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Some Results on a Two Variables Pell Polynomials

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ABSTRACT

New Pell polynomials in two dimensions together with many important properties are presented in this work. The two dimensions Pell polynomials expansion coefficients of a first partial derivative of a differentiable function in terms of their original expansion coefficients, is included in the present article. The main advantage of the presented formulas is that the new properties of Pell polynomials in two variables greatly simplify the original problems and the result will lead to easy calculation of the coefficients of expansion. A direct spectral method along with the presented two variables Pell polynomials is proposed to solve partial differential equation. Illustrated examples are included to demonstrate the validity of the technique.

1. Introduction

In the recent years, there has been an increase usage among scientists and engineers to apply special polynomials technique along with an approximate solution to solve linear and nonlinear problems. The main advantage of the polynomial technique is its ability to transfer complicated problems into a system of algebraic equations. In [3], a numerical method to solve some types of differential equations is presented using Lagrange polynomial. A pseudo spectral technique for solving multi term time fractional differential equations is proposed in [5]. This technique is

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based on the Lagrange polynomials. The authors in [19] suggested a method depending on using of third kind Chebyshev polynomials some problems arising in fluid dynamics. See [4, 13, 18] for other applications of orthogonal polynomials.

Some approaches is depended on reducing the original differential equation into an integral equations with the aid of integration by truncated polynomial series and utilizing this operational matrix of integration to eliminate the integral operations. As an overview of this method can be found in [15] where an approximate solution is formulated for solving integro differential equations of fractional variable order. This method consists of transforming the original problem to a set of algebraic equation. In [18, 7], the computation of approximate solution for nonlinear singular initial value problems was considered using Hermite wavelets operational matrix of integration and Chebyshev wavelets operational matrix of integration respectively while in [9], the optimal control problems was solved with the aid of Legendre mother wavelets operational matrix of integration. A general expression for the operational matrix of integration of Bessel functions was derived in [16] to solve some examples in optimal control, for more works, see [8, 14, 15, 16, 17, 18, 19, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

The aim of the present work deals with newly defined of a two variable polynomials for Pell polynomials, then a Pell polynomial approximation for the numerical solution of a class of partial differential equation is suggested which based on truncated Pell polynomial series in the equation along with its operational matrices of partial derivatives into such partial differential equation. The study of numerical solution of partial differential equations has provided an important field for mathematical sciences researchers. In particular, a polynomial approximation technique was used in [7, 9, 16, 10, 20, 4]. The numerical solution of elliptic partial differential inhomogeneous equations was introduced in [4] based on Chebyshev polynomial scheme. The authors in [12, 16] used the operational matrices method partial differential equations in two independent variables. The Bernoulli wavelets operational matrices of integration and product applied in [17] to solve classes of problems in calculus of variations. A fast curvelet based finite difference algorithm was proposed in [6] for approximate solution of nonlinear partial differential equations in which finite difference approximations is utilized for differential operators contained in the partial differential equation. For more studies, see [5, 18, 20].

2. The Contribution and The Structure of The Article

The aim of the present work deals with newly defined of a two variable polynomials for Pell polynomials, then a Pell polynomials approximation for the approximate solution of a class of partial differential equation is suggested which is based on truncated Pell polynomials series in the equation along with its operational matrixes of partial derivatives into such partial differential equation. By defining special formula for two dimensions Pell polynomials, new important properties of Pell polynomials are enabled to derive by matrices. The majority of the research paper dealing with four closed analytical formulas for two dimensions Pell polynomials.

These motivate our interest in Pell polynomials in two variables.

The four explicit new formulae are:

1. An explicit formula for the expansion coefficients of the two dimension Pell polynomials derivative with respect to the variable x in terms of original polynomials expansion coefficients of the function.
2. An explicit formula for the expansion coefficients of the two dimension Pell polynomials derivative with respect to the variable y in terms of original polynomials expansion coefficients of the function.
3. Operation matrix of two dimension Pell polynomials with respect to x.
4. Operation matrix of two dimension Pell polynomials with respect to y.

