.9528 | 0.7164 | | $k_{HSL}$ | 0.9610 | 1.2273 | 1.2307 | 0.8571 |
| | $k_{HMO}$ | 0.9113 | 0.9519 | 0.9528 | 0.6754 | | $k_{HMO}$ | 0.9610 | 1.2283 | 1.2307 | 0.8486 |
| | $k_{AM}$ | 0.9113 | 0.9521 | 0.9528 | 0.6687 | | $k_{AM}$ | 0.9610 | 1.2277 | 1.2307 | 0.8534 |
| | $k_{GM}$ | 0.9113 | 0.9532 | 0.9528 | 0.6300 | | $k_{GM}$ | 0.9610 | 1.2315 | 1.2308 | 0.8684 |
| | $k_{MED}$ | 0.9113 | 0.9533 | 0.9528 | 0.6291 | | $k_{MED}$ | 0.9610 | 1.2316 | 1.2308 | 0.8794 |
| | $k_{KS}$ | 0.9113 | 0.9532 | 0.9528 | 0.6300 | | $k_{KS}$ | 0.9610 | 1.2315 | 1.2308 | 0.8663 |
| | $k_{S_{arith}}$ | 0.9113 | 0.9533 | 0.9528 | 0.6288 | | $k_{S_{arith}}$ | 0.9610 | 1.2317 | 1.2308 | 0.8847 |
| | $k_{SMD}$ | 0.9113 | 0.9533 | 0.9528 | 0.6289 | | $k_{SMD}$ | 0.9610 | 1.2317 | 1.2308 | 0.8831 |
| | $k_{MU1}$ | 0.9113 | 0.9478 | 0.9527 | 0.7939 | | $k_{MU1}$ | 0.9610 | 1.2230 | 1.2306 | 0.8884 |
| | $k_{MU2}$ | 0.9113 | 0.9490 | 0.9527 | 0.7651 | | $k_{MU2}$ | 0.9610 | 1.2259 | 1.2307 | 0.8687 |
| | $k_{MU3}$ | 0.9113 | 0.9532 | 0.9528 | 0.6300 | | $k_{MU3}$ | 0.9610 | 1.2315 | 1.2308 | 0.8663 |
| | $k_{MU4}$ | 0.9113 | 0.9533 | 0.9528 | 0.6289 | | $k_{MU4}$ | 0.9610 | 1.2317 | 1.2308 | 0.8831 |

Table 15: Estimated MSE when $n = 150$ ,$\emptyset = .95$ ,$p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | $k_{HK}$ | 1.8743 | 1.6685 | 1.6496 | 0.3563 | 1 | $k_{HK}$ | 1.7040 | 1.4579 | 1.4423 | 0.4036 |
| | $k_{HKB}$ | 1.8743 | 1.6684 | 1.6496 | 0.3563 | | $k_{HKB}$ | 1.7040 | 1.4553 | 1.4422 | 0.3965 |
| | $k_{LW}$ | 1.8743 | 1.6685 | 1.6496 | 0.3563 | | $k_{LW}$ | 1.7040 | 1.4579 | 1.4423 | 0.4036 |
| | $k_{HSL}$ | 1.8743 | 1.6251 | 1.6466 | 0.3916 | | $k_{HSL}$ | 1.7040 | 1.4102 | 1.4388 | 0.4382 |
| | $k_{HMO}$ | 1.8743 | 1.6653 | 1.6494 | 0.3587 | | $k_{HMO}$ | 1.7040 | 1.4420 | 1.4412 | 0.4039 |
| | $k_{AM}$ | 1.8743 | 1.6623 | 1.6492 | 0.3610 | | $k_{AM}$ | 1.7040 | 1.4143 | 1.4391 | 0.4337 |
| | $k_{GM}$ | 1.8743 | 1.6685 | 1.6496 | 0.3563 | | $k_{GM}$ | 1.7040 | 1.4581 | 1.4424 | 0.4046 |
| | $k_{MED}$ | 1.8743 | 1.6684 | 1.6496 | 0.3563 | | $k_{MED}$ | 1.7040 | 1.4587 | 1.4424 | 0.4083 |
| | $k_{KS}$ | 1.8743 | 1.6683 | 1.6496 | 0.3564 | | $k_{KS}$ | 1.7040 | 1.4581 | 1.4424 | 0.4046 |
| | $k_{S_{arith}}$ | 1.8743 | 1.6683 | 1.6496 | 0.3564 | | $k_{S_{arith}}$ | 1.7040 | 1.4589 | 1.4424 | 0.4098 |
| | $k_{SMD}$ | 1.8743 | 1.6684 | 1.6496 | 0.3563 | | $k_{SMD}$ | 1.7040 | 1.4588 | 1.4424 | 0.4093 |
| | $k_{MU1}$ | 1.8743 | 1.5033 | 1.6379 | 0.5050 | | $k_{MU1}$ | 1.7040 | 1.3327 | 1.4329 | 0.5242 |
| | $k_{MU2}$ | 1.8743 | 1.6067 | 1.6453 | 0.4081 | | $k_{MU2}$ | 1.7040 | 1.3491 | 1.4342 | 0.5061 |
| | $k_{MU3}$ | 1.8743 | 1.6683 | 1.6496 | 0.3564 | | $k_{MU3}$ | 1.7040 | 1.4581 | 1.4424 | 0.4046 |
| | $k_{MU4}$ | 1.8743 | 1.6684 | 1.6496 | 0.3563 | | $k_{MU4}$ | 1.7040 | 1.4588 | 1.4424 | 0.4093 |

