Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 1 Article 1

1-7-2021

Harmonic Multivalent Functions Defined by General Integral Operator

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Recommended Citation

Abdul Ameer, Mays S.; Juma, Abdul Rahman S.; and Al-Saphory, Raheam A. (2021) "Harmonic Multivalent Functions Defined by General Integral Operator," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 1, Article 1.
DOI: 10.29350/qjps.2021.26.1.1234
Available at: https://qjps.researchcommons.org/home/vol26/iss1/1

Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (1) (2021) pp. Math 1–12

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

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Article History

Received on: 2/11 /2020

Revised on: 19/ 11 /2020

Accepted on: 24/ 11 /2020

Keywords:

volution,Distortion theorem,
eral integral operator,
tivalent harmonic functions,
ii of starlikeness and convexity.

DOI:https://doi.org/10.29350/qjps2021.26.1.1234

ABSTRACT

The main aim of the present work is to introduce the class of multivalent harmonic functions defined by the general integral operator. Thus, We get some geometric properties, like coefficients estimate, extreme point and distortion theorem, convolution property, radii of starlikeness, and convexity.

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1. Introduction

The function $f = u + iv$ is said to be continuous in the complex domain $F \subset C$ harmonic if real harmonic is u and v in F, we can write $f = h + g$, In any simply connected domain F, where h and g are analytic in F. See Clunie and Sheil-Small [3].

Denote by $H(m)$ the family of all multivalent harmonic functions $f = h + \bar{g}$, that are sense-preserving in the open unit disc $U = \{z: |z| < 1\}$, where

$$h(z) = z^m + \sum_{k=2}^{\infty} a_{k+m-1} z^{k+m-1} \ , \ \ g(z) = \sum_{k=1}^{\infty} b_{k+m-1}\, z^{k+m-1} \, . \qquad (1)$$

Recently Mohammed and Darus [5] defined by

$I(b_i; d_j; n)f(z): A \to A$:

$$I(b_i; d_j; n)f(z) = z^m + \sum_{k=2}^{\infty} \frac{(b_1,n)_{k-1} \ldots (b_t,n)_{k-1}}{(n,n)_{k-1} \ldots (d_1,n)_{k-1} \ldots (d_r,n)_{k-1}} \, a_{k+m-1} z^{k+m-1} \, . \qquad (2)$$

The Srivastava-Attiya operator $H_{c,b}: A \to A$ is defined in [6]:

$$H_{c,b} f(z) = z^m + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^c a_{k+m-1} z^{k+m-1} \, , \qquad (3)$$

where $z \in U, b \in \mathbb{C}/\{0, -1, -2, \ldots\}, c \in \mathbb{C}$ and $f \in A$. This linear operator $H_{c,a}$ written

$$H_{c,b} f(z) = G_{c,b} * f(z) = (1 + b)^c (\emptyset(z, c, b) - b^{-c}) * f(z),$$

the Hadamard product (convolution). Here,

$$\Phi(z, c, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^c} \, ,$$

the well-known Hurwitz –Lerch zeta function (see[6], [7]), defined by :

$$H_{ic}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k)^c} = z\Phi(z, c, 1).$$

The linear operator $H_n^{c,b}(b, d_j)(f): A \to A$ and given by [4] as

$$H_n^{c,b}(b_i, d_j) f(z) = z^m + \sum_{k=2}^{\infty} \frac{(b_1,n)_{k-1} \ldots (b_t,n)_{k-1}}{(n,n)_{k-1} \ldots (d_1,n)_{k-1} \ldots (d_r,n)_{k-1}} \left(\frac{1+b}{k+b}\right)^c a_{k+m-1} z^{k+m-1} \, , \qquad (4)$$

$(z \in U, b \in \mathbb{C}/\{0, -1, -2, \ldots\}, c \in \mathbb{C}, d_j \in \mathbb{C}/\{0, -1, -2, -3, \ldots\}, |n| < 1$ and

$t = r + 1, r \in \mathbb{N}_0 = \{0,1,2,3, \ldots\})$.

