Al-Qadisiyah Journal of Pure Science

Volume 26 | Number 1 Article 9

1-7-2021

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Dr.Salim Dawood
College of Education, Dept. of Mathematics, University of Al -Mustinseryah, Iraq, dr_salim2015@yahoo.com

Ali Qassim Jabur
College of Education, Dept. of Mathematics, University of Al -Mustinseryah, Iraq, alsrayly842@gmail.com

Al-Qadisiyah Journal of Pure Science Vol.(26) Issue (1)(2021) pp Math. 112-123

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

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Authors Names

a.Dr.Salim Dawood

b.Ali Qassim Jabur

Article History

Received on: 4/12/2020
Revised on: 17/12/2020
Accepted on: 20/12/2020

Keywords:

Fuzzy soft set
fuzzy soft Hilbert space
fuzzy soft adjoint operator
fuzzy soft projection

DOI:https://doi.org/10.29350/jops.2021.26. 1.1251

ABSTRACT

In this paper, we study and explore a new type of fuzzy soft linear operator in fuzzy soft Hilbert space which is fuzzy soft projection operator and fuzzy soft perpendicular projection operator. More precisely, we present some definitions and characterizations of such operator in connection with fuzzy soft notions in Hilbert space. to the related on fuzzy soft projection operator have been given. In addition, we introduce the relation between fuzzy soft projection operator and fuzzy soft perpendicular projection operator and other types in fuzzy soft Hilbert space.

1.Introduction

The mathematical models related to real-world is too problematical and we cannot usually to find the exact solutions [4]. Then may be interested to use the concept of approximate approach to compute their solutions by using some mathematical tools in Hilbert space such as, fuzzy, soft, or fuzzy soft of set theory [7].

Thus, Zadeh [14] suggested in 1965 an expansion of the theory of the set, which is idea of fuzzy sets to cope with insecurity. A fuzzy set on a domain X is defined by membership function From X to [0,1]. Also In 1999, Molodtsov [9] introduce new types of sets is said to be soft set. The soft set is a mathematical instrument for uncertainty in modeling by associating a collection with a set of parameters, i.e., it is a parameterized family of Universal Set sub-sets. Many after that, researchers also presented new, expanded ideas based on soft sets where introduced by researchers, and examples were given for them and researched their characteristics, such as soft point [1], Soft normed Spaces [13], Soft Inner Product Spaces [3], Soft Hilbert space [12] and Projection operators on soft inner product spaces[11].

[a] College of Education, Dept. of Mathematics, University of Al -Mustinseryah, Iraq, E-Mail: dr_salim2015@yahoo.com

[b] College of Education, Dept. of Mathematics, University of Al -Mustinseryah, Iraq, E-Mail: alsrayly842@gmail.com

And many searchers, for example Maji [8] in 2001, made mixd between fuzzy set and soft set to get definition fuzzy soft set. This definition has been applied and some concepts are being given, like as Fuzzy soft point [10] and fuzzy soft normed spaces [2].

Currently, the fuzzy soft inner product spaces and fuzzy soft Hilbert space [6] were introduced by Faried et al. Additionally the Fuzzy soft linear operator [5]. Finally, they defined the fuzzy soft self-adjoint operator [4] and studied its properties.

The purpose of this work, submitted new type of fuzzy soft linear operator is called fuzzy soft projection operator and fuzzy soft perpendicular projection operator. Therefore, the main reason behind this study is that, played a good role to find an optimal solution depending on the orthogonality and closeness in for many concret-real problems in classical and fuzzy soft Hilbert space.

The outline of this paper is organized as: Section 2 concerns to basic needed concepts and definitions. Section 3 deals with fuzzy soft projection operator and fuzzy soft perpendicular projection operator and some related examples and theorems. Section 4 provides conclusions.

2.Basic concepts

Definition 2.1 [14]: If $\hat{A}$ be a **fuzzy set** over universe set $\mathcal{X}$ is a set characterized by a membership function $\mu_{\hat{A}}: \mathcal{X} \longrightarrow \mathfrak{T}$, where $\mathfrak{T} = [0,1]$ and $\hat{A}$ represented by an ordered pairs $\hat{A} = \{(x, \mu_{\hat{A}}(x)) | x \in \mathcal{X},\ \mu_{\hat{A}}(x) \in \mathfrak{T}\}$ or $\hat{A} = \left\{\frac{\mu_{\hat{A}}(x)}{x} : x \in \mathcal{X}\right\}$

$\mu_{\hat{A}}(x)$ is said to be degree of membership of x in $\hat{A}$

And $\mathfrak{T}^{\mathcal{X}} = \{\hat{A} : \hat{A} is\ a\ function\ from\ \mathcal{X}\ into\ \mathfrak{T}\}$

Definitions 2.2 [9]: If $\mathcal{X}$ be a universe set, and E be a set of parameters, $\mathcal{P}(\mathcal{X})$ the power set of $\mathcal{X}$ and $A \subseteq E$. Suppose that $\mathcal{G}$ is a mapping given by $\mathcal{G}: A \to \mathcal{P}(\mathcal{X})$, where $\mathcal{G}_A = \{\mathcal{G}(e) \in \mathcal{P}(\mathcal{X}): e \in A\}$.

The pair $(\mathcal{G}, A)\ or\ \mathcal{G}_A$ is called **soft set** over $\mathcal{X}$ with respect to A.