This works approximates the unknown function based on the two-dimensional Pell polynomials operational matrix method for the numerical solution of partial differential equation. The original equation is transformed into the products of several matrices by using the operational matrix. The reminder of the paper is organized as follows: sections 3-5 present the definition of two dimension Pell polynomials with some important concepts. Section 6 states and proves an analytical formula which expressing the derivative of Pell polynomials in two variables with respect to x and y in terms of Pell polynomials themselves in a matrix called operational matrix of derivative. The relationship between the coefficients of the two expansions a_{ij} and c_{ij} and d_{ij} are given by recurrence relations and included in sections 7-8. The spectral approximate solution of partial differential equation using the obtained operational matrices is discussed in section 9. Some numerical examples of the proposed method are also listed in section 9. Finally, a discussion and conclusion is appeared in section 10.

3. The New Definition of Pell Polynomials in Two Variables

Pell polynomials with two variables in $L^2(R)$ over $[-1,1]\times[-1,1]$ with respect to one dimensional Pell polynomials can be defined as

$$PE_{i+1,j+1}(x,y) = PE_{n+1}(x)PE_{m+1}(y) \tag{1}$$

where: $PE_n(x)$ and $PE_n(y)$ are given by

$$PE_{n+1}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} (2x)^{n-2k} \tag{2}$$

$$PE_{m+1}(y) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m-k}{k} (2y)^{m-2k} \tag{3}$$

Using Eqns. 2-3, one can get

$$PE_{n+1,m+1}(x,y) = PE_{n+1}(x) \sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m-k}{k} (2y)^{m-2k} \tag{4}$$

or

$$PE_{n+1,m+1}(x,y) = PE_{m+1}(y) \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} (2x)^{n-2k} \tag{5}$$

By the help of properties of gamma functions, Pell polynomials $PE_{n+1,m+1}(x,y)$can be represented as

$$PE_{n+1,m+1}(x,y) = PE_{m+1}(y) \sum_{k=0}^{\left[\frac{n}{2}\right]} 2^{n-2k} \frac{\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n-2k+1)} (x)^{n-2k}, n > 0$$

$$\text{or } PE_{n+1,m+1}(x,y) = PE_{n+1}(x) \sum_{k=0}^{\left[\frac{m}{2}\right]} 2^{m-2k} \frac{\Gamma(m-k+1)}{\Gamma(k+1)\Gamma(m-2k+1)} (x)^{m-2k}, m > 0$$

For $n = 3$ and $m = 3$, the following polynomials are obtained

$$PE_{0,0}(x,y) = 0$$

$$PE_{2,0}(x,y) = 0$$

$$PE_{1,0}(x,y) = 0$$

$$PE_{3,0}(x,y) = 0$$

$$PE_{1,1}(x,y) = 1$$

$$PE_{2,1}(x,y) = 2x$$

$$PE_{3,1}(x,y) = 4x^2 + 1$$

$$PE_{1,2}(x,y) = 2y$$

$$PE_{2,2}(x,y) = 4xy$$

$$PE_{3,2}(x,y) = 2y(2x^2 + 1)$$

$$PE_{1,3}(x,y) = 4y^2 + 1$$

$$PE_{2,3}(x,y) = 2x(4y^2 + 1)$$

$$PE_{3,3}(x,y) = (4x^2 + 1)(4y^2 + 1)$$

Pell polynomials in two variables can be also defined using the following recurrence relation

$$PE_{n,m}(x,y) = 2xPE_{n-1,m}(x,y) + PE_{n-2,m}(x,y)$$

and

$$PE_{n,m}(x,y) = 2yPE_{n,m-1}(x,y) + PE_{n,m-2}(x,y)$$

4. Function Approximation

The function $f(x,y) \in L^2(R)$ defined over [-1, 1] can be expressed as

$$f(x,y) = X(x)Y(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} PE(x,y) \qquad (6)$$

By truncating the series in Eq. 6, one can obtain

$$f(x,y) = X(x)Y(y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} PE(x,y) \qquad (7)$$

Eq. 7 can be rewritten in vector form

$$f(x,y) = a^T PE(x,y)$$

The two vectors a and $PE(x,y)$ are respectively the coefficients matrix and Pell vector matrix, they are given by

$$a = [a_{00} \quad a_{01} \quad \dots \quad a_{n0} \quad a_{10} \quad a_{11} \quad \dots \quad a_{1n}, \quad \dots \quad a_{n1} \quad a_{n2} \quad \dots \quad a_{nm}] \qquad (8)$$

$$PE(x,y) = [PE_{00} \quad PE_{10} \quad \dots \quad PE_{n0} \quad PE_{10} \quad PE_{11} \quad \dots \quad PE_{1n} \quad \dots \quad PE_{n1} \quad PE_{n2} \quad \dots \quad PE_{nm}$$
(9)