Table 16: Estimated MSE when $n = 150$ ,$\emptyset = .95$ ,$p = 4$

| $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ | $\sigma$ | $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d_{CLASS}}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | $k_{HK}$ | 0.9112 | 1.0040 | 1.0025 | 0.7413 | 15 | $k_{HK}$ | 0.9307 | 1.1384 | 1.1388 | 0.8129 |
| | $k_{HKB}$ | 0.9112 | 1.0036 | 1.0024 | 0.7110 | | $k_{HKB}$ | 0.9307 | 1.1334 | 1.1385 | 0.8263 |
| | $k_{LW}$ | 0.9112 | 1.0031 | 1.0024 | 0.6915 | | $k_{LW}$ | 0.9307 | 1.1300 | 1.1382 | 0.8346 |
| | $k_{HSL}$ | 0.9112 | 1.0032 | 1.0024 | 0.6957 | | $k_{HSL}$ | 0.9307 | 1.1268 | 1.1380 | 0.8419 |
| | $k_{HMO}$ | 0.9112 | 1.0034 | 1.0024 | 0.7028 | | $k_{HMO}$ | 0.9307 | 1.1298 | 1.1382 | 0.8352 |
| | $k_{AM}$ | 0.9112 | 1.0033 | 1.0024 | 0.7003 | | $k_{AM}$ | 0.9307 | 1.1248 | 1.1379 | 0.8463 |
| | $k_{GM}$ | 0.9112 | 1.0042 | 1.0025 | 0.7601 | | $k_{GM}$ | 0.9307 | 1.1409 | 1.1390 | 0.8110 |
| | $k_{MED}$ | 0.9112 | 1.0043 | 1.0025 | 0.7748 | | $k_{MED}$ | 0.9307 | 1.1413 | 1.1390 | 0.8170 |
| | $k_{KS}$ | 0.9112 | 1.0040 | 1.0025 | 0.7395 | | $k_{KS}$ | 0.9307 | 1.1408 | 1.1390 | 0.8103 |
| | $k_{S_{arith}}$ | 0.9112 | 1.0044 | 1.0025 | 0.7909 | | $k_{S_{arith}}$ | 0.9307 | 1.1415 | 1.1390 | 0.8223 |
| | $k_{SMD}$ | 0.9112 | 1.0043 | 1.0025 | 0.7876 | | $k_{SMD}$ | 0.9307 | 1.1415 | 1.1390 | 0.8214 |
| | $k_{MU1}$ | 0.9112 | 0.9979 | 1.0021 | 0.6689 | | $k_{MU1}$ | 0.9307 | 1.1187 | 1.1375 | 0.8588 |
| | $k_{MU2}$ | 0.9112 | 0.9974 | 1.0020 | 0.6699 | | $k_{MU2}$ | 0.9307 | 1.1270 | 1.1380 | 0.8415 |
| | $k_{MU3}$ | 0.9112 | 1.0040 | 1.0025 | 0.7395 | | $k_{MU3}$ | 0.9307 | 1.1408 | 1.1390 | 0.8103 |

### 3.1 The discussion of simulation results

In this section , we present the results of simulation study of some restricted estimator to determine the good estimator, by using the mean square error MSE criterion. From Table 1 to Table 16, the performance of RLS, RRR, (k – d) class and RMUR estimators for all cases of the sample size $n$ , the coefficient correlation ∅ and the variance $\sigma^2$. We can observe that the following results.

