Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 1 Article 6

1-7-2021

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Recommended Citation
Hameed, Raad Awad; Taha, Wafaa M.; and Talab, Sarah Majeed (2021) "EXISTENCE OF PERIODICSOLUTIONS OF A NONLINEAR ALLEN-CAHN EQUATION WITH NEUMANN CONDITION," *Al-Qadisiyah Journal of Pure Science*: Vol. 26: No. 1, Article 6.
DOI: 10.29350/qjps.2021.26.1.1248
Available at: https://qjps.researchcommons.org/home/vol26/iss1/6

Al–Qadisiyah Journal of Pure Science
ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514
DOI : /10.29350/jops.
http://qu.edu.iq/journalsc/index.php/JOPS

EXISTENCE OF PERIODIC SOLUTIONS OF A NONLINEAR ALLEN-CAHN EQUATION WITH NEUMANN CONDITION

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Article History

Received on: 27/11/2020
Revised on:10/12/2020
Accepted on:13/12/2020

Keywords:
Nonlinear diffusion Allen-Cahn equation; Topological degree theory, Isolated Neumann condition.

DOI: https://doi.org/10.29350/qjps2021.26.1.1248

ABSTRACT

This research is investigated the periodic solutions to a nonlinear diffusion Allen-Cahn equation with a homogenous Neumann boundary condition. The theory of Leray-Schauder fixed point degree was adopted and the study is showed that there exist nontrivial periodic solutions to the nonlinear diffusion Allen-Cahn equation with isolated Neumann condition by the topological degree theory.

1. Introduction

The main objective of this study is to establish the existence of time periodic solutions for the following nonlinear parabolic system

$$\frac{\partial v}{\partial t} - \Delta v^m = m(t)(v^3 - v), \qquad (x,t) \in S_T \qquad (1.1)$$

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$$\frac{\partial v}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \tag{1.2}$$

$$v(x,0) = v(x,T), \qquad x \in \Omega \tag{1.3}$$

where the constant $m \geq 1$, $S_T = \Omega \times (0,T)$ is a space-time cylinder with Ω is a two or three-dimensional domain of sufficient smooth. The domain Ω is an open bounded regular subset of $\mathbb{R}^n$. Here, we denote for the outward unit normal vector $\frac{\partial}{\partial n}$ on $\partial\Omega$, where $\frac{\partial v}{\partial n} = 0$ that is mean no flux from the-boundary. The equations (1.1) might be used to characterize a population dynamics; you can see $[12]$. Where $v(x,t)$ represents the density of the species at position x and time t. Let (∇v^m) be a diffusion term. The logistic growth term $m(t)(v^3 - v)$, supposes that too low population densities $[13]$. In the last two decades, many authors investigated the periodic problems with nonlocal terms $[1-5]$. A perfect model was published by Nistri and Allegretto and they suggested the following form

$$\frac{\partial v}{\partial t} - \Delta v = f(x,t,a,\Phi[v],v),$$

with first type of Dirichlet conditions. As well, according to the actual needs, many researches interesting to nonlinear diffusion equations with nonlocal terms, like the porous equation $[6-7]$ with typical form

$$\frac{\partial v}{\partial t} = \Delta v^m + (v - \Phi[v])v, \tag{1.4}$$

and a class of doubly degenerate parabolic equation $[8]$ with typical form

$$\frac{\partial v}{\partial t} = \mathrm{div}(|\nabla v^m|^{p-2}\nabla v^m) + (a - \Phi[v])v. \tag{1.5}$$

the equation (1.4) is singular whenever $0 < m < 1$ and degenerate whenever $m > 1$. Equation (1.5) is degenerate only when $v = 0$, or ∇v. This degenerate equation exhibiting a doubly nonlinearity generalizes the porous medium equation $p = 2$ and the parabolic $p-$Laplace equation $m = 1$. If $m = 1, p = 2$, then (1.5) is a nondegenerate parabolic equation and a special case for this case is the heat equation [4].