5. New Fundamental Relation

To find the numerical solution of partial differential equation with Pell polynomials method, it is necessary to evaluate the Pell coefficients of the approximate solution, therefore; for convenience the following relation

$$u(x,y) = \sum_{r=1}^{N}\sum_{s=1}^{N} a_{r,s} PE_{r,s}(x,y)$$

where: $PE_{r,s}(x,y) = PE_r(x)PE_s(y)$ and $a_{r,s}$'s are unknown constants to be determined.

can be written in the matrix form as follows

here $P_r(x)$ and $P_s(y)$denote the Pell polynomials of degree r and s respectively

The Pell series solution

$$u(x,y) = \sum_{r=1}^{N}\sum_{s=1}^{N} a_{r,s} PE_{r,s}(x,y)$$

can be written as a matrix form

$$u(x,y) = PE(x)Q(y)A$$

where: $PE(x) = [PE_1(x) \quad PE_2(x) \quad \dots \quad PE_N(x)]_{\,1x(N+1)}$

and

$$Q(y) = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}_{(N+1)\times(N+1)^2}$$

where:

$$R = [PE_1(y) \quad PE_2(y) \quad \dots \quad PE_N(y)]$$

$$O = [0 \quad 0 \quad \dots \quad 0]$$

Here A is the unknown Pell coefficients matrix

$$A = [a_{00} \quad a_{01} \quad \dots \quad a_{0N} \quad a_{10} \quad a_{11} \quad \dots \quad a_{1N} \quad \dots \quad a_{N0} \quad a_{N1} \quad \dots \quad a_{NN}]^T$$

6. Operation Matrix D_x

Theorem 1

Let $PE(x, y)$ be the Pell polynomials in two variables into $[-1,1] \times [-1,1]$, then we have

$$\frac{\partial PE_{n,m}(x,y)}{\partial x} = D_x PE_{k,j}(x, y)) \tag{10}$$

where
$$D_x = \begin{pmatrix} R_1 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & R_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R_m \end{pmatrix}$$

$$D_x = I \otimes R_m$$

where the operational matrix D_x is of dimension $nm \times nm$ and the two matrices R and O are $n \times m$ matrices, in which the element of the matrix R are defined as follows

$$R_m = 2\sum_{k=0}^{n-1} (-1)^k \, k \, PE_{k,m}(x, y)$$

$$R_{ij} = 2\begin{cases} j-1 & i-j \quad \text{even} \\ 1-j & \dfrac{i-j}{4} \quad \text{integer} \end{cases}$$

Proof

Suppose that the Pell polynomials expansion function $f(x, y)$ be as

$$f_{n,m}(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} PE_{ij}(x, y) \tag{11}$$

Then $\frac{\partial f_{n,m}(x,y)}{\partial x}$ can be constructed as

$$\frac{\partial f_{n,m}(x, y)}{\partial x} = \sum_{i=0}^{n} \sum_{j=1}^{m} b_{ij} \, PE_{ij}(x, y)$$

where:

$$b_{ij} = 2i\sum_{r=i+1\ (r+i)odd}^{n} (-1)^r\, a_{ij},\ i \geq 0$$

Take $f(x,y) = PE_{n,m}(x,y)$ in Eq. 11, then $a_{n,m} = 1$ and $a_{i,m} = 0$ for $i \neq m$, the result will be

$$b_{ij} = \begin{cases} (-1)^i\, 2(i+1) & n+i \text{ even}, \quad i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, Eq. 11 becomes

$$\frac{\partial PE_{i,j}(x,y)}{\partial x} = 2\sum_{k=1\ (n+k)even}^{n} (-1)^{k+1}\,(k+1)PE_{k,j}(x,y)$$

This is the required result.