Definition 2.3 [8]: The soft set $(\mathcal{G}, A)$ is called **fuzzy soft set** $(\mathcal{FS} - set)$ over a universe set $\mathcal{X}$,whenever $\mathcal{G}$ is a mapping $\mathcal{G}: A \to \mathfrak{T}^{\mathcal{X}}$,and $\{\mathcal{G}(e) \in \mathfrak{T}^{\mathcal{X}}: e \in A \}$

The family of all $\mathcal{FS} - sets$, symbolized by $\mathcal{FSS}(\widetilde{\mathcal{X}})$

Definition 2.4 [10]: The $\mathcal{FS} - set$ $(\mathcal{G}, A) \in \mathcal{FSS}(\widetilde{\mathcal{X}})$ is called **fuzzy soft point** over $\mathcal{X}$, symbolized by $((x_{\mu_{\mathcal{G}(e)}}, A)$ or $\tilde{x}_{\mu_{\mathcal{G}(e)}}$), if $e \in A$ and $x \in \mathcal{X}$,

$$\mu_{\mathcal{G}(e)}(x) = \begin{cases} \lambda & , if\ x = x_o \in \mathcal{X}\ and\ \ e = e_0 \in A \\ 0 & , if\ x \in \mathcal{X} - \{x_0\}\ or\ \ e \in A - \{e_0\} \end{cases} \text{, where } \lambda \in (0,1]$$

Remark 2.5 [10]: $\mathcal{C}(A)$ is the family of all $\mathcal{FS} - Complex$ numbers and $\mathcal{R}(A)$ is also the family of all $\mathcal{FS} - Real$ numbers

Definition 2.6 [2]: A vector space $\mathcal{X}$ over a field $\mathcal{R}$ and E be the parameter set, let $\mathcal{R}$ be the set of all real numbers and $A \subseteq E$.The $\mathcal{FS}-set\ (\mathcal{G},A) \in \mathcal{FSS}(\widetilde{\mathcal{X}})$ is called a **fuzzy soft vector** $(\mathcal{FS}-vector)$ over $\mathcal{X}$, symbolized by $\tilde{x}_{\mu_{\mathcal{G}(e)}}$, if there is exactly one $e \in A$ such that $\mu_{\mathcal{G}(e)}(x) = \alpha$ for some $x \in \mathcal{X}$ and $\mu_{\mathcal{G}(e')}(x) = 0$ for all $e' \in A - \{e\}$ where $\alpha \in (0,1]$ is the value of the membership degree.

The set of all $\mathcal{FS}-vector$ over $\mathcal{X}$ is symbolized by $\mathcal{FSV}(\widetilde{\mathcal{X}})$.

Definition 2.7 [2]: If $\widetilde{\mathcal{X}}$ be $\mathcal{FS}-vector\ space$. Then, a mapping $\widetilde{\|.\|}: \widetilde{\mathcal{X}} \to \mathcal{R}(A)$ is called fuzzy soft norm on $\widetilde{\mathcal{X}}$ $(\mathcal{FSN})$ if $\widetilde{\|.\|}$ satisfies the following:

(1) $\left\| \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}} \right\| \geq \tilde{0}, \forall\ \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$,and $\left\| \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}} \right\| = \tilde{0} \Leftrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{\theta}$

(2) $\left\| \widetilde{\tilde{r} \cdot \tilde{x}_{\mu_{\mathcal{G}(e)}}} \right\| = |\tilde{r}| \left\| \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}} \right\|, \forall\ \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}, \forall\ \tilde{r} \in \mathcal{C}(A)$.

(3) $\left\| \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} + \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \right\| \leq \left\| \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}} \right\| + \left\| \widetilde{\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \right\|, \forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \widetilde{\mathcal{X}}$.

The $\mathcal{FS}-vector\ space$ with $\mathcal{FSN}$ $\widetilde{\|.\|}$ is called **fuzzy soft normed vector space** $(\mathcal{FSN}-space)$, and symbolized by $(\widetilde{\mathcal{X}}, \widetilde{\|\ .\|})$.

Definition 2.8 [6]: If $\widetilde{\mathcal{X}}$ be $\mathcal{FSV}-space$. Then, the mapping $\widetilde{\langle .,. \rangle} : \widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} \to (\mathcal{C}(A)\ or\ \mathcal{R}(A))$ is called fuzzy soft inner product on $\widetilde{\mathcal{X}}$ $(\mathcal{FS}\mathfrak{T})$ if $\widetilde{\langle .,. \rangle}$ satisfies:

(1) $\langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}} \rangle \geq \tilde{0}$, for all $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} \in \widetilde{\mathcal{X}}$ and

$\langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}} \rangle = \tilde{0} \Leftrightarrow \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} = \tilde{\theta}$

(2) $\langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle = \overline{\langle \widetilde{\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}, \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}} \rangle}$ $,\forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \widetilde{\mathcal{X}}$, where bar denotes complex conjugate of $\mathcal{FS}-complex\ number$.

(3) $\langle \widetilde{\tilde{\alpha}\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle = \tilde{\alpha} \langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle$, for all $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \widetilde{\mathcal{X}}$ and for all $\tilde{\alpha} \in \mathcal{C}(A)$.

(4) $\langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} + \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}, \tilde{z}_{\mu_{3_{\mathcal{G}(e_3)}}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{z}_{\mu_{3_{\mathcal{G}(e_3)}}}} \rangle + \langle \widetilde{\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}, \tilde{z}_{\mu_{3_{\mathcal{G}(e_3)}}}} \rangle$

For all $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}, \tilde{z}_{\mu_{3_{\mathcal{G}(e_3)}}} \in \widetilde{\mathcal{X}}$

The $\mathcal{FS}-vector\ space$ $\widetilde{\mathcal{X}}$ with $\mathcal{FS}\mathfrak{T}$ $\widetilde{\langle .,. \rangle}$ is called **fuzzy soft inner product space** $(\mathcal{FS}\mathfrak{T}-space)$, and symbolized by $(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle})$

Definition 2.9 [6]: If $(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle})$ be $\mathcal{FS}\mathfrak{T}-space$ and $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \widetilde{\mathcal{X}}$ then $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}$ is called **fuzzy soft orthogonal** to $\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}$ written $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} \perp \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}$, if $\langle \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle = \tilde{0}$

Theorem 2.10 [6]: Let $(\tilde{X}, \widetilde{\langle .,. \rangle})$ be $\mathcal{FS}\mathfrak{T}-space$ and let $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \widetilde{\mathcal{X}}$ if $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} \perp \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}$, then $\left\| \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}} + \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \right\|^2 = \left\| \widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}} \right\|^2 + \left\| \widetilde{\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \right\|^2$.