The authors study a periodic solutions of nonlinear diffusion Allen-Cahn equation with homogenous Neumann condition. If we compare between two type of boundary condition the first type the Dirichlet condition and second types the isolated Neumann condition, when using the second type of boundary condition to providing some a priori estimates in this case the proof is more difficult. On the other hand, the different from the cases of first type of boundary condition, an auxiliary problem for the system$(1.1) - (1.3)$ is considered for using the topological degree theory. We make out that this problem for $(1.1) - (1.3)$ is considered for using the theory of Leray-Schauder fixed point degree [15]. We show that this system $(1.1) - (1.3)$ admits a non-trivial periodic solution.

The rest of the paper is organized as follows. In Section 2 , we introduce some necessary preliminaries including the auxiliary problem. In Section 3, we show that a number of necessary priori estimations of the solutions of the auxiliary problem. Additionally, we prove the main result of this article is related to non-trivial nonnegative periodic solution.

2. Preliminaries

In this paper, we assume that:

(C1) $\Psi[\cdot]: L_+^p(\Omega) \to \mathbb{R}^+$ is a continuous and bounded functional satisfying

$$\Psi[\upsilon] \leq C\|\upsilon\|_{L^p(\Omega)}^p, \qquad p = 1,2,\ldots$$

where $\Psi[\upsilon] = m(\upsilon^3 - \upsilon)$ and C is a positive constant independent of υ where,

$\mathbb{R}^+ = [0,+\infty), L_+^p(\Omega) = \{\upsilon \in L^p(\Omega) | \upsilon \geq 0, \text{a.e. in } \Omega\}$

(C2) $m(t+T) = m(t), m \in W^{1,\infty}[0,T]$ and $\inf_{t\in[0,T]} m(t) = m_0 > 0$,

$\|m\|_{L^\infty[0,T]} := 0$

Since the equation (1.1) is degenerate at points where $\upsilon = 0$, the problem (1.1) – (1.3)has no classical solutions in general, so we focus on the discussion of weak solutions in the following sense:

Definition 1 A function υ is said to be a weak solution of the problem (1.1) – (1.3), if $\upsilon \in L^\infty(S_T) \cap C_T(\overline{S}_T)$, $\upsilon^m \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C_T(\overline{S}_T)$ and υ satisfies [3]

$$\iint_{S_T} \left(-\upsilon\frac{\partial \eta}{\partial t} + \nabla \upsilon^m \nabla \eta - m(t)(\upsilon^2 - 1)\upsilon\eta\right) dxdt = 0, \qquad (2,1)$$

for any $\eta \in C^1(\overline{S}_T)$ with $\eta(x,0) = \eta(x,T)$.

by using the topological degree theory, we introduce a map by studying the following auxiliary system

$$\frac{\partial \upsilon_\delta}{\partial t} - \operatorname{div}((m\upsilon_\delta^{m-1} + \delta)\nabla \upsilon_\delta) = m(t)(\upsilon_\delta^2 - 1)\upsilon_\delta^+ \quad (x,t) \in S_T \qquad (2.2)$$

$$\frac{\partial v_\delta}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \qquad (2.3)$$

$$v_\delta(x,0) = v_\delta(x,T), \qquad x \in \Omega, \qquad (2.4)$$

where $v^+ = \max\{0, v\}$, δ is a sufficiently small positive constant. Then, the desired solution will be obtained as the limit point of the solutions of the problem $(1.1) - (1.3)$[3]. In the following, we present a map by the following problem

$$\frac{\partial v_\delta}{\partial t} - \mathrm{div}((m v_\delta^{m-1} + \delta)\nabla v_\delta = f, \qquad (x,t) \in S_T \qquad (2.5)$$

$$\frac{\partial v_\delta}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \qquad (2.6)$$

