On n-Convergence of Filters

Raad Aziz Hussain Al-Abdulla

College of Science, Department of Mathematics, University of Al-Qadisiyah, Iraq, raad.hussain@qu.edu.iq

Follow this and additional works at: https://qjps.researchcommons.org/home

Part of the Mathematics Commons

Recommended Citation
Available at: https://qjps.researchcommons.org/home/vol26/iss1/10

This Article is brought to you for free and open access by Al-Qadisiyah Journal of Pure Science. It has been accepted for inclusion in Al-Qadisiyah Journal of Pure Science by an authorized editor of Al-Qadisiyah Journal of Pure Science. For more information, please contact bassam.alfarhani@qu.edu.iq.
On n-Convergence of Filters

1. Introduction

The main idea of this analysis is to define the generalization of some forms of topological concepts. (Leven, 1963 [7]) define semi open(s-open) set, semi closed (s-closed) set and study their characterization. He has defined a set A called (s-open)set in topological space $X$ if find an open set $O : O \subseteq A \subseteq \overline{O}$ where $\overline{O}$ denotes by the closure of $O$ in $X$, the complement semi-open (s-open) set named semi-closed (s-closed) set. (Maheswari and Tapi, 1978 [8]) defined the feebly closed $(f$-closed), feebly open $(f$-open) set. A set $A \subseteq X$ where $(X, t)$ topological space called feebly open $(f$-open) set in $X$ if there is a $V \subseteq \mathfrak{t}$ such as $V \subseteq A \subseteq \overline{V}^s$. As known, convergence is a basic concept in analysis. General topology contains important type of convergence, that reverting to Cartan, 1937 [3]; Willard, 1970 [15]; and Bourbaki, 1989 [1] depends on the filter's concept. Also, through [11] and [14], we can get some of the concept that were dealt with in this research. Through the previous concepts we were able to get some theories, properties and important relations among them. We also in this work found some examples of some reverse cases that are generally incorrect.
**Notation:** Whenever in the research the symbol (tp-s) is mentioned it, will be used to denote the topological space, (s-open) to semi open set, (s-closed) to semi closed set, (f-open) to feebly open set and (f-closed) to feebly closed set.

### 2. Preliminaries

**Definition (2.1):** [7] Assume that \((X, \tau)\) is a tp-s \& \(A \subseteq X\). Then \(A\) is named s-open in \(X\) if there exists \(w \in \tau : w \subseteq A \subseteq \overline{w}\). Or equivalent [7], A called s-open in \(X\) if \(A \subseteq \overline{A}\), equivalent \(\overline{A} = \overline{\overline{A}}\), the complement of s-open is named s-closed. Then \(A\) is called s-closed in \(X\) if there is a closed set \(B\) so that \(B^s \subseteq A \subseteq B\), or equivalent [4], \(A\) is s-closed in \(X\) if \(\overline{A}^s \subseteq A\), equivalent \(A^s = \overline{A}^s\). It is clear that when \(A\) open set then it is s-open and when \(A\) closed set then it is s-closed.

**Proposition (2.2):** [7] Let \(\{A_{\lambda}\}_{\lambda \in \Lambda}\) be a collection of s-open in a tp-s \(X\) then \(\bigcup_{\lambda \in \Lambda} A_{\lambda}\) is s-open.

**Remark (2.3):** [7] Let \(A, B\) s-open in tp-s \(X\), then \(A \cap B\) does not need to be s-open in \(X\).

**Example (2.4):** Assume that \(X = \{a, b, c\}, \tau = \{\{a\}, \{b\}, \{a, b\}, X, \emptyset\}\) then each of \(\{a, c\}, \{b, c\}\) are s-open, however \(\{a, c\} \cap \{b, c\} = \{c\}\) not s-open.

**Lemma (2.5):** [5] For every singleton \(\{x\}\) in a tp-s \(X\) is either \(\overline{\{x\}}^s = \emptyset\) or \(\{x\} \subseteq \overline{\{x\}}^s\).

**Proposition (2.6):** Let \(\overline{A}^s = \emptyset\) in a tp-s \(X\). Then \(A\) is s-closed.

**Definition (2.7):** [6] Assume that \(X\) is a tp-s \& \(B \subseteq X\). An s-neighborhood of \(B\) is any subset of \(X\) that includes an s-open set that contains \(B\) the s-neighborhood of a subset \(\{x\}\) is named s-neighborhood of point \(x\).

We always denoted by \(N_s(x)\) to the family of all s-neighborhoods of \(x\).

**Definition (2.8):** [8] Let \((X, \tau)\) be tp-s \& \(A \subseteq X\), then \(\bigcap \{B; B\text{ s-closed subset of } X\text{ and } A \subseteq B\}\) named (s-closure) of \(A\).

