On Some Properties of Pell Polynomials

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ABSTRACT

This work starts by reviewing the Pell polynomials, its definition and some basic properties. Afterward, some new properties of such polynomials are investigated. A novel generalization analytical formula is constructed explicitly the first derivative of Pell polynomials in terms of Pell polynomials themselves. Another explicit formula is concerned with the connection between the Pell polynomials expansion coefficients; this motivates our interest in such polynomials. These formulas are utilized to derive some mainly relationship related with power basis coefficients and Pell polynomials. With the Pell polynomials expansion technique, the powers $1, x, x^2, \cdots, x^n$ are expressed in terms of Pell polynomials and an interesting formula is presented with some detail in the proof. An important general formulation for the product of two Pell polynomials is also included in this article. Explicit computations obtain all the representations in this work. Finally, two examples concern boundary value problems and singular initial value problems are included for applications of the proposed interesting properties of Pell polynomials

1. Introduction

Orthogonal functions and polynomial series have attention in dealing with dynamic systems' various problems, theory of elasticity, automation, and remote control [1-9]. Special class of orthogonal functions are wavelets functions, for more details, see [10-12]. The techniques' opinion is that it reduces the dynamic system problem to solving a system of algebraic equations, which simplifies the original problem. Some approaches are based on reducing the underlying differential equation into a system of algebraic

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equations through differentiation, approximating the unknown function in the equation by truncated orthogonal series, $\beta(x) = [\beta_1(x)\ \ \beta_2(x)\ \ \cdots \beta_n(x)]^T$ and using the operational matrix of derivative to eliminate the derivative operations. This operation is calculated based on particular orthogonal polynomials [13-20]. For examples, Legendre Polynomials and Shifted Legendre Polynomial algorithms are used for the numerical solution of stochastic differential equations and the dynamic analysis of viscoelastic pipes conveying fluid by [21] and [22], respectively.

The Pell polynomials are important in numerical analysis. Many research papers are dealing with Pell polynomials contains mainly results of such polynomials. In [23], the authors investigated some properties of Pell polynomials. They obtained optimal second-order bounds of Pell polynomials. The bivariate Fibonacci and Lucas p–polynomials are studied in [24]. Some properties of the Lucas p–polynomials and bivariate Fibonacci are obtained. The recurrence relations of Vieta-Pell and Vieta-Pell-Lucas polynomials are given by [25]. The Binet form and generating functions of Vieta-Pell and Vieta-Pell-Lucas polynomials and some differentiation rules and finite summation formulas are presented. The specific values of Pell numbers and Pell–Lucas numbers are of Pell polynomials $p_n(x)$ and Pell–Lucas polynomials $q_n(x)$, respectively, this result are investigated in [26]. In [27-28], the sequences of generalized Pell numbers are obtained.

The present article is arranged as follows: Pell polynomials' necessary definitions are described in section 2. Some fundamental properties of Pell polynomials are listed in sections 3-5. Section 3 states with the proof the formula are explicitly expressing the derivative of Pell polynomials in terms of Pell polynomials themselves. The exact expression representing the Pell polynomials coefficients of a first order derivative of a differentiable function in terms of its original coefficients is also included in section 3. Section 4 reveals the relationship between the powers and Pell polynomials throughout an interesting general formula. An analytical formula for the product of two Pell polynomials is explained in section 5 and the applications of the presented Pell properties are applied for solving some examples listed in section 6. Finally, a discussion and conclusion appear in section 7.

2. Some Properties of Pell Polynomials

Pell functions are a sequence of orthogonal polynomials, and they are expressed recursively. For $n \geq 2$, Pell polynomials sequence $P_n(x)$ is defined by the following recurrence relation

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),$$

with initial conditions $P_0(x) = 0, P_1(x) = 1$.

These polynomials were introduced by [4].

From the definition of Pell polynomials, the following table is given

Table 1 Pell Polynomials for Different Orders

n	$P_n(x)$
0	0
1	1
2	2x
3	$4x^2 + 1$
4	$8x^3 + 4x$
5	$16x^4 + 12x^2 + 1$
6	$32x^5 + 32x^3 + 6x$
7	$64x^6 + 80x^4 + 24x^2 + 1$
⋮	

From Table 1, the following properties can be noticed

- Pell polynomials have not the same degree.
- The leading coefficients of Pell polynomials are 2^{n-1}.
- The coefficients of Pell polynomials are even numbers if n is odd, except for constant term.
- The degree of Pell polynomials is $n - 1$, for all n.