The class of multivalent harmonic functions denotes by $H_k^*(m, \alpha_1, \mu)$, satisfying

$$\operatorname{Re}\left\{\frac{z(H_n^{c,b}(b_i,d_j)h(z))' - z(H_n^{c,b}(b_i,d_j)g(z))'}{\left(H_n^{c,b}(a_i,b_j)h(z)\right)+\left(H_n^{c,b}(b_i,d_j)g(z)\right)}\right\} \geq m\mu\,, \qquad (5)$$

for $m \geq 1,\ 0 \leq \mu < 1,\ |z| = r < 1.$

The multivalent harmonic functions f in $H_k^*(m,\alpha_1,\mu)$ such that f and g are function of the from

$$h(z) = z^m - \sum_{k=2}^{\infty}|a_{k+m-1}|z^{k+m-1}\ ,\quad g(z) = \sum_{k=1}^{\infty}|b_{k+m-1}|\,z^{k+m-1} \qquad (6)$$

2. The Main Results

In this section, we prove that sufficient coefficient conditions for the class $H_k^*(m,\alpha_1,\mu)$.

Theorem 2.1. Let $f \in H_k^*(m,\alpha_1,\mu)$ given by (1). If

$$\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|$$

$$+\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(b_1,n)_{k-1}\ldots(b_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|b_{k+m-1}| \leq m(1-\mu), \quad (7)$$

$(z \in U, b \in \mathbb{C}/\{0,-1,-2,\ldots\},\ c \in \mathbb{C}, d_j \in \mathbb{C}/\{0,-1,-2,-3,\ldots\}, |n| < 1$ and

$t = r+1, r \in \mathbb{N}_0 = \{0,1,2,3,\ldots\})$, then $f \in H_n^*(m,\alpha_1,\mu)$.

Proof: We must prove that if (7) holds , then

$$\operatorname{Re}\left\{\frac{z(H_n^{c,b}(b_i,d_j)h(z))' - \overline{z(H_n^{c,b}(b_\iota,d_\jmath)g(z))'}}{\left(H_n^{c,b}(b_i,d_j)h(z)\right)+\overline{\left(H_n^{c,b}(b_\iota,d_\jmath)g(z)\right)}}\right\} = Re\,\frac{T(z)}{N(z)},$$

where

$$T(z) = z(H_n^{c,b}(b_i,d_j)h(z))' - \overline{z(H_n^{c,b}(b_\iota,d)g(z))'}\,,$$

$$N(z)\left(H_n^{c,b}(b_i,d_j)h(z)\right) + \overline{\left(H_n^{c,b}(b_\iota,d_\jmath)g(z)\right)}.$$

Now,

$$|T(z) + m\,(1-\mu)N(z)| - |T(z) - m\,(1+\mu)N(z)|$$

$$\geq (2m-m\mu)|z^m| - \sum_{k=2}^{\infty}(2m+k-m\mu-1)\ \frac{(b_1,n)_{n-1}\ldots(b_t,n)_{n-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|z^{k+m-1}$$

$$-\sum_{k=1}^{\infty}(m\mu+k-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c\overline{|b_{k+m-1}|z^{k+m-1}} - m\mu\,|z^m|$$

$$-\sum_{k=2}^{\infty}(k-m\mu-1)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|z^{k+m-1}$$
$$-\sum_{k=1}^{\infty}(2m+k+m\mu-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c\overline{|b_{k+m-1}|z^{k+m-1}}$$

$$\geq 2m(1-\mu)|z^m|-\sum_{k=2}^{\infty}(2m+2k-2m\mu-2)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|z^{k+m-1}$$
$$-\sum_{k=1}^{\infty}(2m+2k+2m\mu-2)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c\ \overline{|b_{k+m-1}|z^{k+m-1}}$$

$$\geq 2m(1-\mu)|z^m|-2\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|z^{k+m-1}$$
$$-2\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c\ \overline{|b_{k+m-1}|z^{k+m-1}}$$

$$> 2m(1-\mu)|z^m|\times\Big\{1-\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{m(1-\mu)\ (n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|$$
$$-\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{m(1-\mu)\ (n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c\left|\bar{b}_{k+m-1}\right|\Big\}.$$

This completes the proof of the theorem.