Definition 2.11 [7]: A sequence of $\mathcal{FS}-$ $vectors$ $\left\{\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\right\}$ in $\mathcal{FSN}-space$ $(\tilde{X},\widetilde{\|\cdot\|})$ is called **fuzzy soft convergent** and converges to $\tilde{x}^{o}_{\mu_{0_{\mathcal{G}(e_0)}}}$ if $\lim_{n\to\infty}\left\|\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\widetilde{-}\tilde{x}^{o}_{\mu_{0_{\mathcal{G}(e_0)}}}\right\|=\tilde{0}$ i.e. $\forall\tilde{\epsilon}>\tilde{0}$, $\exists n_o\in N$ such that $\left\|\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\widetilde{-}\tilde{x}^{o}_{\mu_{0_{\mathcal{G}(e_0)}}}\right\|<\tilde{\epsilon}$, $\forall n\geq n_o$.
It if denoted by $\lim_{n\to\infty}\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}=\tilde{x}^{o}_{\mu_{0_{\mathcal{G}(e_0)}}}$ or $\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\to\tilde{x}^{o}_{\mu_{0_{\mathcal{G}(e_0)}}}$ as $n\to\infty$

Definition 2.12 [7]: A sequence of $\mathcal{FS}-$ $vectors$ $\left\{\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\right\}$ in $\mathcal{FSN}-space$ $(\tilde{X},\widetilde{\|\cdot\|})$ is called **fuzzy soft Cauchy sequence**, if $\forall\tilde{\epsilon}>\tilde{0}$, $\exists n_o\in N$ such that

$$\left\|\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\widetilde{-}\tilde{x}^{m}_{\mu_{m_{\mathcal{G}(e_m)}}}\right\|<\tilde{\epsilon}\ ,\forall n,m\geq n_o\ ,n>m\ .$$

That is to say that $\left\|\tilde{x}^{n}_{\mu_{n_{\mathcal{G}(e_n)}}}\widetilde{-}\tilde{x}^{m}_{\mu_{m_{\mathcal{G}(e_m)}}}\right\|\to\tilde{0}$ as $n,m\to\infty$

Definition 2.13 [7]: The $\mathcal{FSN}-space$ $(\tilde{X},\widetilde{\|\cdot\|})$ is called **fuzzy soft complete** $(\mathcal{FS}-complete)$ if every $\mathcal{FS}-$ Cauchy sequence is $\mathcal{FS}-$ convergent sequence in it.

Definition 2.14 [6]: The $\mathcal{FS}\mathfrak{T}-space$ $(\widetilde{\mathcal{X}},\widetilde{\langle.,.\rangle})$, if it is $\mathcal{FS}-complete$ in the induced $\mathcal{FSN}$ $\left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|=\sqrt{\langle\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}},\tilde{x}_{\mu_{\mathcal{G}(e)}}}\rangle}$ is called fuzzy soft Hilbert space $(\mathcal{FSH}-space)$, and symbolized by $(\widetilde{\mathcal{H}},\widetilde{\langle.,.\rangle})$.

Definition 2.15 [5]: If $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-space$ and $\tilde{\mathcal{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ be a fuzzy soft operator $(\mathcal{FS}-operator)$ .Then $\tilde{\mathcal{T}}$ is called **fuzzy soft linear operators** $(\mathcal{FSL}-operator)$ if:

1. $\tilde{\mathcal{T}}\left(\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}+\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\right)=\tilde{\mathcal{T}}\left(\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\right)+\tilde{\mathcal{T}}\left(\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\right)$, $\forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}},\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\in\widetilde{\mathcal{H}}$

2. $\tilde{\mathcal{T}}\left(\tilde{\beta}\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\right)=\tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\right)$,$\forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\in\widetilde{\mathcal{H}}$ and $\tilde{\beta}\in\mathcal{C}(A)$

i.e. $\tilde{\mathcal{T}}\left(\tilde{\alpha}\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}+\tilde{\beta}\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\right)=\tilde{\alpha}\tilde{\mathcal{T}}\left(\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\right)+\tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\right)$, for all $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}},\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\in\widetilde{\mathcal{H}}$ and $\tilde{\alpha},\tilde{\beta}$ any fuzzy soft scalars

Definition 2.16 [5]: If $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-space$ and $\tilde{\mathcal{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ be $\mathcal{FS}-$operator is called **fuzzy soft bounded operator** $(\mathcal{FSB}-operator)$,if $\exists\ \tilde{m}\in\mathcal{R}(A)$ such that $\left\|\tilde{\mathcal{T}}\left(\widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}}\right)\right\|\leq\tilde{m}\left\|\widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}}\right\|$, for all $\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\in\widetilde{\mathcal{H}}$

Now, the collection of all $\mathcal{FS}-bounded\ linear$ operators denoted by $\tilde{B}(\widetilde{\mathcal{H}})$.