$$v_\delta(x,0) = v_\delta(x,T), \qquad x \in \Omega, \qquad (2.7)$$

Then we can define a map $\upsilon_\delta = \mathrm{H}f$ with $\mathrm{H} : \mathrm{C}_\mathrm{T}(\overline{\mathrm{S}}_\mathrm{T}) \to \mathrm{C}_\mathrm{T}(\overline{\mathrm{S}}_\mathrm{T})$. The-authors have using classical estimates (see [9]). We can show that $\|v_\delta\|_{L^\infty(S_T)}$ is bounded by $\|f\|_{L^\infty(S_T)}$ and v_δ is Hölder continuous in S_T. Then by the Arzela-Ascoli theorem, the map H is compact. So the map is a compact continuous map. Let $f(v) = m(t)((v_\delta^2 - 1)v_\delta^+)$ where $v_\delta^+ = \max\{v_\delta, 0\}$, we can see that the nonnegative solution of problem $(1.1) - (1.3)$ is also a nonnegative solution solves $v_\delta = H(m(t)((v_\delta^2 - 1)v_\delta^+))$. So we will study the existence of the nonnegative fixed points of the map $v_\delta = H(m(t)((v_\delta^2 - 1)v_\delta^+))$ instead of the nonnegative solutions of problem $(1.1) - (1.3)$.

3 The main results

First, by the same way as in [5], we can get the non-negativity of the-solutions of problem (2.2)-(2.4).

Lemma 1 If a non-trivial function $\upsilon_\delta \in C(\overline{S}_T)$ solves $\upsilon_\delta = H\big(1, \big(\upsilon_\delta^2 - 1\big)\upsilon_\delta^+\big)$, then $\upsilon_\delta(x,t) \geq 0 \quad \forall (x,t) \in \overline{S}_T$

In the following, by the Moser iterative technique, we will show the a priori estimate for the upper bound of nonnegative periodic solutions of problem $(2.5) - (2.7)$. Here and below we denote by $\|\cdot\|_p, (1 \leq p \leq \infty)$ then $L^p(\Omega)$ norm in Banach space.

Lemma 2 Let $\in [0,1]$, $\upsilon_\delta(x,t)$ be a nonnegative periodic solution solving $\upsilon_\delta = T\,(1, \sigma f(\upsilon_\delta))$, Then there exists a positive constant K independent of σ and δ, such that

$$\|\upsilon_\delta\|_\infty < K \tag{3.1}$$

where $\upsilon_\delta(t) = \upsilon_\delta(.,t)$

Proof Suppose υ_δ is a nontrivial periodic solution, Multiplying Eq (2.5) by υ_δ^{r+1} where $(r \geq 0)$ and integrating over Ω, we get

$$\frac{1}{r+2}\frac{d}{dt}\|\upsilon_\delta(t)\|_{r+2}^{r+2} + \frac{4m(r+1)}{(r+m+1)^2}\left\|\nabla(\upsilon_\delta^{\frac{r+m+1}{2}}(t))\right\|_2^2 \leq \|m(x,t)\|_{L^\infty(\Omega\times(0,T))}\|\upsilon_\delta(t)\|_{r+2}^{r+4},$$

where $\big(m(x,t)\big(\upsilon_\delta^4 - \upsilon_\delta^2\big) \leq M\upsilon_\delta^4\big)$ and $M = \sup_{(x,t)} a(x,t) \in \overline{S}_T$

$$\frac{d}{dt}\|\upsilon_\delta(t)\|_{r+2}^{r+2} + C\left\|\nabla(\upsilon_\delta^{\frac{r+m+1}{2}}(t))\right\|_2^2 \leq M(r+1)\|\upsilon_\delta(t)\|_{r+2}^{r+4}, \tag{3.2}$$

and hance

$$\frac{d}{dt}\|v_\delta(t)\|_{r+2}^{r+2} + C_1\left\|\nabla(v_\delta^{\frac{r+m+1}{2}}(t))\right\|_2^2 \le C_2(r+1)\|v_\delta(t)\|_{r+2}^{r+4}, \qquad (3.3)$$

where $C_j(j=1,2)$ is a positive constants independent of v_δ, k and m.

Assume that $\|v_\delta(t)\|_\infty \neq 0$ and set

$$r_k = 2^k - 2, \qquad \alpha_k = \frac{2(r_k+2)}{r+m+1}, \qquad v_k(t) = v_\delta^{\frac{r+m+1}{2}}(t) \text{ where } (k=0,1,\dots)$$

then $\alpha_k < 2, m_k = 2^{k-1} + m_{k-1}$. For convenience, we denote by C a positive constant independent of v_δ, k and m, which may take different values. From (3.3) we obtain

$$\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} + C\|\nabla v_k(t)\|_2^2 \le C(r+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}, \qquad (3.4)$$