Theorem 2.2: Suppose $f\ \in H_k^*(m,\alpha_1,\mu)$ if and only if

$$\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\ \frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|a_{k+m-1}|+$$
$$\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ \ldots(d_1,n)_{k-1}\ \ldots(d_r,n)_{k-1}}\left(\frac{1+b}{k+b}\right)^c|b_{k+m-1}|\leq m(1-\mu), \qquad (8)$$

$(z\in U, b\ \in\mathbb{C}/\{0,-1,-2,\ldots\}, c\in\mathbb{C}, d_j\in\mathbb{C}/\{0,-1,-2,-3,\ldots\}, |n|<1$ and

$t=r+1, r\in\mathbb{N}_0=\{0,1,2,3,\ldots\})$.

Proof: Given a necessary and sufficient condition for f by (5) and we have

$$Re\left\{\frac{m(1-\mu)z^m-\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,m)_{n-1}}{(n,n)_{k-1}\ldots(b_1,n)_{k-1}\ldots(b_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k+m-1}}{z^m-\sum_{k=2}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(b_1,n)_{k-1}\ldots(b_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k+m-1}+\sum_{k=1}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|b_{k+m-1}|z^{k+m-1}}-\right.$$

$$\left.\frac{\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|b_{k+m-1}|z^{m+k-1}}{z^m-\sum_{k=2}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k+m-1}+\sum_{k=1}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|b_{k+m-1}|z^{k+m-1}}\right\}\geq 0 \quad (9).$$

The condition be due above all values z, when choosing z the values on the positive true axis where $0 \leq z = r < 1$ we should have

$$\frac{m(1-\mu)-\sum_{k=2}^{\infty}(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{n-1}\ldots(b_1,n)_{k-1}\ldots(b_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k-1}+\sum_{k=1}^{\infty}\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(b_1,n)_{n-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c\overline{|b_{k+m-1}|z^{k-1}}}$$

$$-\frac{\sum_{k=1}^{\infty}(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{n-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c|b_{k+m-1}|z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{(b_1,m)_{n-1}\ldots(b_t,m)_{n-1}}{(n,n)_{k-1}\ldots(b_1,n)_{k-1}\ldots(b_r,n)_{k-1}}(\frac{1+b}{k+b})^c|a_{k+m-1}|z^{k-1}+\sum_{k=1}^{\infty}\frac{(b_1,m)_{n-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c\overline{|b_{k+m-1}|z^{k-1}}}\geq 0 . \quad (10)$$

The condition (8) doesn't hold, at that point the numerator in (10), when it moves to 1 is negative. This contention with the condition for $f(z) \in H_k^*(m,\alpha_1,\mu)$, and consequently the verification is finished.

3. Extreme Point

Theorem 3.1: Suppose $f(z)$ given by (6). Then $f \in H_k^*(m,\alpha_1,\mu)$ if and only if

$$f(z) = \sum_{k=1}^{\infty}(X_{k+m-1}\, h_{k+m-1}(z) + Y_{k+m-1}g_{k+m-1}(z)),$$

where

$$h_m(z) = z^m,$$

$$h_{k+m-1}(z) = z^m - \sum_{k=2}^{\infty}\frac{m(1-\mu)}{(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}z^{k+m-1},\ (k = 2,3,\ldots)$$

and

$$g_{k+m-1}(z) = z^m - \sum_{n=1}^{\infty}\frac{m(1-\mu)}{(n+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}z^{k+m-1},\ (k = 2,3,\ldots)$$

$$h_{k+m-1}(z) \geq 0 \ , g_{k+m-1}(z) \geq 0 \ , x_m = 1 - \sum_{k=2}^{\infty} X_{k+m-1} + \sum_{k=1}^{\infty} Y_{k+m-1} .$$

The extreme points of $f \in H_k^*(m, \alpha_1, \mu)$ are $\{ h_{k+m-1}\}$ and $\{g_{k+m-1}\}$.