Example 2.17 [5]: The $\mathcal{FS}-operator$ $\tilde{I}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ defined by $\tilde{I}\left(\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\right)=\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}$, $\forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}}\in\widetilde{\mathcal{H}}$
And it is called fuzzy soft identity operator

Definition 2.18 [5]: If $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-space$ and $\tilde{\mathcal{T}}:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ be $\mathcal{FSB}-operator$, then The **fuzzy soft adjoint operator** $\tilde{\mathcal{T}}^*$ is defined by

$$\langle\widetilde{\tilde{\mathcal{T}}\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}},\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}}\rangle=\langle\widetilde{\tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}},\tilde{\mathcal{T}}^*\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}}\rangle\ ,\ \forall\ \tilde{x}_{\mu_{1_{\mathcal{G}(e_1)}}},\tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}\in\widetilde{\mathcal{H}}$$

Theorem 2.19 [5]: Let $\tilde{\mathcal{T}},\tilde{R} \in \tilde{B}(\widetilde{\mathcal{H}})$,where $\widetilde{\mathcal{H}}$ is $\mathcal{FSH}-space$ and $\tilde{\beta} \in \mathcal{C}(A)$, then $\tilde{\mathcal{T}}^{**} = \tilde{\mathcal{T}}$, $\left(\tilde{\beta}\tilde{\mathcal{T}}\right)^* = \bar{\tilde{\beta}}\tilde{\mathcal{T}}^*$, $\left(\tilde{\mathcal{T}}+\tilde{R}\right)^* = \tilde{\mathcal{T}}^* + \tilde{R}^*$ and $\left(\tilde{\mathcal{T}}\tilde{R}\right)^* = \tilde{R}^*\tilde{\mathcal{T}}^*$.

Theorem 2.20 [5]: if $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$,where $\widetilde{\mathcal{H}}$ is $\mathcal{FSH}-space$, then $\widetilde{\|\tilde{\mathcal{T}}^*\|} = \widetilde{\|\tilde{\mathcal{T}}\|}$ and $\widetilde{\|\tilde{\mathcal{T}}^*\tilde{\mathcal{T}}\|} = \widetilde{\|\tilde{\mathcal{T}}\|}^2$.

Definition 2.21 [4]: The $\mathcal{FS}-operator$ $\tilde{\mathcal{T}}$ of $\mathcal{FSH}-space$ $\widetilde{\mathcal{H}}$ is called **fuzzy soft self-adjoint** $(\mathcal{FS}-self\ adjoint\ operator)$ if $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^*$.

3.Main results

This section introduce the previous results as in [3-6] to case of the fuzzy soft projection operator and fuzzy soft perpendicular projection operator in fuzzy soft Hilbert space. In addition, this work is devoted to an extension for the original new results.

Definition 3.1: Let $\widetilde{\mathcal{X}}$ be $\mathcal{FSH}-space$ and let $\tilde{\mathcal{P}}$ be $\mathcal{FSL}-operator$ such that $\tilde{\mathcal{P}}:\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is called **fuzzy soft projection operator** on $\widetilde{\mathcal{H}}$ $(\mathcal{FS}-projection)$ if $\tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}$ i.e. $\tilde{\mathcal{P}}$ is an idempotent.

Example 3.1: fuzzy soft identity operator and zero operators are $\mathcal{FS}-$ projection

Theorem 3.1: Let $\widetilde{M_1}\ and\ \widetilde{M_2}$ be two subspace of $\mathcal{FSH}-space$ $\widetilde{\mathcal{H}}$ such that $\widetilde{\mathcal{H}} = \widetilde{M_1} \oplus \widetilde{M_2}$. Define $\tilde{\mathcal{P}}:\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ by $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$, then $\tilde{\mathcal{P}}$ is $\mathcal{FS}-projection.$

Proof: (1) Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{\mathcal{H}}$ and $\tilde{\alpha},\tilde{\beta} \in \mathcal{C}(A)$

$\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}$, $\tilde{y}_{\sigma_{\mathcal{F}(e)}} = \tilde{y}_{1_{\sigma_{1\mathcal{F}(e_1)}}} + \tilde{y}_{2_{\sigma_{2\mathcal{F}(e_2)}}}$

$\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}, \tilde{y}_{1_{\sigma_{1\mathcal{F}(e_1)}}} \in \widetilde{M_1}$ and $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}, \tilde{y}_{2_{\sigma_{2\mathcal{F}(e_2)}}} \in \widetilde{M_2}$

$$\tilde{\alpha}\tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{\beta}\tilde{y}_{\sigma_{\mathcal{F}(e)}} = \tilde{\alpha}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\right) + \tilde{\beta}\left(\tilde{y}_{1_{\sigma_{1\mathcal{F}(e_1)}}} + \tilde{y}_{2_{\sigma_{2\mathcal{F}(e_2)}}}\right)$$

$$= \left(\tilde{\alpha}\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{\beta}\tilde{y}_{1_{\sigma_{1\mathcal{F}(e_1)}}}\right) + \left(\tilde{\alpha}\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}} + \tilde{\beta}\tilde{y}_{2_{\sigma_{2\mathcal{F}(e_2)}}}\right)$$

$$\tilde{\mathcal{P}}\left(\tilde{\alpha}\tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{\beta}\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{\alpha}\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{\beta}\tilde{y}_{1_{\sigma_{1\mathcal{F}(e_1)}}} = \tilde{\alpha}\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \tilde{\beta}\tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)$$

$\Longrightarrow$ $\tilde{\mathcal{P}}$ is $\mathcal{FSL}-operator$

(2) Let $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}} \Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}$, where $\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} \in \widetilde{M_1}$ and $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}} \in \widetilde{M_2}$

$$\tilde{\mathcal{P}}^2\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{0}\right) = \tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$$

$= \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \Longrightarrow \tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}$. Therefor $\tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$

Theorem 3.2: $\mathcal{FSL}-operator\ \tilde{\mathcal{P}}$ on $\widetilde{\mathcal{H}}$, then $\mathcal{FS}-projection$ on $\widetilde{\mathcal{H}}=\widetilde{M_1}\oplus\widetilde{M_2}$ subspace **if and only if** it is idempotent .