By using the Gagliardo-Nirenberg inequality, we have

$$\|v_k(t)\|_{\alpha_k} + D\|\nabla v_k(t)\|_p^\theta \|v_k(t)\|_1^{1-\theta}, \qquad (3.5)$$

with

$$\theta_k = \frac{N}{N+2} \in (0,1)$$

By inequalities (3.4), (3.5) and the fact that $\|v_k(t)\|_1 = \|v_{k-1}(t)\|_{\alpha_k-1}^{\alpha_k-1}$, we obtain the following differential inequality

$$\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} \le -C\|v_k(t)\|_{\alpha_k}^{\frac{2}{\theta}}\|v_k(t)\|_1^{\frac{2(\theta-1)}{\theta}} + C(r_k+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}$$

$$\le -C\|v_k(t)\|_{\alpha_k}^{\frac{2}{\theta}}\|v_{k-1}(t)\|_{\alpha_{k-1}}^{2\frac{(\theta-1)}{\theta}\alpha_{k-1}} + C(r_k+1)\|v_k(t)\|_{\alpha_k}^{\alpha_k+2}$$

Let

$$\zeta_k = \max\{1, \sup_t \|v_k(t)\|_{\alpha_k}\},$$

we have

$$\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\alpha_k} \le \|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k(m_k+1)}{m_k+2}}\{-C\|v_k(t)\|_{\alpha_k}^{\frac{2}{\theta}-\frac{\alpha_k(m_k+1)}{m_k+2}}\zeta_{k-1}^{2\frac{(\theta-1)}{\theta}\alpha_k-1}$$
$$+C(r_k+1)\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}}\|v_k(t)\|_{\alpha_k}^2\}. \qquad (3.6)$$

By young's inequality

$$cd \leq \delta c^{p'} + \delta^{-\frac{q'}{p'}} d^{q'},$$

where $p' > 1,\ q' > 1,\ c > 0,\ d > 0, \delta > 0$ and $\frac{1}{p'} = 1 - \frac{1}{q'}$. Set

$$c = \|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k+2(m_k+2)}{m_k+2}}, \qquad d = s_k + 1, \qquad \delta = \frac{1}{2}\zeta_{k-1}^{2\frac{(\theta-1)}{\theta}\alpha_k - 1},$$

$$p' = l_k = \frac{2(m_k+2)}{\theta_k(\alpha_k + 2(m_k+2))} - \frac{\alpha_k(m_k+2)}{\alpha_k + 2(m_k+2)},$$

Then we obtain

$$(s_k+1)\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k+2(m_k+2)}{m_k+2}} \leq \frac{1}{2}\|v_k(t)\|_{\alpha_k}^{\frac{2}{\theta} - \frac{\alpha_k(m_k+1)}{m_k+2}} \zeta_{k-1}^{2\frac{(\theta-1)}{\theta}\alpha_k - 1}$$
$$+C(s_k+1)^{\frac{l_k}{l_{k-1}}} \zeta_{k-1}^{2\frac{(1-\theta)}{\theta}\alpha_k - 1\frac{1}{l_{k-1}}}. \tag{3.7}$$

Here we have used the fact that $p' = l_k > r > 1$ for some r independent of k. In fact, it is easy to verify that

$$\lim_{k\to\infty} l_k = +\infty$$

Denote

$$\alpha_k = \frac{l_k}{l_k - 1}, \qquad b_k = \frac{1-\theta}{\theta}\frac{2\alpha_k - 1}{l_k - 1},$$

and combining (3.7) with (3.6) we have

$$\frac{d}{dt}\|v_k(t)\|^{\alpha_k} \leq \|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k(m_k+1)}{m_k+2}} \{\frac{-C}{2}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{\theta} - \frac{\alpha_k(m_k+1)}{m_k+2}} \zeta_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_k - 1p}$$
$$+C(s_k+1)^{\alpha_k}\zeta_{k-1}^{b_k}\}. \tag{3.8}$$

