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On Projection Invariant Semisimple Modules

<p>Authors Names</p> <p>Yeliz KARA</p> <p>Article History</p> <p>Received on: 1/10 /2020 Revised on: 2 / 10 /2020 Accepted on: 30/11 / 2020</p> <p>Keywords:</p> <p>Exchange property Extending module Projection invariant submodule Semisimple module</p> <p>DOI:https://doi.org/10.29350/jops.2021.26.1.1210</p>	<p>ABSTRACT</p> <p>In this paper, we introduce and investigate the notion of projection invariant semisimple modules. Some structural properties of aforementioned class of modules are studied. We obtain indecomposable decompositions of former class of modules under some module theoretical conditions. Moreover, we explore when the finite exchange property implies full exchange property for the class of projection invariant semisimple modules. Finally, we obtain that the endomorphism ring of a projection invariant semisimple modules is a π-Baer ring.</p>
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1.Introduction

Throughout this paper all rings are associative with unity and modules are unital right R -modules. R will denote such a ring and M_R will denote such a module. Recall that a module M is called *extending* if every submodule of M is essential in a direct summand of M . This condition is an important generalization of injective, semisimple and uniform modules. There have been several generalizations of extending modules with respect to special subsets.

Recall that a submodule N of M is called *projection invariant* (resp., *fully invariant*), if $e(N) \subseteq N$ for all $e^2 = e \in \text{End}(M_R)$ (resp., $e \in \text{End}(M_R)$). Torsion subgroup of a group, the singular (or, second singular) submodule of a module and the radical of a ring are the examples of projection invariant submodules in different algebraic constructions. Notice that every fully invariant submodule is projection invariant. Recall from [3] and [2], a module M is called π -*extending* (resp., *FI-extending*) if every projection invariant (resp., fully invariant) submodule of M is essential in a direct summand of M . It is shown that extending condition implies π -extending condition implies FI-extending condition.

In this paper, we introduce and investigate the notion of projection invariant semisimple modules which is a generalization of semisimple modules. We call a module M is *projection invariant semisimple*, denoted by π -*semisimple*, provided that for each projection invariant submodule N of M , there exists a direct summand K of M such that K is essential in N . Observe that the class of π -semisimple modules is contained in the class of π -extending modules. We focus on module theoretical properties of π -semisimple modules such as direct sums and direct summands. Moreover, we prove that a π -semisimple module with an Abelian endomorphism ring over a ring with ascending chain condition on the right annihilators has an indecomposable decomposition. In particular, we obtain that the finite exchange property implies full exchange property. Finally, we conclude that the endomorphism ring of a π -semisimple module is a π -Baer ring.

Let $X \subseteq M$, then $X \leq M, X \leq_{ess} M, X \leq^{\oplus} M, X \leq_p M, Z_2(M)$ and $\text{End}(M_R)$ denote X is a submodule of M , X is an essential submodule of M , X is a direct summand of M , X is a projection invariant right submodule of M , the second singular submodule of M and the endomorphism ring of M_R , respectively. Recall that a module M over a ring R is said to have *(finite) exchange property* if for any (finite) index set I , whenever $M \oplus N = \bigoplus_{i \in I} A_i$ for modules N and A_i , then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$ for submodules $B_i \leq A_i$. A ring R is called *Abelian* if every idempotent of R is central. A family $(N_i)_{i \in I}$

of independent submodules of a module M is said to be a *local summand* if for any finite subset $F \subseteq I$, $\bigoplus_{i \in F} N_i$ is a direct summand of M . For unknown terminology and notation, we refer to [2] and [7].

We conclude this section to recall the following results which are used implicitly throughout this paper.

Lemma 1.1. [5, p.50] (i) Let $X_R \leq N_R \leq M_R$. Then $X \leq_p N \leq_p M$ implies that $X \leq_p M$.