Proof: Let $\widetilde{\mathcal{H}}=\widetilde{M_1}\oplus\widetilde{M_2}$ and $\tilde{\mathcal{P}}$ be $\mathcal{FS}-projection$. To prove $\tilde{\mathcal{P}}^2=\tilde{\mathcal{P}}$

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{\mathcal{H}}\implies\tilde{x}_{\mu_{\mathcal{G}(e)}}=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}+\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}$,where $\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\in\widetilde{M_1}$ and $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\in\widetilde{M_2}\implies$ $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$

$\tilde{\mathcal{P}}^2\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right)=\tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\right)=\tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}+\tilde{0}\right)=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$

$=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\implies\tilde{\mathcal{P}}^2=\tilde{\mathcal{P}}$

$\implies$ Therefor $\tilde{\mathcal{P}}$ is idempotent

Conversely, let $\tilde{\mathcal{P}}^2=\tilde{\mathcal{P}}$ to prove $\tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$

Let $\widetilde{M_1}=\left\{\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{\mathcal{H}}:\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{x}_{\mu_{\mathcal{G}(e)}}\right\}$,and

$\widetilde{M_2}=\left\{\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{\mathcal{H}}:\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{0}\right\}$

$\implies\widetilde{M_1},\widetilde{M_2}$ are subspace of $\widetilde{\mathcal{H}}$. To prove $\widetilde{\mathcal{H}}=\widetilde{M_1}\oplus\widetilde{M_2}$

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{\mathcal{H}}\implies\tilde{x}_{\mu_{\mathcal{G}(e)}}=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)+\left[\tilde{x}_{\mu_{\mathcal{G}(e)}}-\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right]$

Put $\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$ and $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}=\tilde{x}_{\mu_{\mathcal{G}(e)}}-\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$

$\tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\right)=\tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right)=\tilde{\mathcal{P}}^2\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$, where $\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\in\widetilde{M_1}$

$\tilde{\mathcal{P}}\left(\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\right)=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}-\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right)=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)-\tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right)$

$=\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)-\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{0}$, where $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\in\widetilde{M_2}$

$\tilde{x}_{\mu_{\mathcal{G}(e)}}=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}+\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\implies\widetilde{\mathcal{H}}=\widetilde{M_1}+\widetilde{M_2}$

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{M_1}\cap\widetilde{M_2}\implies\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{M_1}$ and $\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{M_2}$

$\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{0}$ and $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)=\tilde{x}_{\mu_{\mathcal{G}(e)}}\implies\tilde{x}_{\mu_{\mathcal{G}(e)}}=\tilde{0}$

$\implies\widetilde{M_1}\cap\widetilde{M_2}=\{\tilde{0}\}\implies\widetilde{\mathcal{H}}=\widetilde{M_1}\oplus\widetilde{M_2}$

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}\in\widetilde{\mathcal{H}}\implies\tilde{x}_{\mu_{\mathcal{G}(e)}}=\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}+\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}$, where $\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\in\widetilde{M_1}$ and $\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\in\widetilde{M_2}$

$\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}} + \tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}\right) + \tilde{\mathcal{P}}\left(\tilde{x}_{2_{\mu_{2\mathcal{G}(e_2)}}}\right) = \tilde{x}_{1_{\mu_{1\mathcal{G}(e_1)}}}$

Remark 3.1: From above theorem, we have the $\mathcal{FS}-projection\ \tilde{\mathcal{P}}$ on $\widetilde{\mathcal{H}}$, determines a pair of subspaces of $\widetilde{M_1}$, $\widetilde{M_2}$ such that $\widetilde{\mathcal{H}} = \widetilde{M_1} \oplus \widetilde{M_2}$, where $\widetilde{M_1}$ is the range of $\tilde{\mathcal{P}}$ i.e. $\widetilde{M_1} = \left\{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}} :\ \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}}\right\}$ and $\widetilde{M_2}$ is the kernel of $\tilde{\mathcal{P}}$ i.e. $\widetilde{M_2} = \left\{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0} : \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}}\right\}$.

Theorem 3.3: Let $\tilde{\mathcal{P}}$ be $\mathcal{FS}-projection$ on $\mathcal{FSH}-space\ \widetilde{\mathcal{H}}$. Then the range of $\tilde{\mathcal{P}}$ is the set of all $\mathcal{FS}-vectors$ which are fixed under $\tilde{\mathcal{P}}$ i.e.

$$Ran(\tilde{\mathcal{P}}) = \left\{\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}} : \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}\right\}$$

Proof: Suppose that $\tilde{A} = \left\{\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}} : \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}\right\}$.

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in Ran(\tilde{\mathcal{P}}) \Longrightarrow$ there exists $\tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{\mathcal{H}}$ such that $\tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$\tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \Longrightarrow \tilde{\mathcal{P}}^2\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$

$\Longrightarrow \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$, because $\tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}$

But $\tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}} \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$\Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \tilde{A} \Longrightarrow Ran(\tilde{\mathcal{P}}) \subseteq \tilde{A}$ .... (1)

But $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} \in Ran(\tilde{\mathcal{P}}) \Longrightarrow \tilde{A} \subseteq Ran(\tilde{\mathcal{P}})$ .... (2)

From (1) and (2), we have $(\tilde{\mathcal{P}}) = \tilde{A}$.