Then

$$(m_k+2)\frac{d}{dt}\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}} \leq -C\|v_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{\theta} - \frac{\alpha_k(m_k+1)}{m_k+2}} \zeta_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_k - 1p} \tag{3.9}$$

$$+C(m_k+2)^{\alpha_k}\zeta_{k-1}^{b_k}.$$

From the periodicity of $\upsilon_k(t)$, we know that there exists t_0 at which $\|\upsilon_k(t)\|_{\alpha_k}$ reaches its maximum and thus the left hand of (3.9) vanishes [3]. Then we obtain

$$\|\upsilon_k(t)\|_{\alpha_k} \le \{C[(m_k+2)^{\alpha_k}\zeta_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1}p}]\}^{\frac{1}{\alpha_k}},$$

where

$$\alpha_k = \frac{(m_k+2)p-(\theta)(\alpha_k(m_k+1))}{(\theta)(m_k+2)} = \frac{\alpha_k l_k}{m_k+2}$$

Therefore we conclude that

$$\|\upsilon_k(t)\|_{\alpha_k} \le \{C(m_k+2)^{\alpha_k}\zeta_{k-1}^{b_k+\frac{(\theta-1)}{\theta}\alpha_{k-1^2}}]\}^{\frac{1}{\alpha_k}} = \{C(m_k+2)^{\alpha_k}\}^{\frac{m_k+2}{\alpha_k l_k}}\zeta_{k-1}^{\frac{(1-\theta)(m_k+2)\alpha_{k-1^2}}{(l_k-1)\theta}}$$

Since $\frac{m_k+2}{(l_k-1)\theta} = \frac{\alpha_k}{1-\theta\alpha_k}$ and $\frac{m_k+2}{\alpha_k l_k}$ and α_k are bounded, we get

$$\|\upsilon_k(t)\|_{\alpha_k} \le C2^{ka'}\zeta_{k-1}^{\frac{(1-\theta)\alpha_{k-1^2}}{(2-\theta\alpha_k)}},$$

where $a'>1$ is a positive constant independent of k. As $\alpha_k = \frac{2(m_k+2)}{m_k+2} < 2$ implies that $\frac{(1-\theta)\alpha_{k-1^2}}{(2-\theta\alpha_k)} \le \frac{(1-\theta)\alpha_{k-1^2}}{(2-\theta 2)} \le 2$ and $\zeta_{k-1}\ge 1$, then we have

$$\|\upsilon_k(t)\|_{\alpha_k} \le CA^k\zeta_{k-1}^2,$$

or

$$\ln\|\upsilon_k(t)\|_{\alpha_k} \le \ln\zeta_k \le \mathrm{In}C + k\ln A + 2\ln\zeta_{k-1},$$

where $A = 2^{a'} > 1.$ Thus

$$\ln\|\upsilon_k(t)\|_{\alpha_k} \le \ln C\sum_{i=0}^{k-2}2^i + 2^{k-1}\ln\zeta + \ln A(\sum_{j=0}^{k-2}(k-j)2^j)$$

$$\le (2^{k-1}-1)lnC + 2^{k-1}\ln\zeta + f(k)\ln A,$$

or

$$\|v_\delta(t)\|_{m_k+2} \le \left\{C^{2^{k-1}} \zeta^{2^{k-1}} A^{f(k)}\right\}^{\frac{2}{m_k+2}},$$

where

$$f(k) = 2^{k+1} - 2 - k - 2^{k-1}.$$

Letting $k \to \infty$, we obtain

$$\|v_\delta(t)\|_\infty \le C\zeta \le C(\max\{1, \sup_t \|v_\delta(t)\|_{n+2}^n\}). \quad (3.10)$$

On the other hand, it follows from (3.3) with $r = n - 1$ that

$$\frac{d}{dt}\|v(t)\|_{n+1}^{n+1} + C_1\|\nabla v(t)\|_2^2 \le C_2\|v(t)\|_{n+1}^{n+3} \quad (3.11)$$

By Hölder's inequality and Sobolev's theorem, we have

$$\|v(t)\|_{n+1}^{2n} + |\Omega|^{\frac{n-1}{n+1}}\|\nabla v_\delta^m(t)\|_{2n}^{2n} \le C\|\nabla v(t)\|_2^2. \quad (3.12)$$