1. From Table 1 to Table 6, when ($n = 50, \emptyset = .85, 95, ,99$ and $\sigma = .1 , 1, 15$ ) the performance the RMUR estimator is better than of any estimator because the RMUR estimator has minimum mean square error MSE, while when $\sigma = 5$ and $\emptyset = .99$ the (k – d) class estimator is better than of any estimator because the (k – d) class has lowest MSE .
2. From Table 7 to Table 12, when ($n = 100, \emptyset = .85$ ,95, .99 ) and ($\sigma = .1 , 1, 5, 15$), the RMUR estimator has minimum mean square error, only in Table 12, when $\sigma = .5$ and $\emptyset = .99$ the RRR estimator has lowest mean square error .
3. From Table 13 to Table 16 for all cases of the coefficient correlation ∅ and the variance $\sigma^2$ the RMUR estimator is the best because the RMUR has lowest MEE comparing of any estimator.

### 4. Numerical example-

To illustrate the performance of RLS, RRR, (k – d) class and RMUR estimators by using the real life data, numerical example is given. We consider the data about the total national product, which is cited by Akdeniz [1], Gruber [6] for comparison of the estimators that given in this study. This data shows the relationship between the dependent variable $Y$ and the percentage that is the united states spent on four independent variables $X_1, X_2, X_3$ and $X_4$ representing the percentage spent by Farance, West Germany , Japan ,and Soviet Union respectively. The goal is to compare the scalar mean square error for RLS , RRR, (k – d) class and RMUR estimators. The scalar mean square of the RRR, (k – d) class and RMUR are given in Eq (2.7), (2.10) and (2.15) respectively . Table 17 show the performance of the RLS, RRR, (k – d) class and RMUR estimators by using scalar mean square error . we use the linear restrictions in (2) $R$ and $r$ as follows

$R = [1\ 1\ 1\ 1\ ; 0\ 1\ 3\ 1\ ]$ , $r = [1.2170\ \ 1.0904]$

Table 17 : The scalar mean square error of the RLS, RRR, (k – d) class and RMUR estimators for different estimated ridge parameter k

| $k$ | $\beta_{RLS}$ | $\beta_{RRR}$ | $\beta_{k,d\,CLASS}$ | $\beta_{RMUR}$ |
|---|---|---|---|---|
| 0.0161 | 135.7475 | 92.9958 | 92.2758 | 83.1722 |
| 0.0243 | 135.7475 | 84.4906 | 81.6447 | 67.6833 |
| 0.050 | 135.7475 | 70.0587 | 64.7167 | 40.0855 |
| 0.020 | 135.7475 | 88.4942 | 86.678 | 75.1888 |
| 0.10 | 135.7475 | 58.2697 | 53.1129 | 20.2068 |
| 0.15 | 135.7475 | 52.8104 | 48.3190 | 13.0448 |
| 0.20 | 135.7475 | 49.5981 | 45.5077 | 9.6146 |

In Table 17, we can observe that, the RMUR estimator has lowest scalar mean square error for all different k comparing of other restricted estimators , that means the RMUR estimator is better than of any restricted estimators, and this is clear by locking to Figure 1,2 and 3.

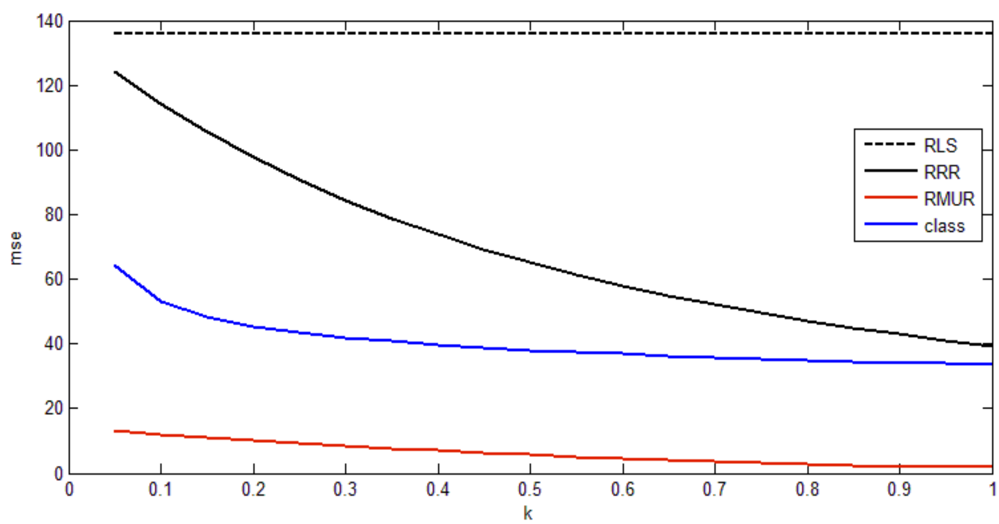


Figure 1: Estimated mse of RLS, RRR, (k – d) class and RMUR estimators for different estimated ridge parameter k