Theorem 3.4: Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-space$ and let $\tilde{\mathcal{P}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be $\mathcal{FSL}-operator$. Then $\tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$ on $\widetilde{\mathcal{H}}$ **if and only if** $\tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FS} -$ projection on $\widetilde{\mathcal{H}}$

Proof: Suppose $\tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$ on $\widetilde{\mathcal{H}}$. To prove $\tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FS} -$ projection

1) To prove $\tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FSL}-operator$. Let $\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{\mathcal{H}}$ and $\tilde{\alpha}, \tilde{\beta} \in C(A)$

$$(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{\alpha}\tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{\beta}\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{\alpha}\left(\tilde{I}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) - \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right) + \tilde{\beta}\left(\tilde{I}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) - \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)\right)$$

$$= \tilde{\alpha}(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \tilde{\beta}(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)$$

2) To prove $\left(\tilde{I} - \tilde{\mathcal{P}}\right)^2 = \left(\tilde{I} - \tilde{\mathcal{P}}\right)$

$\left(\tilde{I} - \tilde{\mathcal{P}}\right)^2 = \tilde{I} - \tilde{\mathcal{P}} - \tilde{\mathcal{P}} + \tilde{\mathcal{P}}^2 = \tilde{I} - \tilde{\mathcal{P}}^2 - \tilde{\mathcal{P}}^2 + \tilde{\mathcal{P}}^2$, because $\tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}$ $\Longrightarrow \tilde{I} - \tilde{\mathcal{P}}^2 = \tilde{I} - \tilde{\mathcal{P}} \Longrightarrow \tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$ on $\widetilde{\mathcal{H}}$

Conversely, let $\tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FS}-projection$ on $\widetilde{\mathcal{H}} \Longrightarrow \left(\tilde{I} - \tilde{\mathcal{P}}\right)^2 = \left(\tilde{I} - \tilde{\mathcal{P}}\right)$

$\Longrightarrow \tilde{I} - \tilde{\mathcal{P}} - \tilde{\mathcal{P}} + \tilde{\mathcal{P}}^2 = \tilde{I} - \tilde{\mathcal{P}} \Longrightarrow \tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}} \Longrightarrow \tilde{\mathcal{P}}$ is $\mathcal{FS} - projection$ on $\widetilde{\mathcal{H}}$.

Definition: Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH} - space$ and let $\tilde{\mathcal{P}} \in \tilde{B}(\widetilde{\mathcal{H}})$, we say that $\tilde{\mathcal{P}}$ is **a fuzzy soft** on $\widetilde{\mathcal{H}}$ $(\mathcal{FS} - perpendicular\ projection)$ if $\tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}^* = \tilde{\mathcal{P}}$.

Example 3.2: fuzzy soft identity operator and zero operators are $\mathcal{FS} -$ perpendicular projection

Theorem 3.5 **:** Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH} - space$, and let $\tilde{\mathcal{P}}$ is $\mathcal{FS} -$ projection on $\widetilde{\mathcal{H}}$. Then $\tilde{\mathcal{P}}$ is $\mathcal{FS} - perpendicular\ projection$ on $\widetilde{\mathcal{H}}$ **if and only if** the range and kernel of $\tilde{\mathcal{P}}$ are orthogonal .

Proof: Let $\widetilde{M}$ be the range of $\tilde{\mathcal{P}}$ and $\widetilde{N}$ is the kernel of $\tilde{\mathcal{P}}$ i.e.

$\widetilde{M} = \left\{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}} : \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}}\right\}$ and $\widetilde{N} = \left\{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0} : \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{H}}\right\}$

$\Longrightarrow \widetilde{\mathcal{H}} = \widetilde{M} \oplus \widetilde{N}$

Suppose that $\tilde{\mathcal{P}}$ is $\mathcal{FS} - perpendicular\ projection$ on $\widetilde{\mathcal{H}}$ .To prove $\widetilde{M} \perp \widetilde{N}$

Let $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M},\ \tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{N} \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}} , \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{0}$

$\langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle = \langle \widetilde{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right), \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{\mathcal{P}}^*\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)} \rangle = \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)} \rangle = \tilde{0} \Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} \perp \tilde{y}_{\sigma_{\mathcal{F}(e)}} \Longrightarrow \widetilde{M} \perp \widetilde{N}$

Conversely, suppose that $\widetilde{M} \perp \widetilde{N}$ .To prove $\tilde{\mathcal{P}}$ is $\mathcal{FS} -$ perpendicular projection

Let $\tilde{z}_{\delta_{\mathcal{K}(e)}} \in \widetilde{\mathcal{H}}$, then $\tilde{z}_{\delta_{\mathcal{K}(e)}}$can be uniquely written as $\tilde{z}_{\delta_{\mathcal{K}(e)}} = \tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{y}_{\sigma_{\mathcal{F}(e)}}$ where $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$, $\tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{N} \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$$\langle \widetilde{\tilde{\mathcal{P}}\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right), \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle$$

$$= \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle + \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle$$

Since $\langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle = \tilde{0} \Longrightarrow \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle$ .... (3)

$$\langle \widetilde{\tilde{\mathcal{P}}^*\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right), \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle = \langle \widetilde{\tilde{z}_{\delta_{\mathcal{K}(e)}}, \tilde{\mathcal{P}}\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right)} \rangle = \langle \widetilde{\tilde{z}_{\delta_{\mathcal{K}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle =$$

$$\langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{y}_{\sigma_{\mathcal{F}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle + \langle \widetilde{\tilde{y}_{\sigma_{\mathcal{F}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle$$

Since $\langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{y}_{\sigma_{\mathcal{F}(e)}}} \rangle = \tilde{0} \Longrightarrow \langle \widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}, \tilde{x}_{\mu_{\mathcal{G}(e)}}} \rangle$ .... (4)

From (3) and (4), we have $\langle \widetilde{\tilde{\mathcal{P}}\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right), \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle = \langle \widetilde{\tilde{\mathcal{P}}^*\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right), \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle$

$\Longrightarrow \langle \widetilde{(\tilde{\mathcal{P}} - \tilde{\mathcal{P}}^*)\left(\tilde{z}_{\delta_{\mathcal{K}(e)}}\right), \tilde{z}_{\delta_{\mathcal{K}(e)}}} \rangle = \tilde{0}$, for all $\tilde{z}_{\delta_{\mathcal{K}(e)}} \in \widetilde{\mathcal{H}}$

$\Longrightarrow \tilde{\mathcal{P}} - \tilde{\mathcal{P}}^* = \tilde{0} \Longrightarrow \tilde{\mathcal{P}}^* = \tilde{\mathcal{P}}$

$\Longrightarrow \tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{\mathcal{H}}$.