Combined with (3.11), it yields

$$\frac{d}{dt}\|v_\delta(t)\|_{n+1}^{n+1} + C_1\|v_\delta(t)\|_{n+1}^{2n} \le C_2\|v_\delta(t)\|_{n+1}^{n+3}. \quad (3.13)$$

By Young's inequality, it follows that

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$$\frac{d}{dt}\|v_\delta(t)\|_{n+1}^{n+1} + C_1\|v_\delta(t)\|_{n+1}^{2n} \le C_2. \quad (3.14)$$

where $C_i (i = 1, 2)$ are constants independent of v. Taking the periodicity of v. into account, we infer from (3.14) that

$$\|v(t)\|_2 \le C.$$

which together with (3.10) implies (3.1).■ .

Corollary 1 There exists a positive constant R independent of δ, such that

$$\deg(I - H(1, m(t)((v_\delta^+)^3 - v_\delta^+)), B_R, 0) = 1,$$

where B_R is a ball centered at the origin with radius R in $L^\infty(S_T)$.

Proof It follows from lemma 2 that there exists a positive constant R independent of δ, such that

$$\upsilon \neq H\left(\sigma\left(m(t)((v_\delta^+)^3 - v_\delta^+)\right)\right), \qquad \forall v \in \partial B_R, \qquad \sigma \epsilon [0,1]$$

So the degree is well defined on B_R. From the homotopy invariance of the Leray- Schauder degree and the existence and uniqueness of the solution of $H(1, 0)$, we can get the following

$$\deg(1 - H(m(t)\left((v_\delta^+)^3 - v_\delta^+\right), B_R, 0) = \deg(1 - H(1, \sigma(m(t)\left((v_\delta^+)^3 - v_\delta^+\right), B_R, 0)$$

$$= \deg(1 - H(1,0), B_R, 0)$$

$$= 1$$

Lemma 3 There exist a constants $r > 0$ and $\delta > 0$, such that for any $r > r_0$ and $\delta > \delta_0$ $H(m(t)(((v_\delta^+)^3 - v_\delta^+))$ admits no nontrivial solution v_δ, satisfying

$$0 < \|v_\delta\|_{L^\infty(S_T)} \leq r.$$

Proof By contradiction, let v_δ be a non-trivial solution of $v_\delta = H(m(t)(((v_\delta^+)^3 - v_\delta^+))$ satisfying $0 < \|v_\delta\|_{L^\infty(S_T)} \leq r$. For any test function $\phi(x) \in C_0^\infty(\Omega)$, multiplying (2.5) by $\frac{\phi^2}{v_\delta}$ and integrating over $S_T^* = B_\delta(x_0) \times (0, T)$, we obtain

$$\iint_{S_T^*} \frac{\phi^2}{v_\delta}\frac{\partial v_\delta}{\partial t}dtdx + \iint_{S_T^*}((rmv_\delta^{m-1}+\delta)\nabla v_\delta \nabla\left(\frac{\phi^2}{v_\delta}\right))dtdx$$

$$\leq \iint_{S_T} \phi^2\left(m(t)\left(\left(v_\delta^2 - 1\right)\right)\right)dtdx. \tag{3.15}$$

From the periodicity of v_δ w.r.t t we have

$$\iint_{S_T^*} \frac{\phi^2}{v_\delta}\frac{\partial v_\delta}{\partial t}dtdx = \int_\Omega \phi^2 \int_0^T \frac{\partial(lnv_\delta)}{\partial t}dtdx = 0. \tag{3.16}$$

The second term on the left-hand side in (3.15) can be rewritten as

Thuthus

$$\iint_{S_T^*}((ms_\delta^{m-1}+\delta)\nabla s_\delta \nabla\left(\frac{\phi^2}{v_\delta}\right))dtdx$$

$$= \iint_{S_T^*}((mv_\delta^{m-1}+\delta)\nabla v_\delta \nabla\left(\phi.\frac{\phi^2}{v_\delta}\right))dtdx$$

$$= \iint_{S_T^*}((mv_\delta^{m-1}+\delta)\nabla v_\delta \left(\frac{\phi}{v_\delta}\nabla\phi + \nabla\left(\frac{\phi}{v_\delta}\right)\phi\right))dtdx$$