The new formulation of Pell polynomials is constructed as below.

The Pell polynomials $P_n(x)$ can be defined in terms of x^n [25],

$$P_{n+1}(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (2x)^{n-2k}$$ (1)

Eq. 1 can be rewritten as

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} 2^{n-2k} \frac{\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n-2k+1)} x^{n-2k}, n > 0$$

with a simple modification, one can define the Pell polynomials as:

For even number n

$$P_n(x) = P_{2l}(x) = \sum_{j=0}^{L} 2^{2j} \binom{L+j}{L-j} x^{2j}$$ (2)

and for odd number n

$$P_n(x) = P_{2l+1}(x) = \sum_{j=0}^{n} 2^{2j+1} \binom{L+j+1}{L-j} x^{2j+1}$$ (3)

From the above relations given by Eqns. 2-3, a general matrix form of Pell polynomials can obtain

$$P(x) = X(x)M^T$$ (4)

where: P(x) and X(x) are two matrices of the form

$$P(x) = [P_1(x) \quad P_2(x) \quad ... \quad P_N(x)], \qquad X(x) = [1 \quad x \quad ... \quad x^n]$$

and the constant matrix M is the following $(n + 1) \times (n + 1)$ lower triangle

$$
\begin{array}{ccccccccc}
0 \\
1 & 0 \\
0 & 2 \\
1 & 0 & 2^2 \\
0 & 2\binom{2}{1} & 0 & 2^3 \\
1 & 0 & 2^2\binom{3}{1} & 0 & 2^4 \\
0 & 2\binom{3}{2} & 0 & 2^3\binom{4}{1} & 0 & 2^5 \\
1 & 0 & 2^2\binom{4}{2} & 0 & 2^4\binom{5}{1} & 0 & 2^6 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 2\binom{L}{L-1} & 0 & 2^3\binom{L+1}{L-2} & 0 & 2^5\binom{L+2}{L-3} & 0 & ... & 2^{2(L+1)}
\end{array}
$$

If n is an odd number, then the last row will be

$$\begin{bmatrix} 1 & 0 & 2^2\begin{pmatrix} L+1 \\ L-1 \end{pmatrix} & 0 & 2^4\begin{pmatrix} L+2 \\ L-2 \end{pmatrix} & 0 & 2^6\begin{pmatrix} L+3 \\ L-3 \end{pmatrix} & \cdots & 2^{2L} \end{bmatrix}.$$

Note that in the last row the matrix M will be

for odd values: n = 2L + 1,

for even values: n = 2L

Theorem 1

The generating function of the Pell polynomials is [26]

$$f(t,x) = \frac{t}{1 - 2xt - t^2}$$

Proof

The generating function can be defined as

$$f(t,x) = \sum_{n=0}^{\infty} P_n(x)t^n$$

Then

$$f(t,x) = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

$$2xtf(t,x) = 2xP_0(x) + 2xP_1(x)t^2 + 2xP_2(x)t^3 + \cdots + 2xP_{n-1}(x)t^n + \cdots$$

and

$$t^2f(t,x) = 2P_0(x)t^2 + P_1(x)t^3 + P_2(x)t^4 + \cdots + P_{n-2}(x)t^n + \cdots$$

Therefore;

$$f(t,x)(1 - 2xt - t^2) = (t + 2x(P_1(x)t^2 + P_2(x)t^3 + \cdots + P_n(x)t^{n+1} + \cdots)$$

$$+(P_0(x)t^2 + P_1(x)t^3 + P_2(x)t^4 + \cdots + P_{n-2}(x)t^n + \cdots$$

$$= (t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

$$-2xP_0(x)t - 2xP_1(x)t^2 - 2xP_2(x)t^3 - \cdots - 2xP_n(x)t^n - \cdots$$

$$-P_0(x)t^2 - P_1(x)t^3 - P_2(x)t^4 - \cdots - P_{n-2}(x)t^n - \cdots$$

$$= (t + (2xP_1(x) + P_0(x))t^2 + \cdots + (2xP_{n-1}(x) + P_{n-2}(x))t^n + \cdots$$

$$-2xP_0(x)t - 2xP_1(x)t^2 - 2xP_2(x)t^3 - \cdots - 2xP_n(x)t^n - \cdots$$

$$-P_0(x)t^2 - P_1(x)t^3 - P_2(x)t^4 - \cdots - P_{n-2}(x)t^n - \cdots$$

$$= t$$

The next sections give the main results.