(ii) Let $M = \bigoplus_{i \in I} M_i$ and $X \leq_p M$. Then $X = \bigoplus_{i \in I} (X \cap M_i)$ and $X \cap M_i \leq_p M_i$ for all $i \in I$.

Lemma 1.2. Let M_R be a module and $N \leq K \leq M_R$. If $N \leq_p M$ and $(K/N)_R \leq_p (M/N)_R$, then $K \leq_p M$.

Proof. Let $f = f^2 \in \text{End}(M_R)$ and consider $\alpha: M/N \xrightarrow{\theta} M \xrightarrow{f} M \xrightarrow{\pi} M/N$ where $\theta: M/N \rightarrow M$ is defined by $\theta(m + N) = f(m)$ for all $m \in M$ and π is the canonical map. Then $\alpha = \pi f \theta \in \text{End}(M_R)$ and $\alpha = \alpha^2$. Thus $\alpha(K/N) \subseteq K/N$. Hence $\pi f \theta(K/N) = f(K)/N \subseteq K/N$ which yields that $f(K) \subseteq K$. Thus $K \leq_p M$. ■

2. Main Results

In this section, we introduce and investigate the class of π -semisimple modules. We study on some structural properties and indecomposable decompositions for the class of π -semisimple modules.

Definition 2.1. We call an R -module M *projection invariant semisimple*, denoted by π -semisimple, if for each projection invariant submodule N of M , there exists a direct summand K of M such that $K \leq_{ess} N$.

Lemma 2.2. (i) M_R is π -semisimple if and only if every projection invariant submodule N of M is a direct summand of M .

(ii) Assume that M_R is an indecomposable module. Then M_R is semisimple if and only if M_R is π -semisimple.

(iii) If M_R is π -semisimple, then M_R is π -extending.

Proof. (i)(\Rightarrow) Let $N \leq_p M$. Then there exists $K \leq^\oplus M$ such that $K \leq_{ess} N$. Hence $M = K \oplus K'$ for some $K' \leq M$. Since $N \leq_p M$, $N = (N \cap K) \oplus (N \cap K') = K \oplus (N \cap K')$ by Lemma 1.1. It follows that $N = K$, as $K \leq_{ess} N$. Therefore $N \leq^\oplus M$. (\Leftarrow) This implication is clear.

(ii) It is clear that every semisimple module is π -semisimple. Observe that every submodule of an indecomposable module is projection invariant. Therefore being π -semisimple implies being semisimple.

(iii) Let M_R be π -semisimple and $N \trianglelefteq_p M$. Then $N \leq^\oplus M$ by part (i). Thus $N \leq_{ess} N \leq^\oplus M$, so M_R is π -extending. ■

Proposition 2.3. (i) Let M_R be π -semisimple and N a projection invariant submodule of M . Then N is π -semisimple.

(ii) Let M_R be π -semisimple and N a projection invariant submodule of M . Then M/N is π -semisimple.

Proof. (i) Let $A \trianglelefteq_p N$ and $N \trianglelefteq_p M$. Then $A \trianglelefteq_p M$ by Lemma 1.1. It follows that $M = A \oplus A'$ for some $A' \leq M$ by Lemma 2.2(i). Thus $N = (N \cap A) \oplus (N \cap A') = A \oplus (N \cap A')$ by Lemma 1.1. Therefore $A \leq^\oplus N$, so N is π -semisimple by Lemma 2.2(i).

(ii) Let $A/N \trianglelefteq_p M/N$ and $N \trianglelefteq_p M$. Then $A \trianglelefteq_p M$ by Lemma 1.2. Since M_R is π -semisimple, $M = A \oplus A'$ for some $A' \leq M$. Hence $M/N \cong A/N \oplus (A' + N)/N$, so $A/N \leq^\oplus M/N$. Thus M/N is π -semisimple by Lemma 2.2(i). ■

Corollary 2.4. Let $f: M \rightarrow M'$ be an R -epimorphism and M has an Abelian endomorphism ring. If M_R is π -semisimple, then M' is π -semisimple.