And \(\cup \{U; U\text{ s-open subset of } X\text{ and } A \subseteq U\}\) named (s-interior) of \(A\) and are shortened by \(\overline{A}^s\), \(A^s\) respectively.

**Proposition (2.9):** Let \(X\) be a (tp-s) \& \(A, B\) are subset of \(X\) with \(B\) s-open set. If \(x \in B\) and \(B \cap A = \emptyset\), thus \(x \notin \overline{A}^s\).

**Proposition (2.10):** Assume that \((X, \tau)\) is a tp-s \& \(A, B\) subsets of \(X\) then:

(i) \(A\) is s-closed if and only if \(A = \overline{A}^s\) [15] (ii) if \(A \subseteq B\), then \(\overline{A}^s \subseteq B^s\) [15] (iii) \(A^{cs} = A^{c-s}\) [8].

**Definition (2.11):** [5] A set \(A\) in a tp-s \(X\) called f-open in \(X\) where is an open set \(V\) so that \(V \subseteq A \subseteq \overline{V}^s\), or equivalent, A set \(A\) named f-open in \(X\) if and only if \(A \subseteq \overline{A}^s\), however \(A^c\) called f-closed that \(\overline{A} \subseteq A\). It is clear that each open is f-open set, each closed is f-closed set.
Example (2.12): Assume that $X = \{s, z, q, r, w\}$ and $\tau = \{\{s\}, \{q, r\}, \emptyset, \{z, q, s\}, X\}$ then, $f$-open sets are $\{X, \{q, s\}, \{z, q, s\}, \emptyset, \{z, q, r, s\}\}$, $f$-closed sets are $\{X, \{q, r\}, \{z, r, w\}, \emptyset, \{r, w\}, \{w\}\}$.

Take $A = \{q, r, s, z\}$ is $f$-open, but $A \nsubseteq \tau$ it set. Also $A^c = \{w\}$ is $f$-closed, but it is not closed.

Proposition (2.13): [12]. Let $X$ be $tp$-s then the union of all $f$-open sets in $X$ is also $f$-open. And the finite intersection of $f$-open sets in $X$ is $f$-open set.

Definition (2.14): [13]. Assume that $X$ is a $tp$-s & $B \subseteq X$. Any subset of $X$ that includes an $f$-open set containing $B$ is named $f$-neighborhood of $B$, the $f$-neighborhood of a subset $\{x\}$ is often named $f$-neighborhood of point $x$.

The family of all $f$-neighborhoods of $x$ always denoted by $N_f(x)$.

Definition (2.15): [2]. Assume that $X$ is a $tp$-s & $A \subseteq X$. Therefore the intersection of all $f$-closed of $X$ which includes $A$ is called $f$-closure of $A$ and shortened by $A^f$, that means $A^f = \cap \{F : F \text{ is } f\text{-closed in } X \text{ such that } A \subseteq F\}$. However $f$-interior of $A$ shortened by $A^{\circ f}$, that means $A^{\circ f} = \cup \{G : G \text{ is } f\text{-open in } X \text{ such that } G \subseteq A\}$.

Example (2.16): Let $(X, \tau)$ be a $tp$-s and $a \in X, \tau = \{A \subseteq X : a \in A\} \cup \{\emptyset\}$. Then the $f$-open and open sets in $X$ are equivalent. Let $D \subseteq X$. Then $D^f = \{X \subseteq D : a \in D\}$ and $D^{\circ f} = \{\emptyset : a \notin D\}$.

Proposition (2.17): [9]. Let $X$ be a $tp$-s & $A, B$ subset of $X$ where $B$ is $f$-open, If $x \in B$ and $A \cap B = \emptyset$ then $x \notin A^f$.

Proposition (2.18): [9]. Assume that $(X, \tau)$ is a $tp$-s & $A, B$ subsets of $X$ then: $A$ is $f$-closed if and only if $A = A^f$.

Definition (2.19): The intersection of every feebly open subset of $tp$-s $X$ contained a set $A$ is named feebly kernel of $A$ and shortened by $(f \ker(A))$. Means that: $f \ker(A) = \cap \{U : U \text{ } f\text{-open and } A \subseteq U\}$.

Lemma (2.20): A subset $A$ of a $tp$-s $X$ is $n$-open if and only if $A^{s} \subseteq f\ker (A)$.

Proof: Let $A$ be $n$-open in $X$, then $A^{s} \subseteq U$ when $A \subseteq U$ and $U$ is $f$-open in $X$, this emplace $A^{s} \subseteq \cap \{U : A \subseteq U \text{ and } U \in f\text{-open}(X)\} = f \ker (A)$. Conversely, assume that $A^{s} \subseteq f \ker (A)$. Then $A^{s} \subseteq \cap \{U : A \subseteq U \text{ and } U \in f\text{-open in } (X)\}$. Therefore $A^{s} \subseteq U$ for all $f$-open set $U$ in $X$.