$$= \iint_{S_T^*}((mv_\delta^{m-1}+\delta)\nabla\left(\frac{\phi}{v_\delta}\right)v_\delta\left(v_\delta\nabla\phi - v_\delta^2\nabla\left(\frac{\phi}{v_\delta}\right)\right))dtdx$$

$$+ \iint_{S_T^*}((mv_\delta^{m-1}+\delta)\phi\nabla v_\delta\nabla\left(\frac{\phi}{v_\delta}\right)dtdx$$

$$= \iint_{S_T^*}((mv_\delta^{m-1}+\delta)|\nabla\phi|^2 dtdx \tag{3.17}$$

$$- \iint_{S_T^*}((mv_\delta^{m-1}+\delta)(v_\delta\nabla\phi - v_\delta\nabla\phi)\nabla\left(\frac{\phi}{v_\delta}\right)dtdx$$

$$= \iint_{S_T^*}((mv_\delta^{m-1}+\delta)|\nabla\phi|^2 dtdx$$

$$- \iint_{S_T^*}(v_\delta^2\left|\nabla\left(\frac{\phi}{v_\delta}\right)\right|^2 dtdx$$

$$\leq \iint_{S_T^*}((mv_\delta^{m-1}+\delta)|\nabla\phi|^2 dtdx$$

$$\iint_{S_T^*} ((mv_\delta^{m-1} + \delta)\nabla v_\delta \nabla\left(\frac{\phi^2}{\delta}\right))dtdx \leq \iint_{S_T^*} ((mv_\delta^{m-1} + \delta)|\nabla\phi|^2 dtdx \quad (3.18)$$

Combining (3.16) with (3.15)(3.18) , we obtain

$$\iint_{S_T} \phi^2(m(t)v_\delta^2 - 1))dtdx \leq \iint_{S_T^*} ((mv_\delta^{m-1} + \delta)|\nabla\phi|^2 dtdx \quad (3.19)$$

By an approximating process, we can choose $\phi = \phi_1$ and ϕ_1 is the corresponding eigenfunction of the principal eigenvalue μ_1. By (3.19)

$$0 \leq \iint_{S_T^*} ((mv_\delta^{m-1} + \delta)|\nabla\phi_1|^2 - \phi_1^2(m(t)v_\delta^2 - 1))dtdx$$

$$\leq \iint_{S_T^*} ((mr^{m-1} + \delta)\phi_1\Delta\phi_1 - \phi_1^2(m(t)(r^2 - 1))dtdx$$

$$\leq \iint_{S_T^*} \phi_1^2(\mu_1(mr^{m-1} + \delta) - m(t)(r^2 - 1))dtdx \quad (3.20)$$

$$\leq \int_{B_\rho(x_0)} \phi_1^2 \int_0^T (\mu_1(mr^{m-1} + \delta) - m(t)(r^2 - 1))dtdx$$

$$\leq \int_{B_\rho(x_0)} \phi_1^2 (\mu_1(mr^{m-1} + \delta) - m_0(r^2 - 1))dx$$

$$0 \leq \int_{B_\rho(x_0)} \phi_1^2 (\mu_1(mr^{m-1} + \delta) - m_0(r^2 - 1))dx \quad (3.21)$$

$$\int_{B_\rho(x_0)} \phi_1^2 m_0(r^2 - 1))dx \leq \int_{B_\rho(x_0)} \phi_1^2 (\mu_1(mr^{m-1} + \delta))dx \quad (3.22)$$

$$m_0(r^2 - 1) \leq \frac{\int_{B_\rho(x_0)} \phi_1^2 (\mu_1(mr^{m-1} + \delta))dx}{\int_{B_\rho(x_0)} \phi_1^2 \, dx} \quad (3.23)$$

we get

$$m_0(r^2 - 1) \leq (\mu_1(mr^{m-1} + \delta)) \frac{\int_{B_\rho(x_0)} \phi_1^2 \, dx}{\int_{B_\rho(x_0)} \phi_1^2 \, dx} \quad (3.24)$$

$$m_0(r^2-1)-\mu_1(mr^{m-1}+\delta)\leq 0 \tag{3.25}$$

and thus get a contradiction.