3. The Derivative of Pell Polynomials

The present section aims to state and prove two lemmas for the derivatives of Pell polynomials and the coefficients. The first lemma expresses the first derivative of Pell polynomials in terms of same polynomials themselves. The relationship between the coefficients expansions and derivatives of Pell polynomials is illustrated through Lemma 2.

Lemma 1

For all $m \geq 1$, we have

$$\dot{P}_m(x) = 2 \sum_{k=0}^{m-1}(-1)^{k+1}(k+1)P_{k+1} \text{ for } (k+n) \text{ even.} \tag{5}$$

Proof

Assume that the Pell polynomials expansion of function $y(x)$ is as below

$$y(x) = \sum_{s=0}^{\infty} a_s(s+1)P_s(x) \tag{6}$$

Then $\dot{y}(x)$ can be constructed as

$$\dot{y}(x) = \sum_{s=0}^{\infty} b_s s(x) \tag{7}$$

where: $b_s = 2s \sum_{r=s+1}^{\infty}(-1)^r a_s, k \geq 0$ (8)

Take $y(x) = P_m(x)$ in Eq. 6, then $a_m = 1$ and $a_i = 0$ for $i \neq m$, the result will be

$$b_s = \begin{cases} (-1)^{s+1}2(s+1) & m+s \text{ is even,} \quad s \le m-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, Eq. 7 becomes

$$\dot{P}_m(x) = 2\sum_{s=0}^{m}(-1)^{s+1}(s+1)P_s(x), \qquad m+s \text{ even}$$

which is the same result in Eq. 5

Lemma 2

Consider a function expanded in Pell series

$$y(x) = \sum_{i=1}^{N} a_i P_i(x) \tag{9}$$

The derivative of Eq. 9 is

$$\dot{y}(x) = \sum_{i=1}^{N-1} c_i P_i(x) \tag{10}$$

Then the relationship between the coefficients of the two expansions, a_i and c_i will be given as follows

$$c_N = 0$$

$$c_{N-1} = 2(N-1)a_N$$

$$c_{N-2} = 2(N-2)a_{N-1}$$

$$c_r = 2ra_{r+1} - \frac{1}{r+2}c_{r+2} , r = N-3, N-4, \dots, 1$$

4. The Relationship between the Power and Pell Polynomials

The relationship between the Power $1, x, x^2, \cdots, x^n$ and Pell Polynomials is investigated in this section.

Lemma 3

S is odd

$$x^S = \frac{1}{2^S}\left[P_{S+1}(x) - \left(\binom{S}{1} - \binom{S}{0}\right)P_{S-1}(x) + \left(\binom{S}{2} - \binom{S}{1}\right)P_{S-3}(x) - \cdots + \left(\binom{S}{S-3} - \right.\right.$$

$$\left.\left.\binom{S}{S-4}\right)P_2(x)\right] \tag{11}$$

S is even

$$x^S = \frac{1}{2^S}\left[P_{S+1}(x) - \left(\binom{S}{1} - \binom{S}{0}\right)P_{S-1}(x) + \left(\binom{S}{2} - \binom{S}{1}\right)P_{S-3}(x) - \cdots - \left(\binom{S}{S-2} - \right.\right.$$

$$\left.\left.\binom{S}{S-1}\right)P_1(x)\right] \tag{12}$$

Proof

The mathematical induction is suggested to prove Eqns. 11 and 12

For $s = 0$, $x^s = 1 = P_1$, therefore Eq. 11 is true for $s = 0$.

For $s = 1$, $x^s = x$, $x = \frac{1}{2}P_2 = x$; therefore Eq. 11 is true for $s = 1$.