Proof: Suppose

$$f(z) = \sum_{k=1}^{\infty} (X_{k+m-1} h_{k+m-1}(z) + Y_{k+m-1} g_{k+m-1}(z))$$

$$= \sum_{k=1}^{\infty} (X_{k+m-1} + Y_{k+m-1}) z^m - \sum_{k=2}^{\infty} \frac{m(1-\mu)}{(k+m(1-\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c} X_{k+m-1} z^{k+m-1}$$

$$+ \sum_{k=1}^{\infty} \frac{m(1-\mu)}{(k+m(1+\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c} Y_{k+m-1} z^{k+m-1}$$

$$= z^m - \sum_{k=2}^{\infty} \frac{m(1-\mu)}{(k+m(1-\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c} X_{k+m-1} z^{k+m-1}$$
$$+ \sum_{k=1}^{\infty} \frac{m(1-\mu)}{(k+m(1+\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,m)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c} Y_{k+m-1} z^{k+-1} .$$

Moreover. we have

$$\sum_{k=2}^{\infty} \frac{(k+m(1-\mu)-1) \frac{(b_1,n)_{k-1} \dots (a_t,n)_{n-1}}{(n,n)_{k-1} \dots (b_1,n)_{k-1} \dots (b_r,n)_{n-1}} (\frac{1+a}{k+a})^c}{m(1-\mu)} |a_{k+m-1}|$$
$$+ \sum_{k=1}^{\infty} \frac{(k+m(1+\mu)-1) \frac{(b_1,n)_{n-1} \dots (b_t,n)_{n-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c}{m(1-\mu)} |b_{k+m-1}| =$$
$$\sum_{k=2}^{\infty} \frac{(k+m(1-\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c}{m(1-\mu)}$$
$$\left(\sum_{k=2}^{\infty} \frac{m(1-\mu)}{(k+m(1-\mu)-1) \frac{(b_1,n)_{k-1} \dots (b_t,n)_{k-1}}{(n,n)_{k-1} \dots (d_1,n)_{k-1} \dots (d_r,n)_{k-1}} (\frac{1+b}{k+b})^c} X_{k+m-1} \right)$$

$$+\sum_{k=1}^{\infty}\frac{(k+m(1+\mu)-1)\frac{(b_1,n)_{n-1}\ldots(b_t,m)_{n-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}{m(1-\mu)}$$

$$\left(\sum_{k=1}^{\infty}\frac{m(1-\mu)}{(k+m(1+\mu)-1)\frac{(b,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}Y_{k+m-1}\right)$$

$$=\sum_{k=2}^{\infty}X_{k+m-1}+\sum_{k=1}^{\infty}Y_{k+m-1}=1-X_m\leq 1.$$

Therefore $f(z)\in H_k^*(m,\alpha_1,\mu)$.

Conversely, if $f(z)\in H_k^*(m,\alpha_1,\mu)$. Suppose

$$X_p=1-\sum_{k=2}^{\infty}X_{k+m-1}+\sum_{n=1}^{\infty}Y_{k+p-1}.$$

Set $X_{k+m-1}=(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^s|a_{k+m-1}|,\ (k=2,3,\ldots)$

$Y_{k+m-1}=(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(\ \ r,n)_{k-1}}(\frac{1+b}{k+b})^s|b_{k+m-1}|,\ (k=2,3,\ldots)$

Now,

$$f(z)=z^m-\sum_{k=2}^{\infty}a_{k+m-1}z^{k+m-1}+\sum_{k=1}^{\infty}b_{k+m-1}\overline{z^{k+m-1}}$$

$$=z^m-\sum_{k=2}^{\infty}\frac{m(1-\mu)X_{k+m-1}}{\frac{(k+m(1-\mu)-1)((b,n)_{k-1}\ldots(b_t,n)_{k-1})}{((n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d,n)_{k-1})}(\frac{1+b}{k+b})^s}z^{k+m-1}+$$

$$\sum_{k=1}^{\infty}\frac{m(1-\mu)Y_{k+m-1}}{\frac{(k+m(1+\mu)-1)((b_1,n)_{k-1}\ldots(b_t,n)_{k-1})}{((n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1})}(\frac{1+b}{k+b})^s}\overline{z^{k+m-1}}$$