Proof. Note that $M/\ker f \cong M'$. Let $e = e^2 \in \text{End}(M_R)$ and $y \in e(\ker f)$. Then $y = e(x)$ for some $x \in \ker f$. Hence $f(y) = f(e(x)) = e(f(x)) = e(0) = 0$, as $\text{End}(M_R)$ is Abelian. It follows that $y \in \ker f$, so $\ker f \trianglelefteq_p M_R$. By Proposition 2.3(ii), M' is π -semisimple. ■

Corollary 2.5. Let M_R be π -semisimple. Then $M/Z_2(M)$ is π -semisimple and $M = Z_2(M) \oplus M'$ where M' is nonsingular π -semisimple.

Proof. Since $Z_2(M) \trianglelefteq M$, $Z_2(M) \trianglelefteq_p M$. Then $M/Z_2(M)$ is π -semisimple by Proposition 2.3(ii). Moreover, $Z_2(M) \leq^\oplus M$ by Lemma 2.2(i). Hence $M = Z_2(M) \oplus M'$ for some $M' \leq M$. It follows that $M' \cong M/Z_2(M)$ is nonsingular π -semisimple. ■

Theorem 2.6. Let $M = \bigoplus_{i \in I} M_i$ where $\{M_i\}_{i \in I}$ is the family of fully invariant submodules in M . Then M is π -semisimple if and only if M_i is π -semisimple for all $i \in I$.

Proof. (\Leftarrow) Let $N_R \trianglelefteq_p M_R$ and $M = \bigoplus_{i \in I} M_i$. Then $N = \bigoplus_{i \in I} (N \cap M_i)$ where $N \cap M_i \trianglelefteq_p M_i$ for all $i \in I$ by Lemma 1.1. Since M_i is π -semisimple, $N \cap M_i \leq^\oplus M_i$ for all $i \in I$. It follows that $N \leq^\oplus M$. Therefore M is π -semisimple. By Lemma 2.2(i). (\Rightarrow) Assume M_i is π -semisimple for all $i \in I$. Note that $\bigoplus_{i \in I} M_i \trianglelefteq M$, so $\bigoplus_{i \in I} M_i \trianglelefteq_p M$. Observe that $M/(\bigoplus_{i \in I} M_i) \cong M_j$ for $j \neq i \in I$. Therefore Lemma 2.2(iii) yields that M_i is π -semisimple for all $i \in I$. ■

Proposition 2.7. (i) Let M_R be π -semisimple. Then every fully invariant direct summand of M_R is π -semisimple.

(ii) Let M_R be a π -semisimple module with an Abelian endomorphism ring. Then every direct summand of M_R is π -semisimple.

Proof. (i) Notice that every fully invariant submodule is projection invariant. Thus, the proof follows from Proposition 2.3(i).

(ii) Let M_R be a π -semisimple module with an Abelian endomorphism ring and $K = eM$ for some $e = e^2 \in \text{End}(M_R)$. Since $\text{End}(M_R)$ is Abelian, $g(eM) \subseteq eM$ for all $g = g^2 \in \text{End}(M_R)$. Thus $K_R \trianglelefteq_p M_R$, so part (i) yields the result. ■

Proposition 2.8. Let $M = M_1 \oplus M_2$ such that $M_2 \trianglelefteq M$. If M_R is π -semisimple, then both M_1 and M_2 are π -semisimple.

Proof. It is clear from Proposition 2.7 (i) that M_2 is π -semisimple. Let $X_1 \trianglelefteq_p M_1$. Then $X_1 \oplus M_2 \trianglelefteq_p M$ by [3, Lemma 4.13]. Since M_R is π -semisimple, $X_1 \oplus M_2 \leq^\oplus M$ by Lemma 2.2 (i). Hence $M = X_1 \oplus M_2 \oplus A$ for some $A \leq M$. Now, $M_1 = M_1 \cap (X_1 \oplus M_2 \oplus A) = X_1 \oplus (M_1 \cap (M_2 \oplus A))$ by modular law. Therefore $X_1 \leq^\oplus M_1$, so M_1 is π -semisimple by Lemma 2.2 (i). ■

Theorem 2.9. Let R be a ring and M an R -module such that R satisfies ascending chain condition on right annihilator of the form $r(m)$, ($m \in M$). If M is π -semisimple with an Abelian endomorphism ring, then M has an indecomposable decomposition.