Assume that Eq. 11 is true for $s = h$, that is

$$x^h = \frac{1}{2^k}\left[P_{h+1}(x) - \left(\binom{h}{1} - \binom{h}{0}\right)P_{h-1}(x) + \left(\binom{h}{2} - \binom{h}{1}\right)P_{h-3}(x) - \cdots\right.$$

$$\left. - \left(\binom{h}{h-2} - \binom{h}{h-1}\right)P_1\right]$$

$$x^{h+1} = \frac{1}{2^h}\left[P_{h+1}(x) - \left(\binom{h}{1} - \binom{h}{0}\right)P_{h-1}(x) + \left(\binom{h}{2} - \binom{h}{1}\right)P_{h-3}(x) - \cdots\right.$$

$$\left. - \left(\binom{h}{h-3} - \binom{h}{h-1}\right)P_2\right]\cdot\frac{1}{2}P_2(x)$$

$$x^{h+1} = \frac{1}{2^{h+1}}\left[P_{h+1}(x)P_2(x) - \left(\binom{h}{1} - \binom{h}{0}\right)P_{h-1}(x)P_2(x) + \left(\binom{h}{2} - \binom{h}{1}\right)P_{h-3}(x)P_2(x)\right.$$

$$\left. - \cdots - \left(\binom{h}{h-3} - \binom{h}{h-1}\right)P_2(x)P_2(x)\right]$$

$$P_{h+1}(x)P_2(x) = P_{h+2}(x) - P_h(x)$$

$$P_{h-1}(x)P_2(x) = P_h(x) - P_{h-2}(x)$$

$$P_{h-3}(x)P_2(x) = P_{h-2}(x) - P_{h-4}(x)$$

$$P_2(x)P_2(x) = P_3(x) - P_1(x)$$

$$x^{h+1} = \frac{1}{2^{h+1}}\left[P_{h+2}(x) - P_h(x) - \left(\binom{h}{1} - \binom{h}{0}\right)P_h(x) + \left(\binom{h}{1} - \binom{h}{0}\right)P_{h-2}(x)\right.$$

$$+ \left(\binom{h}{1} - \binom{h}{0}\right)P_{h-2}(x) + \left(\binom{h}{2} - \binom{h}{1}\right)P_{h-2}(x) - \left(\binom{h}{2} - \binom{h}{1}\right)P_{h-4}(x)$$

$$\left. - \cdots \mp \left(\binom{h}{h-3} - \binom{h}{h-1}\right)P_3(x) \mp \left(\binom{h}{h-3} - \binom{h}{h-1}\right)P_1(x)\right]$$

Since $\;-\left(1 + \left(\binom{h}{1} - \binom{h}{0}\right)\right) = h = \binom{h+1}{1} - \binom{h+1}{0}$,

and $\binom{h}{i+2} - \binom{h}{i} = \binom{h+1}{i+2} - \binom{h+1}{i+1}$ for $i = 0, 1, \dots$

Therefore;

$$x^{h+1} = \frac{1}{2^{h+1}}\left[P_{h+2}(x) - \left(\binom{h+1}{1} - \binom{h+1}{0}\right)P_h(x) + \cdots\right]$$

5. Product of Pell Polynomials

The formula of two Pell polynomials is evaluated at a fixed x with different indices and given in the following lemma.

Lemma 4: The product of two Pell polynomials satisfies the following relationship

$$P_n(x)P_m(x) = P_{|n-m|+1}(x) - P_{|n-m|+3}(x) + P_{|n-m|+5}(x) - \cdots + P_{n+m-1}(x), \; n, m \geq 1 \quad (13)$$

Proof: Eq. 13 is an identity for $m = 1$, since $P_1 = 1$, it then follows that

$$P_n(x)P_1(x) = P_n(x) \tag{14}$$

Multiply both sides of Eq. 14 by $2x$, yields

$$P_n(x)\big(2xP_1(x)\big) = 2xP_n(x)$$

Using the basic relationship recurrence for Pell polynomials

$$2xP_{n-1}(x) = P_n(x) - P_{n-2}(x)$$

This will lead to $P_n(x)P_2(x) = P_{n+1}(x) - P_{n-1}(x)$ which proves Eq. 13 for $m = 2$.