$$=z^m-\sum_{k=2}^{\infty}[z^m-h_{k+m-1}(z)]X_{k+m-1}+\sum_{k=1}^{\infty}[z^m-g_{k+m-1}(z)]Y_{k+m-1}$$

$$=\ [1-\sum_{k=2}^{\infty}X_{k+m-1}-\sum_{n=1}^{\infty}Y_{k+m-1}]z^m$$

$$+\sum_{k=2}^{\infty}X_{k+m-1}h_{k+m-1}(z)+\sum_{k=1}^{\infty}Y_{k+m-1}g_{k+m-1}(z)$$

$$=\sum_{k=2}^{\infty}X_{k+m-1}h_{k+m-1}(z)+\sum_{k=1}^{\infty}Y_{k+m-1}g_{k+m-1}(z).$$

4. The Distortion Theorem

Theorem 4.1: Let $f(z)\in H_k^*(m,\alpha_1,\mu)$. Then for $|z|=r<1$, let

$$\psi_n=\frac{(b_1,n)_{k-1}\ldots(b_t,n)_{k-1}}{(n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c \tag{11}$$

$$|f(z)|\leq(1+|b_m|)r^m+r^{m+1}\left(\frac{m(1-\mu)}{(m(1-\mu)+1)|\psi_2|}-\frac{m(1+\mu)|b_m|}{(m(1-\mu)+1)|\psi_2|}\right)$$

and

$$|f(z)|\geq(1-|b_m|)r^m-r^{m+1}\left(\frac{m(1-\mu)}{(m(1-\mu)+1)|\psi_2|}-\frac{m(1+\mu)|b_m|}{(m(1-\mu)+1)|\psi_2|}\right).$$

Proof: Since

$$\frac{(m(1-\mu)+1}{m(1-\mu)}|\psi_2|\sum_{k=2}^{\infty}(a_{k+m-1}+b_{k+m-1})$$

$$\leq \sum_{k=2}^{\infty}\frac{n+m(1-\mu)-1}{m(1-\mu)}(a_{k+m-1}+b_{k+m-1})|\psi_2|$$

$$\leq \sum_{k=2}^{\infty}\left(\frac{k+m(1-\mu)-1}{m(1-\mu)}|a_{k+m-1}|+\frac{\frac{k+m(1-\mu)-1}{m}}{m(1-\mu)}|b_{k+m-1}|\right)|\psi_2|,$$

the result of Theorem 2.2 we get

$$\sum_{k=2}^{\infty}(|a_{k+m-1}|+|b_{k+m-1}|)\leq \frac{m(1-\mu)}{(m(1-\mu)+1)|\psi_2|}(1-|b_m|). \quad (12)$$

Since $f \in H_k^*(m,\alpha_1,\mu)$, and $|z|=r$

$$|f(z)|=|z^m|-\sum_{k=2}^{\infty}a_{k+m-1}\,z^{k+m-1}+\sum_{k=1}^{\infty}b_{k+m-1}\,\bar{z}^{k+m-1}$$

$$\leq |z^m|+\sum_{k=2}^{\infty}|a_{k+m-1}|\,|z|^{k+m-1}+\sum_{k=1}^{\infty}|b_{k+m-1}|\,|\bar{z}|^{k+m-1}$$

$$=r^m+\sum_{k=2}^{\infty}|a_{k+m-1}|\,r^{k+m-1}+\sum_{k=1}^{\infty}|b_{k+m-1}|\,r^{k+m-1}$$

$$\leq (1+|b_m|)r^m+\left(\sum_{k=2}^{\infty}a_{k+m-1}+b_{k+p-1}\right)r^{m+1}$$

$$\leq (1+|b_m|)r^m+r^{m+1}\left(\frac{m(1-\mu)}{(m(1-\mu)+1)|\psi_2|}-\frac{m(1+\mu)|b_m|}{(m(1-\mu)+1)|\psi_2|}\right).$$

It gives the first result.
Likewise, we get the following lower bound.