Proof. Let $\{X_\lambda \mid \lambda \in I\}$ be an independent family of submodules of M and $X = \bigoplus_{\lambda \in I} X_\lambda$ be a local summand of M . Define the canonical projection $\pi_k: X \rightarrow \bigoplus_{k \in I, k \neq \lambda} X_k$. Thus $f(X) = f(\bigoplus_{\lambda \in I} X_\lambda) = \bigoplus_{\lambda \in I} f(X_\lambda) = \bigoplus_{\lambda \in I} f(\ker \pi_\lambda)$ where $f^2 = f \in \text{End}(M_R)$. Since $\text{End}(M_R)$ is Abelian, $f(\ker \pi_\lambda) \subseteq \ker \pi_\lambda$. Then $f(X) \subseteq X$, so $X \trianglelefteq_p M_R$. It follows that $M = X \oplus A$ for some $A \leq M$. It follows from [7, Theorem 2.17] that M has an indecomposable decomposition. ■

Corollary 2.10. Let R be a ring and M an R -module such that R satisfies ascending chain condition on right annihilator of the form $r(m)$, ($m \in M$). If M is π -semisimple with an Abelian endomorphism ring, then M is a direct sum of uniform submodules.

Proof. Observe from Lemma 2.2(iii) that being π -semisimple implies π -extending, and an indecomposable π -extending module is uniform by [3, Proposition 3.8]. Therefore the proof is a consequence of Theorem 2.9 and Proposition 2.7(ii). ■

Corollary 2.11. Let R be a right Noetherian ring and M an R -module with an Abelian endomorphism ring. If M is π -semisimple, then finite exchange property implies full exchange property.

Proof. It follows from Theorem 2.9 and [8, Corollary 6]. ■

Recall that a module M is called *locally Noetherian*, if every finitely generated submodule is Noetherian.

Corollary 2.12. Let M be a locally Noetherian module with an Abelian endomorphism ring. If M is π -semisimple, then finite exchange property implies full exchange property.

Proof. Let $m \in M$. Then $R/r(m) \cong mR$ is Noetherian right R -module. It follows that R satisfies ascending chain condition on right annihilator of the form $r(m)$, ($m \in M$). Therefore Theorem 2.9 yields the proof. ■

Recall from [6] and [1] that a ring R is *Baer* (resp., π -*Baer*) if the right annihilator of a nonempty subset (resp., projection invariant left ideal) of R is of the form eR for some $e = e^2 \in R$. Observe that the endomorphism ring of a semisimple module is a Baer ring [2, Theorem 3.1.3]. In the following result, we obtain that the endomorphism ring of a π -semisimple module is a π -Baer ring.

Proposition 2.13. Assume that M is π -semisimple module. Then the endomorphism ring of M is a π -Baer ring.

Proof. Let $S = \text{End}(M_R)$ and I be a projection invariant left ideal of S . We claim that $r_S(I) = eS$ for some $e = e^2 \in S$. It can be checked that $r_M(I) \trianglelefteq_p M_R$. Hence $r_M(I) = eM$ for some $e = e^2 \in S$. Thus $IeM = 0$, so $Ie = 0$, as ${}_S M$ is faithful. Then $eS \subseteq r_S(I)$. Now, let $a \in r_S(I)$. Thus $Ia = 0$, so $I(aM) = 0$. It follows that $aM \subseteq r_M(I) = eM$. Therefore $a \in eS$, so $r_S(I) \subseteq eS$. Hence S is a π -Baer ring. ■

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