Since Eq. 13 is true for $m = 1$ and $m = 2$, Assume that Eq. 13 is true for m, then we want to prove that it is true for $m + 1$ to multiply both sides of Eq. 13 by 2x to get

$$P_n(x)\big(2xP_m(x)\big) = 2xP_{|n-m|+1}(x) - 2xP_{|n-m|+3}(x) + \cdots$$

$$P_n(x)\big(P_{m+1}(x) - P_{m-1}(x)\big)$$

$$= P_{|n-m|+2}(x) - P_{|n-m|}(x) + 2xP_{n+m-1}(x) - \big(P_{|n-m|+4}(x) - P_{|n-m|+2}(x)\big)$$

$$+ \cdots + P_{n+m}(x) - P_{n+m-2}(x)$$

$$P_n(x)P_{m+1}(x) = P_n(x)P_{m-1}(x) + P_{|n-m|+2}(x) - P_{|n-m|}(x) - P_{|n-m|+4}(x) + P_{|n-m|+2}(x)$$

$$+ \cdots + P_{n+m}(x) - P_{n+m-2}(x)$$

$$= P_{|n-m|+2}(x) - P_{|n-m|+4}(x) + P_{|n-m|+6}(x) + \cdots + P_{n+m+2}(x) - P_{|n-m|}(x) - P_{|n-m|+4}(x)$$

$$+ P_{|n-m|+2}(x) + \cdots + P_{n+m}(x) - P_{n+m-2}(x)$$

which leads to

$$P_n(x)P_{m+1}(x) = P_{|n-m|}(x) - P_{|n-m|+2}(x) + P_{|n-m|+4}(x) - \cdots + P_{n+m}(x)$$

6. Methodology

Consider the B.V.P.

$$\ddot{y}(x) + f(y, \dot{y}, \ddot{y}) = g(x), \ \ 0 \le x \le 1 \tag{15}$$

Subject to

$$a_1 y(0) + a_2 \dot{y}(0) + a_3 \ddot{y}(0) = \alpha \tag{16}$$

$$b_1 y(0) + b_2 \dot{y}(0) + b_3 \ddot{y}(0) = \beta \tag{17}$$

where: α, β, a_1, a_2, a_3, b_1, b_2 and b_3 be constants.

For solving Eq. 15 together with Eq. 16 and Eq. 17, suppose that the approximate solution is

$$y(x) = a^T P(x) \tag{18}$$

where: a and P(x) are given in the following expression

$$a = [a_1 \quad a_2 \quad \dots \quad a_n]^T, P(x) = [P_1(x) \quad P_2(x) \quad \dots \quad P_n(x)]$$

Using Eq. 18, yields

$$a_3^T \ddot{P}(x) + f(a^T P(x), a_1^T \dot{P}(x), a_2^T \ddot{P}(x)) = g(x) , 0 \le x \le 1 \tag{19}$$

and $a_1 c^T P(0) + a_2 c_1^T \dot{P}(0) + a_3 c_2^T \ddot{P}(0) = \alpha$ \hfill (20)

$$b_1 c^T P(1) + b_2 c_1^T \dot{P}(1) + b_3 c_2^T \ddot{P}(1) = \beta. \tag{21}$$

Utilizing the spectral method, one can obtain n algebraic equations with n unknown coefficients.

Solving this system gives the unknown coefficients a.

EXAMPLE 1

Consider the following differential equation

$$\ddot{y}(x) + 2(\dot{y}(x))^2 + 8y(x) = 0, 0 < x < 1 \text{ with } y(0) = 0, y(1) = -1 \tag{22}$$

Approximate $y(x) = a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x)$

and hence $\dot{y}(x) = 2a_2 P_1(x) + 4a_3 P_2(x), \ddot{y}(x) = 8a_3 P_1(x)$

In this case,

$$a^T = [a_1 \quad a_2 \quad a_3]^T, a_1^T = [2a_2 \quad 4a_3 \quad 0]^T, a_2^T = [8a_3 \quad 0 \quad 0]^T$$

The boundary conditions are approximated by Pell polynomials to get

$$y(0) = 0, \text{ implies } a_1 + a_3 = 0 \tag{23}$$

$$y(1) = -1, \text{ Implies } 2a_2 + 8a_3 = -1 \tag{24}$$

Put $y(x), \dot{y}(x), \ddot{y}(x)$ into Eq. 22, yields

$$a_2^T P(x) + 2(a_1^T P(x))^2 + 8a^T P(x) = 0$$

With the aid of the spectral method, one can obtain

$$4a_3^2 + a_3 = 0$$

$$a_1 + a_3 = 0$$

$$2a_2 + 8a_3 = -1$$

Then $a^T = [0.25 \quad 0.5 \quad -0.25]$. Consequently the approximate solution will be