$$|f(z)|\geq r^m-\sum_{k=2}^{\infty}|a_{k+m-1}|\,r^{k+m-1}+\sum_{k=1}^{\infty}|b_{k+m-1}|\,r^{k+m-1}$$

$$=(1-|b_m|)r^m-\sum_{k=2}^{\infty}(|a_{k+m-1}|+\sum_{k=1}^{\infty}|b_{k+p-1}|)\,r^{k+p-1}$$

$$\geq (1-|b_m|)r^m-r^{m+1}\left(\frac{m(1-\mu)}{(m(1-\mu)+1)|\psi_2|}-\frac{m(1+\mu)|b_m|}{(m(1-\mu)+1)|\psi_2|}\right).$$

5. The Convolution Property

We show that prove two theories, the first theorem about convolution for the class $H_k^*(m,\alpha_1,\mu)$.

Let

$$f(z)=z^m-\sum_{k=2}^{\infty}a_{k+m-1}\,z^{k+m-1}+\sum_{k=1}^{\infty}b_{k+m-1}\bar{z}^{k+m-1}$$

$$g(z)=z^m-\sum_{k=2}^{\infty}c_{k+m-1}\,z^{k+m-1}+\sum_{k=1}^{\infty}d_{k+m-1}\bar{z}^{k+m-1}\,.$$

The convolution of f and g define by

$$(f*g)(z)=f(z)*g(z)=z^m-\sum_{k=2}^{\infty}a_{k+m-1}\,c_{k+m-1}z^{k+m-1}+\sum_{k=1}^{\infty}b_{k+m-1}d_{k+m-1}\bar{z}^{k+m-1}$$

Theorem 5.1: Suppose $f(z)\in H_n^*(m,\alpha_1,\mu)$ and $g(z)\in H_k^*(m,\alpha_1,\mu)$.

Then $f*g\in H_k^*(m,\alpha_1,\mu)\subset H_k^*(m,\alpha_2,\mu)$.

Proof: Let

$$f(z)=z^m-\sum_{k=2}^{\infty}a_{k+m-1}\,z^{k+m-1}+\sum_{k=1}^{\infty}b_{k+m-1}\bar{z}^{k+m-1},$$

be in the class $H_k^*(m,\alpha_1,\mu)$ and

$$g(z)=z^m-\sum_{k=2}^{\infty}c_{k+m-1}\,z^{k+m-1}+\sum_{k=1}^{\infty}d_{k+m-1}\bar{z}^{k+m-1},$$

be in $H_k^*(m,\alpha_2,\mu)$.

Consider convolution functions $f*g$ the following :

$$\sum_{k=2}^{\infty}\frac{(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\dots(b_t,n)_{k-1}}{(n,n)_{k-1}\dots(d_1,n)_{k-1}\dots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}{m(1-\mu)}a_{k+m-1}\,c_{k+m-1}$$

$$+\sum_{k=2}^{\infty}\frac{(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\dots(b_t,n)_{k-1}}{(n,n)_{k-1}\dots(d_1,n)_{k-1}\dots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}{m(1-\mu)}b_{k+m-1}\,d_{k+m-1}$$

$$\le\sum_{k=2}^{\infty}\frac{(k+m(1-\mu)-1)\frac{(b_1,n)_{k-1}\dots(b_t,n)_{k-1}}{(n,n)_{k-1}\dots(d_1,n)_{k-1}\dots(d_r,n)_{k-1}}(\frac{1+b}{n+b})^c}{m(1-\mu)}a_{k+m-1}$$

$$+\sum_{k=2}^{\infty}\frac{(k+m(1+\mu)-1)\frac{(b_1,n)_{k-1}\dots(b_t,n)_{n-1}}{(n,n)_{k-1}\dots(d_1,n)_{k-1}\dots(d_r,n)_{k-1}}(\frac{1+b}{k+b})^c}{m(1-\mu)}b_{k+m-1}\le 1.$$