For example, Take $n = m = 3$

$$PE_{n,m}(x,y) = [PE_{11} \quad PE_{21} \quad PE_{31} \quad PE_{12} \quad PE_{22} \quad PE_{32} \quad PE_{13} \quad PE_{23} \quad PE_{33}]^T$$

$$\frac{\partial PE_{n,m}(x,y)}{\partial x} = \left[\frac{\partial PE_{11}}{\partial x} \quad \frac{\partial PE_{21}}{\partial x} \quad \frac{\partial PE_{31}}{\partial x} \quad \frac{\partial PE_{12}}{\partial x} \quad \frac{\partial PE_{22}}{\partial x} \quad \frac{\partial PE_{32}}{\partial x} \quad \frac{\partial PE_{13}}{\partial x} \quad \frac{\partial PE_{23}}{\partial x} \quad \frac{\partial PE_{33}}{\partial x}\right]^T$$

$$\frac{\partial PE_{11}}{\partial x} = 0\,, \frac{\partial PE_{21}}{\partial x} = 2PE_{11}, \frac{\partial PE_{31}}{\partial x} = 4PE_{21},$$

$$\frac{\partial PE_{12}}{\partial x} = 0, \frac{\partial PE_{22}}{\partial x} = 2PE_{12}, \frac{\partial PE_{32}}{\partial x} = 4PE_{22},$$

$$\frac{\partial P_{13}}{\partial x} = 0, \frac{\partial P_{23}}{\partial x} = 3P_{13}, \frac{\partial P_{33}}{\partial x} = 4P_{23},$$

$$\frac{\partial PE}{\partial x} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix} PE(x,y)$$

$$R_m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} PE_{1m} \\ PE_{2m} \\ PE_{3m} \end{pmatrix}$$

$$R_m = \begin{bmatrix} \frac{\partial PE_{1m}}{\partial x} \\ \frac{\partial PE_{2m}}{\partial x} \\ \frac{\partial PE_{3m}}{\partial x} \end{bmatrix}$$

7. Derivative of $PE_{n,m}(x, y)$ With Respect to x

Consider a function of two variables expanded in Pell aeries

$$f_{n,m}(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} PE_i(x) PE_j(y) \tag{12}$$

The partial derivative of Eq. 12 with respect to x is

$$\frac{\partial f_{n,m}(x,y)}{\partial x} = \sum_{i=1}^{n-1}\sum_{j=0}^{m} c_{ij} PE_i(x) PE_j(y) \tag{13}$$

The relationship between the coefficients of the two expansions a_{ij} and c_{ij} are given by the following recurrence relation

$$c_{n,j=0}$$

$$c_{n-1,j=2(n-1)a_{n,j}}$$

$$c_{n-2,j=2(n-2)a_{n-1,j}}$$

and

$$c_{r,j=2ra_{r+1,j}} - \frac{1}{r+1} c_{r+2,j}\ , r = n-3, n-4, \dots, 1$$

8. Derivative of $PE_{n,m}(x, y)$ With Respect to y

Take the partial derivative of Eq. 12 with respect to y, yields

$$\frac{\partial f_{n,m}(x,y)}{\partial y} = \sum_{i=1}^{n-1}\sum_{j=0}^{m} d_{ij} PE_i(x) PE_j(y) \tag{14}$$

The relationship between the coefficients of the two expansions a_{ij} and d_{ij} are given by the following recurrence relation

$$\frac{\partial f_{n,m}(x,y)}{\partial x} = \sum_{i=0}^{n}\sum_{j=1}^{m-1} d_{ij}$$

$$d_{i,m} = 0$$

$$d_{i,m-1} = 2(m-1)a_{i,m}$$

$$d_{i,m-2} = 2(m-2)a_{i,m-1}$$

$$d_{i,r} = 2ra_{i,r+1} - \frac{1}{r+1} d_{i,r+2}$$

9. Problem Statement and Method of Solution

Consider the following form of partial differential equation

$$u = F(u_x, u_y, u_{xy}, u_{xx}, u_{yy}) \tag{15}$$

Subject to some initial conditions

Let $n = m = 3$,

$$u = A^T PE(x, y) \tag{16}$$

Differentiate Eq. 16 with respect to both x and , yields

$$u_x = D_x APE(x, y) \tag{17}$$

$$u_y = D_y APE(x, y) \tag{18}$$

$$u_{xx} = D_x^2 APE(x, y) \tag{19}$$

$$u_{yy} = D_y^2 APE(x, y) \tag{20}$$

$$u_{xy} = D_x D_y APE(x, y) \tag{21}$$

where: $PE(x, y) = [PE_{11} \quad PE_{21} \quad PE_{31} \quad PE_{12} \quad PE_{22} \quad PE_{32} \quad PE_{13} \quad PE_{23} \quad PE_{33}]^T$

$$D_y = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix}, R_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, D_x = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}, s = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Put $y = 0$ on Eq. 18

$$\frac{\partial u(x,0)}{\partial y} = [2u_{12} \quad 2u_{32} \quad 2u_{32} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]PE^T \tag{22}$$

Put $y = 0$ in Eq. 16

$$u(x, 0) = [u_{11} + u_{13} \quad u_{21} + u_{23} \quad u_{31} + u_{33} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]PE^T \tag{23}$$

Substituting Eqns. 18-21 into Eq. 15

The effectiveness of the suggested two-dimensional Pell polynomials method is illustrated throughout the following two examples.