$$y(x) = 0.25P_1(x) + 0.5P_2(x) - 0.25P_3(x)$$

The compression of the approximate solution using Pell polynomials with the exact solution is illustrated in Figure 1.
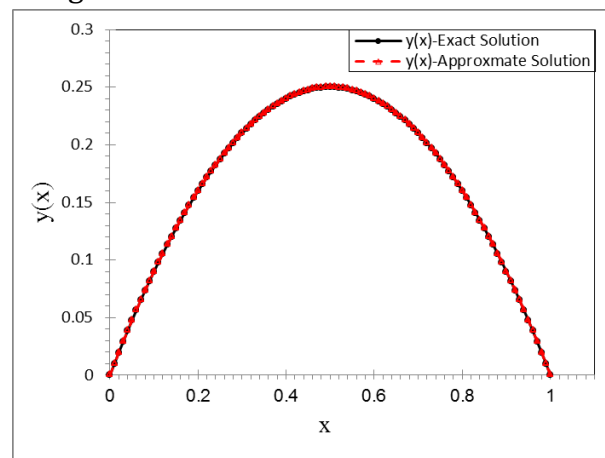

Figure 1: The compression between the approximate Pell solutions with the exact solution for example 1.

EXAMPLE 2

Consider the following singular initial value problem $\ddot{y}(x) = \frac{1}{x}y(x) + \dot{y}(x)$ together with the conditions $y(0) = 0, \dot{y}(0) = 1, \ddot{y}(0) = 2$

The exact solution to this problem is $y(x) = xe^x$.

According to the Pell polynomials algorithm, the following approximate solution is obtained for $n = 3, 4, 5$

$$y_3(x) = -\frac{1}{4}P_1(x) + \frac{3}{8}P_2(x) + \frac{1}{4}P_3(x) + \frac{1}{16}P_4(x)$$

$$y_4(x) = -\frac{11}{48}P_1(x) + \frac{3}{8}P_2(x) + \frac{7}{32}P_3(x) + \frac{1}{16}P_4(x) + \frac{1}{96}P_5(x)$$

$$y_6(x) = -\frac{11}{16}P_1(x) + \frac{293}{768}P_2(x) + \frac{7}{32}P_3(x) + \frac{11}{192}P_4(x) + \frac{1}{96}P_5(x) + \frac{1}{768}P_6(x)$$

Table 1 lists the some values for the solution obtained by the exact solution and by Pell technique with n = 5 while Fig. 1 illustrates a comparison between Pell method for different values of n against the exact one.

Table 1: A comparison between Pell method with n = 5 against the exact one for example 2.

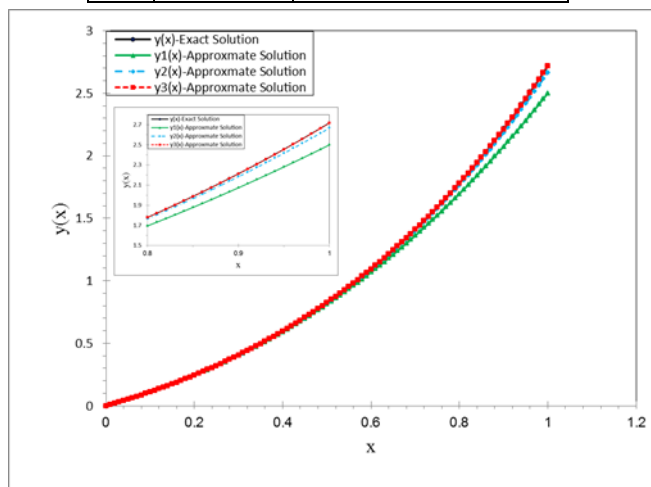| x	Exact	Approximate (n=5)
0	0	0
0.2	0.244281	0.2442801
0.4	0.596730	0.596727
0.6	1.093271	1.0932288
0.8	1.780433	1.7801045
1	2.718282	2.7166667

Figure 2: A comparison between Pell method for different values of n against the exact one for example 2.

7. CONCLUSION

This word concerns with the some new interesting properties of Pell polynomials. An exact expression for the first derivative of Pell polynomials is given in terms of Pell polynomials themselves. Two other analytical formulas, which relate the coefficients in the first differentiated expansions of Pell polynomials with the coefficients of the original expansion and with the power basis, have been proposed. The product of two

Pell polynomials is contained in this article in explicit form. These formulas can be easy, efficient and computationally attractive for solving many problems in differential equations and control theory. Two examples of concern differential equations are solved with the aid of the presented properties.

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