6. The Radii of Starlikeness and Convexity

Theorem 6.1: Assume that the function f defined by (1) be in the class $H_k^*(m,\alpha_1,\mu)$. Then f is

multivalent starlike of order η in the disk $|z|<r_1H_k^*(m,\alpha_1,\mu)$,where

$$r_1(m,\alpha_1,\mu,\eta)=\inf\left\{\sum_{k=2}^{\infty}\frac{(1-\eta)(k+m(1-\mu)-1)\frac{((b_1,n)_{k-1}\dots(b_t,m)_{k-1})(\frac{1+b}{k+b})^c}{((n,n)_{k-1}\dots(d_1,n)_{k-1}\dots(d_r,n)_{k-1})}}{(k+m+\eta)m(1-\mu)}\right\}^{\frac{1}{k-1}}$$

Proof: Show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\le 1-\eta,$$

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{k=2}^{\infty}(k+m-1)a_{k+m-1}z^{k+m-1}}{z^m+\sum_{k=2}^{\infty}a_{k+m-1}z^{k+m-1}}\right| \le \frac{\sum_{k=2}^{\infty}(k+m-1)a_{k+m-1}|z|^{n-1}}{1-\sum_{k-=2}^{\infty}a_{k+m-1}|z|^{n-1}}.$$

Will be bounded by $1-\eta$,

$$\frac{\sum_{k=2}^{\infty}(k+m-1)a_{k+m-1}|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_{k+m-1}|z|^{k-1}} \le 1-\eta,$$

$$\sum_{k=2}^{\infty}(k+m+\eta)a_{k+m-1}|z|^{k-1},$$

by theorem 2.1, we get

$$\sum_{k=2}^{\infty}\frac{(k+m(1-\mu)-1)\frac{((b_1,n)_{k-1}\ldots(b_t,n)_{k-1})\left(\frac{1+b}{k+b}\right)^s}{((n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{n-1})}}{m(1-\mu)}a_{k+m-1} \le 1.$$

Hence

$$|z|^{k-1} \le \sum_{k=2}^{\infty}\frac{(1-\eta)(n+m(1-\mu)-1)\frac{((b_1,n)_{k-1}\ldots(b_t,n)_{k-1})\left(\frac{1+b}{k+b}\right)^s}{((n,n)_{k-1}\ldots(d-_1,n)_{k-1}\ldots(d_r,n)_{k-1})}}{(k+m+\eta)m(1-\mu)},$$

$$|z| \le \left\{\sum_{k=2}^{\infty}\frac{(1-\eta)(k+m(1-\mu)-1)\frac{((b_1,n)_{k-1}\ldots(b_t,n)_{k-1})\left(\frac{1+b}{k+b}\right)^s}{((m,m)_{k-1}\ldots(d_1,m)_{k-1}\ldots(d_r,m)_{k-1})}}{(k+m+\eta)m(1-\mu)}\right\}^{\frac{1}{k-1}}.$$

This completes the proof of the theorem .

Theorem 6.2: The function $f(z)$ defined by (1) be in the class $H_k^*(m,\alpha_1,\mu)$. Then f is multivalent convex of order η in the disk $|z| < r_2(m,\alpha_1,\mu,\eta)$, where

$$r_2(p,\alpha_1,\mu,\delta) = \inf\left\{\sum_{k=2}^{\infty}\frac{(1-\delta)(k+m(1-\mu)-1)\frac{((b_1,n)_{k-1}\ldots(b_t,n)_{k-1})\left(\frac{1+b}{k+b}\right)^c}{((n,n)_{k-1}\ldots(d_1,n)_{k-1}\ldots(d_r,n)_{k-1})}}{(k+m+\delta)m(1-\mu)}\right\}^{\frac{1}{k-1}}$$

Proof: Using the same method to proof of theorem 6.1 we can show this

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \le 1-\delta, (0 \le \delta < 1).$$

Relative to $|z| < r_2$ with the help of theorem 2.1, we have the confirmation of theorem 6.2.

ACKNOWLEDGMENTS

Our thanks in advance to the editors and experts for considering this paper to publish in this estimated journal and for their efforts.

7. CONCLUSION

We have shown that a class of harmonic multivalent functions, interesting results concerning the harmonic multivalent functions defined by general integral operator. Some geometric properties like coefficients conditions, extreme points, distortion theorem, convolution property, radii of starlikeness are investigated and examined. Finally, Many problems still opened, for example, the extension of these results to the case of subclasses for various linear operator [11-13].

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