Example 1

Consider the following partial differential equation

$$u_x(x,y) + u_y(x,y) = 1 + 2xy + x^2$$

with $u(x,0) = x, u_x(x,0) = x^2$

Rewrite x, x^2 and $1 + 2xy + x^2$ in terms of $PE_{n,m}(x,y)$using the following formulas

$$1 = PE_{11}$$

$$x = \frac{1}{2}PE_{21}$$

$$y = \frac{1}{2}PE_{12}$$

$$xy = \frac{1}{4}PE_{22}$$

$$x^2 = \frac{1}{4}(PE_{31} - PE_{11})$$

$$y^2 = \frac{1}{4}(PE_{13} - PE_{11})$$

$$xy^2 = \frac{1}{8}(PE_{23} - PE_{21})$$

$$x^2y = \frac{1}{8}(PE_{32} - PE_{12})$$

$$x^2y^2 = \frac{1}{16}(PE_{33} - PE_{31} - PE_{13} + 2PE_{11})$$

$$x^ny^m = \frac{1}{2^{n+m}}(PE_{n+1,m+1} - PE_{n-1,m-1} - PE_{n-1,m-1} - \left(PE_{n+1,m-1} - PE_{n-1,m-1}\right) - \cdots)$$

$$u_x + u_y = \tfrac{3}{4}PE_{11} + \tfrac{1}{2}PE_{22} + \tfrac{1}{4}PE_{31} \qquad (24)$$

In this example the two Eqns. 22-23 will be

$$2u_{11}PE_{11} + 2u_{22}PE_{21} + 2u_{32}PE_{31} = \frac{1}{4}(2PE_{31}PE_{11}) \qquad (25)$$

$$(u_{11}+u_{13})PE_{11} + (u_{21} + u_{23})PE_{21} + (u_{31} + u_{33})PE_{31} = \frac{1}{2}PE_{21} \qquad (26)$$

Put 18 and 19 into Eq. 24, yields

$$D_x APE^T + D_y APE^T = cPE^T \qquad (27)$$

Here $c = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$

From 25-27, the Pell coefficients A can be calculated for $n = m = 3$

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 \end{bmatrix}$$

The solution is $u(x,y) = \frac{1}{2}PE_{21}(x,y) - \frac{1}{8}PE_{12}(x,y) + \frac{1}{8}PE_{32}(x,y)$

$$u(x,y) = x + x^2y$$

Example 2

Consider the second test problem

$$u_{xx} + u_{yy} - 2u_{xy} = -1$$

Subject to the initial conditions

$$u(x,0) = u_y(x,0) = x$$

$$(AD_x^2PE(x,y) + AD_y^2PE(x,y) - 2AD_xD_yPE(x,y) = -PE_{11}(x,y)$$

or

$$(AD_x^2 + AD_y^2 - 2AD_xD_y)PE(x,y) = cPE^T(x,y)$$

Here $c = [-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$

By equating the coefficients, one can obtain

$$A = \left[-\frac{1}{8} \quad \frac{1}{2} \quad 0 \quad 0 \quad \frac{1}{4} \quad 0 \quad \frac{1}{8} \quad 0 \quad 0\right]$$

The solution is

$$u(x,y) = \frac{1}{2}PE_{21}(x,y) + \frac{1}{4}PE_{22}(x,y) + \frac{1}{8}PE_{13}(x,y) - \frac{1}{8}PE_{11}(x,y)$$

that is $u(x,y) = x + xy + \frac{1}{2}y^2$

10. CONCLUSION

This paper presents two explicit formulae associated with differentiation of Pell polynomials in two variables with respect to the two variables in terms of the Pell polynomials themselves. In addition, an interesting formulae is constructed which relating the coefficients in the differentiated expansions of two dimension Pell polynomials. To those original expansions, spectral techniques Pell polynomials with the aid of these formulae are proposed as an application to solve partial differential equation. The method is shown to be highly accurate and yields exact solutions in few terms.

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