Remarks 3.2:

(1) From above theorem if $\widetilde{M} \perp \widetilde{N}$, we have $\widetilde{N} = \widetilde{M}^{\perp}$ and hence $\widetilde{\mathcal{H}} = \widetilde{M} \oplus \widetilde{M}^{\perp}$

(2) If $\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{\mathcal{H}}$, then $\widetilde{M}$ is closed subspace of $\widetilde{\mathcal{H}}$. If $\widetilde{N}$ is kernel of $\tilde{\mathcal{P}}$, then is also closed subspace of $\widetilde{\mathcal{H}}$,and $\widetilde{N} = \widetilde{M}^{\perp}$. Further if $\widetilde{M}$ is closed subspace of $\widetilde{\mathcal{H}}$,then $\widetilde{\mathcal{H}} = \widetilde{M} \oplus \widetilde{M}^{\perp}$. Therefor there exists $\mathcal{FS}-projection$ $\tilde{\mathcal{P}}$ on $\widetilde{\mathcal{H}}$ with range $\widetilde{M}$. This $\mathcal{FS}-projection$ $\tilde{\mathcal{P}}$ is defined by $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$, where $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$ and $\tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{M}^{\perp}$.

Theorem 3.6: Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-space$ and let $\tilde{\mathcal{P}} \in \tilde{B}(\widetilde{\mathcal{H}})$. Then $\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on closed subspace $\widetilde{M}$ of $\widetilde{\mathcal{H}}$ **if and only if** $\tilde{I}-\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{M}^{\perp}$.

Proof: Suppose $\tilde{\mathcal{P}}$ is $\mathcal{FS}-$ perpendicular projection on closed subspace $\widetilde{M}$ of $\widetilde{\mathcal{H}}$ $\Longrightarrow$ $\tilde{\mathcal{P}}^{*} = \tilde{\mathcal{P}}$, $\tilde{\mathcal{P}}^{2} = \tilde{\mathcal{P}}$

1) $\left(\tilde{I}-\tilde{\mathcal{P}}\right)^{*} = \tilde{I}^{*} - \tilde{\mathcal{P}}^{*} = \tilde{I} - \tilde{\mathcal{P}}$

2) $\left(\tilde{I}-\tilde{\mathcal{P}}\right)^{2} = \left(\tilde{I}-\tilde{\mathcal{P}}\right)\left(\tilde{I}-\tilde{\mathcal{P}}\right) = \tilde{I} - \tilde{\mathcal{P}} - \tilde{\mathcal{P}} + \tilde{\mathcal{P}}^{2} = \tilde{I} - \tilde{\mathcal{P}} - \tilde{\mathcal{P}} + \tilde{\mathcal{P}} = \tilde{I} - \tilde{\mathcal{P}}$

$\Longrightarrow$ $\tilde{I}-\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{\mathcal{H}}$

Now, we show that if $\widetilde{M}$ is the range of $\tilde{\mathcal{P}}$, then $\widetilde{M}^{\perp}$ is the range of $\tilde{I}-\tilde{\mathcal{P}}$

Let $\widetilde{N}$ be the range $\tilde{I}-\tilde{\mathcal{P}}$, then $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{N}$ $\Longrightarrow \left(\tilde{I}-\tilde{\mathcal{P}}\right)\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$\Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{\mathcal{P}}(\tilde{x}_{\mu_{\mathcal{G}(e)}}) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$ $\Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0}$

$\Longrightarrow$ $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \ker(\tilde{\mathcal{P}})$ $\Longrightarrow$ $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}^{\perp} \Longrightarrow$ $\widetilde{N} \subseteq \widetilde{M}^{\perp}$ ....(5)

Again $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}^{\perp} \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0}$ $\Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{\mathcal{P}}(\tilde{x}_{\mu_{\mathcal{G}(e)}}) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$\Longrightarrow \left(\tilde{I}-\tilde{\mathcal{P}}\right)\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

$\Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{N}$ $\Longrightarrow \widetilde{M}^{\perp} \subseteq \widetilde{N}$ ....(6)

From (5) and (6) we get $\widetilde{N} = \widetilde{M}^{\perp}$

Hence $\tilde{I}-\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{M}^{\perp}$

Conversely, suppose $\tilde{I}-\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{M}^{\perp}$

To prove $\tilde{\mathcal{P}}$ is $\mathcal{FS}-perpendicular\ projection$ on $\widetilde{M}$,we have

$\tilde{I}-\left(\tilde{I}-\tilde{\mathcal{P}}\right)$ is $\mathcal{FS}-perpendicular\ projection$ on $\left(\widetilde{M}^{\perp}\right)^{\perp}$, by hypothesis

$\Longrightarrow \tilde{\mathcal{P}}$ is $\mathcal{FS} - perpendicular\ projection$ on $\left(\widetilde{M}^{\perp}\right)^{\perp} \Longrightarrow \tilde{\mathcal{P}}$ is $\mathcal{FS} - perpendicular\ projection$ on $\widetilde{M}$

Theorem 3.7: Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH} - space$ and let $\tilde{\mathcal{P}}$ be $\mathcal{FS}$ − perpendicular projection on the closed subspace $\widetilde{M}$ of $\widetilde{\mathcal{H}}$. Then $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$ **if and only if** $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$.