Corollary 2 There exists a small positive constant r which is independent of δ and satisfies $r < R$ such that

$$\deg(I-H(1,m(t)((v_\delta^+)^3-v_\delta^+)),B_r,0)=0,$$

where B_r is a ball centered at the origin with radius r in $L^\infty(S_T)$.

Proof *Same* way for lemma 3, we can see that there exists a positive constant $0 < r < R$ independent of δ, such that

$$\upsilon_\delta \neq H\big(\tau,\big(m(t)((v_\delta^+)^3-v_\delta^+)\big)+1-\tau\big), \qquad \forall v_\delta \in \partial B_r, \qquad \tau\in[0,1]$$

thus the degree is well defined on B_r. By lemma 3 , we can easy to infer that $v = H\big(0,\big(m(t)((v_\delta^+)^3-v_\delta^+)\big)+1\big)$ admits no solution in B_r. Then by the homotopy invariance of the Leray-Schauder degree, we get

$$\begin{aligned}\deg(I-H(1,(m(t)((v_\delta^+)^3-v_\delta^+)),B_r,0)\\ =\deg(1-H(0,(m(t)((v_\delta^+)^3-v_\delta^+)+1),B_r,0)\\ =0.\ \blacksquare\end{aligned}$$

.

Now we show the proof of the next main result that presents the sufficient condition that guarantee the systems $(1.1)-(1.3)$ process a non-trivial nonnegative periodic solution.

Theorem 1 If assumptions $(C1),(C2)$ hold, then the problem $(1.1)-(1.3)$ admits a non-trivial nonnegative periodic solution υ_δ.

Proof Using corollaries 1 and 2, we have

$$\deg(1-H\big(f(.)\big),\Pi,0)=1,$$

where $\Pi = B_R \setminus B_r, B_\alpha$ is a ball centered at the origin with radius $\alpha \in L^\infty(S_T)$, R and r are positive constants and $R > r$. By the theory of the Leray-Schauder fixed point degree and lemma 1, we can conclude that problem $(2.2) - (2.4)$ admits a non-trivial nonnegative periodic solution v_δ. By lemma 3 and a similar method to that in $[11]$, we can obtain

$$\|\nabla v_\delta\|_{L^p(S_T)} \le C, \qquad \left\|\frac{\partial v_\delta}{\partial t}\right\| \le C$$

Combining with the regularity results $[10]$ a similar argument to that in $[11]$, we can prove that the limit function of is a nonnegative non-trivial periodic solution of problem $(1.1) - (1.3)$.

Example 1.

Consider $\Omega = (-1,1)$ and $G(x) = x^3$. The value $y = 0$ is a singular value, but any neighboring value $y_1 = \delta$ is regular.

Then $\deg(G, \Omega, y_1) = \text{sign } G'\left(\delta^{\frac{1}{3}}\right) = 1$.

If $G(x) = x^2$then similarly

$\deg(G, \Omega, y_1) = \text{sign } G'(-\sqrt{\delta}) + \text{sign } G'(\sqrt{\delta}) = 0$ [14].

Example 2.

Consider $\Omega = \{x_1^2 + x_2^2 < 1)$ and $G(x_1, x_2) = (x_1^3 - x_1 x_2^2, x_2^3)$ The value $y = (0,0)$ is a singular value, but any neighboring value $y_1 = (0, \delta^3)$ with$\delta > 0$ is regular. The preimage $G^{-1}(y_1)$ consists of the three points $(0, \delta), (\delta, \delta)$ and $(-\delta, \delta)$. In the first point G^{-1}has a negative and in the last two points a positive determinant. Hence deg $(G, \Omega, y) = 1$ [14].

Example 3. Let $F(x) = -x^2 - x$ for $x \in \mathbb{R}$. The only solution of $x - F(x) = 0$ is $x_0 = 0$. Then $Id - F'(x_0) = 0$. The index of x_0 vanishes [14].

Conclusions :

In this work, we have considered a nonlinear diffusion Allen-Cahn equation with isolated Neumann boundary condition. For this problem, we have shown that there is an existence of nontrivial periodic solutions and by the topological degree theory.Finally we get a fixed point for equation (1.1), which represents the solution to the equation in infinite Banach space such that $v \in L^{\infty}$.

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