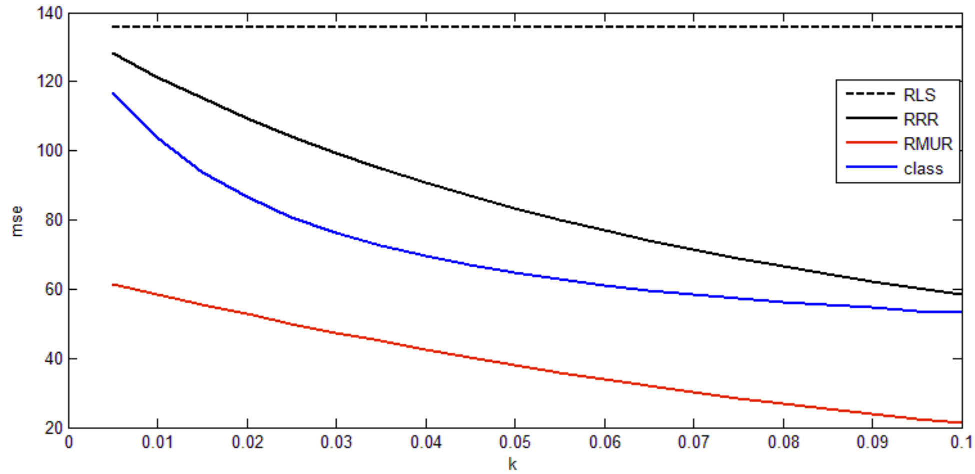


Figure 2: Estimated mse of RLS, RRR, (k – d) class and RMUR estimators for different estimated ridge parameter k

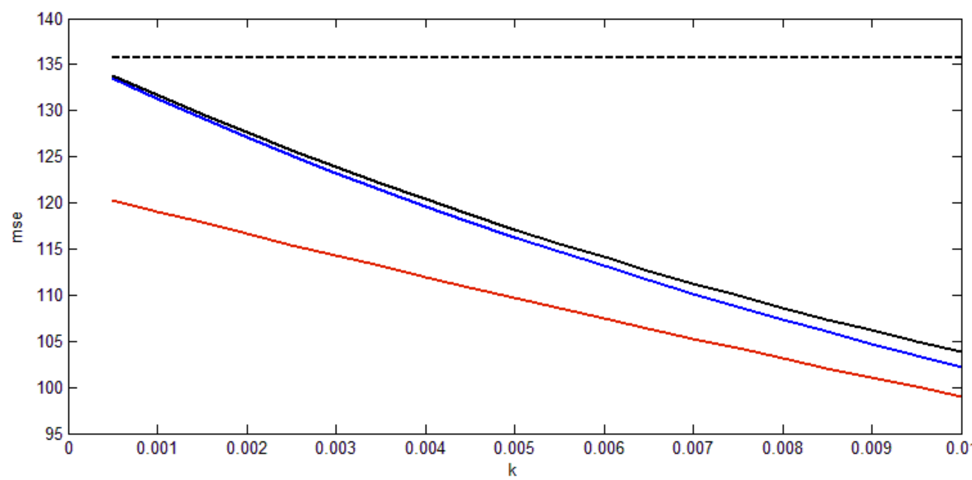


Figure 3: Estimated mse of RLS, RRR, (k – d) class and RMUR estimators for different estimated ridge parameter k

## 5 .Conclusion

When the multicollinearity problem and high variance there exits in linear regression model, the researchers tried to take advantage from prior information of the parameters through introduced the restricted biased estimators. In this paper, we make review of some restricted estimators through simulation study of this estimators. According to the simulation study, we observe that, the RMUR estimator is better than others in the science that it has minimum mean square error MSE comparing of other restricted estimators, that mean the RMUR estimator has good properties, also, we were able to illustrate that through a numerical example and some figures

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