Proof: Suppose $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$. To prove $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

Let $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{y}_{\sigma_{\mathcal{F}(e)}}$, then we must prove $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{y}_{\sigma_{\mathcal{F}(e)}}$

We have $\tilde{\mathcal{P}}\left(\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right) = \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) \Longrightarrow \tilde{\mathcal{P}}^2\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right)$

$\Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{y}_{\sigma_{\mathcal{F}(e)}}\right) = \tilde{0}$

$\Longrightarrow\ \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \ker(\tilde{\mathcal{P}}) \ \Longrightarrow\ \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{M}^{\perp}$

$\Longrightarrow\ \tilde{z}_{\delta_{\mathcal{K}(e)}} = \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{y}_{\sigma_{\mathcal{F}(e)}}$, where $\tilde{z}_{\delta_{\mathcal{K}(e)}} \in \widetilde{M}^{\perp} \Longrightarrow\ \tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{y}_{\sigma_{\mathcal{F}(e)}} + \tilde{z}_{\delta_{\mathcal{K}(e)}}$

Since $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{y}_{\sigma_{\mathcal{F}(e)}} \Longrightarrow \tilde{y}_{\sigma_{\mathcal{F}(e)}}$ in the range of $\tilde{\mathcal{P}}$ i.e. $\tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{M}$

Thus we have $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{y}_{\sigma_{\mathcal{F}(e)}} + \tilde{z}_{\delta_{\mathcal{K}(e)}}$, where $\tilde{y}_{\sigma_{\mathcal{F}(e)}} \in \widetilde{M}, \tilde{z}_{\delta_{\mathcal{K}(e)}} \in \widetilde{M}^{\perp}$

But $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$, so we can write $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{x}_{\mu_{\mathcal{G}(e)}} + \tilde{0}$,where $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$, $\tilde{0} \in \widetilde{M}^{\perp}$

Since $\widetilde{\mathcal{H}} = \widetilde{M} \oplus \widetilde{M}^{\perp}$. Therefor we must have $\tilde{z}_{\delta_{\mathcal{K}(e)}} = \tilde{0}$, $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{y}_{\sigma_{\mathcal{F}(e)}}$

Conversely, suppose that $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$. To prove $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$

Since $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \in \widetilde{M} \Longrightarrow \tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$

Theorem 3.8: Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH} - space$ and let $\tilde{\mathcal{P}}$ be $\mathcal{FS}$ − perpendicular projection on the closed subspace $\widetilde{M}$ of $\widetilde{\mathcal{H}}$. Then $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$, where $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$ **if and only if** $\left\|\widetilde{\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\| = \left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|$.

Proof: Suppose $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$, where $\tilde{x}_{\mu_{\mathcal{G}(e)}} \in \widetilde{M}$.

$\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$,then $\left\|\tilde{\mathcal{P}}\left(\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right)\right\| = \left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|$

Conversely, suppose $\left\|\tilde{\mathcal{P}}\left(\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right)\right\| = \left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|$. To prove $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$

Since $\tilde{x}_{\mu_{\mathcal{G}(e)}} = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \tilde{x}_{\mu_{\mathcal{G}(e)}} - \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \left(\tilde{I} - \tilde{\mathcal{P}}\right)\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$

$\Longrightarrow\ \left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|^2 = \left\|\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \widetilde{\left(\tilde{I} - \tilde{\mathcal{P}}\right)}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right\|^2$ .... (7)

Now, $\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \in \widetilde{M}$. Also $\tilde{\mathcal{P}}$ is $\mathcal{FS} -$ perpendicular projection on $\widetilde{M}$

$\Longrightarrow \tilde{I} - \tilde{\mathcal{P}}$ is $\mathcal{FS} - perpendicular\ projection$ on $\widetilde{M}^{\perp}$ by theorem (3.6)

Therefor $(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) \in \widetilde{M}^{\perp} \Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$ and $(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)$ are orthogonal

Now, by theorem (2.10) we have

$$\left\|\tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) + \widetilde{(\tilde{I} - \tilde{\mathcal{P}})}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)\right\|^2 = \left\|\tilde{\mathcal{P}}\widetilde{\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\|^2 + \left\|\widetilde{(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\|^2 \ldots\ldots (8)$$

From (7) and (8), we get $\left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\|^2 = \left\|\tilde{\mathcal{P}}\widetilde{\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\|^2 + \left\|\widetilde{(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\|^2$

Since $\left\|\tilde{\mathcal{P}}\widetilde{\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\| = \left\|\widetilde{\tilde{x}_{\mu_{\mathcal{G}(e)}}}\right\| \Longrightarrow \left\|\widetilde{(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\|^2 = \tilde{0}$

$$\Longrightarrow \left\|\widetilde{(\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right)}\right\| = \tilde{0} \Longrightarrow (\tilde{I} - \tilde{\mathcal{P}})\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0}$$

$$\Longrightarrow \tilde{I}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) - \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{0}$$

$$\Longrightarrow \tilde{\mathcal{P}}\left(\tilde{x}_{\mu_{\mathcal{G}(e)}}\right) = \tilde{x}_{\mu_{\mathcal{G}(e)}}$$

4.Conclusion

The combination of fuzzy and soft sets gives us original results that are more extended, generalized and precised. Few researchers have studied some of the principles of these general extensions such as fuzzy soft normed spaces, fuzzy soft Hilbert space and fuzzy soft linear operators. In our study, a special type of fuzzy soft linear operators, which is the fuzzy soft projection operator and fuzzy soft perpendicular projection operator has been introduced and explored

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