Definition (2.21): Assume that $X$ is a $tp$-s then a subset $A$ in a space $X$ is named narrow open (n-open) set in a space $X$ if $A \subseteq U$ where $U$ feebly open set in $X$ then $A^{s} \subseteq U$. The complement of narrow open is named narrow closed (n-closed) it is as follows $U \subseteq A^{s}$, where $U$ $f$-closed set in $X$.

Remark (2.22): Each $s$-closed is $n$-open. However the converse is generally not true.

Proof: Suppose $A$ is $s$-closed set in a $tp$-s $X$ and $A \subseteq U$ such that $U$ be $f$-open set. We have $A$ is $s$-closed set, then $A = A^{s}$ and $A = A^{s} \subseteq U$. Therefore $A$ is $n$-open set.
Example (2.23): Assume that \((X, t)\) is a tp-s when \(X = \{z, q, r, s, w\}, \ t = \{X, \emptyset, \{z\}, \{q, s\}, \{z, q, s\}\}\) and \(A = \{q, r, w\}, \) Then \(A\) is \(n\)-open but not \(s\)-closed.

Proposition (2.24): Let \(X\) be tp-s, then the arbitrary intersection of \(n\)-open sets in \(X\) is also \(n\)-open set.

Proof: Assume that \(\{A_\lambda: \lambda \in \Lambda\}\) is any collection of \(n\)-open sets in a space \(X\) and let \(A = \bigcap A_\lambda, \) suppose \(x \in \overline{A}\) thus by Lemma 2.5, there are two cases.

Case 1: \(\{x\} = \emptyset.\) If \(x \notin A\) then \(x \notin A_\alpha\) for some \(\alpha.\) By Proposition 2.6, \(\{x\}\) is \(s\)-closed. Then \(\{x\}^c\) is \(s\)-open set such that \(A_\alpha \subseteq \{x\}^c.\) Then \(x \notin \text{fker} \ (A_\alpha)\). On the other hand by Lemma 2.20, \(A_\alpha\) is \(n\)-open.

Then \(x \in \overline{A} \subseteq \overline{A_\alpha} \subseteq \text{fker} \ (A_\alpha)\) paradoxically \(x \in A\) and hence \(x \in \text{fker} \ (A).\) Hence \(A\) is \(n\)-open set.

Case 2: \(\{x\} \subseteq \overline{\{x\}}.\) Let \(F = \overline{\{x\}}\) and \(x \notin \text{fker} \ (A),\) \(\exists\) feebly closed set \(C\) does contain \(x,\) so that \(C \cap A = \emptyset.\) Then \(x \in F = \overline{\{x\}} \subseteq \overline{C} \subseteq C.\) As \(F\) is an open set containing \(x,\) then \(F\) is \(s\)-open set and \(x \in \overline{A}\)

therefore, \(F \cap A \neq \emptyset.\) Since \(F \subseteq C,\) then \(C \cap A \neq \emptyset\) paradoxically. Then \(x \in \text{fker} \ (A)\) and hence \(is\) \(A\) \(n\)-open set.

In Example (2.25) we can show that the union of \(n\)-open sets is not necessarily \(n\)-open:

Example (2.25): Let \(X = \{z, q, r\}\) with \(t = \{X, \emptyset, \{z\}, \{q\}, \{z, q\}\}\) be a topology on \(X\). Then \(\{z\}, \{q\}\) are \(n\)-open sets but \(\{z\} \cup \{q\} = \{z, q\}\) is not \(n\)-open set.

Definition (2.26): Assume that \(X\) is a tp-s \& \(B \subseteq X.\) Any subset of \(X\) that contains an \(n\)-open set containing \(B\) is named \(n\)-neighborhood of \(B,\) the \(n\)-neighborhood of a subset \(\{x\}\) is named \(n\)-neighborhood of point \(x.\)

The family of all \(n\) -neighborhoods of \(x\) are always denoted by \(N_n(x).\)

Definition (2.27): Let \(X\) be a tp-s, \(p \in X \& A \in X.\) The point \(p\) is named \(n\)-limit point of \(A\) if each \(n\)-open set containing \(U,\) includes a point of \(A\) distinct from \(p.\) We are going to name the family of all \(n\)-limit point of \(A\) the \(n\)-derivative set of \(A\) and shortened by \(A^n.\) Then \(p \in A^n\) if for each \(n\)-open set \(U\) in \(X\) such \(p \in U\) this indicates that \((U \cap A) \setminus \{p\} \neq \emptyset.\)

Definition (2.28): Assume that \(X\) is a tp-s \& \(A \subseteq X,\) then \(\bigcap \{F: F \text{ is } n\text{-closed in } X \text{ such that } A \subseteq F\} \) is named \(n\)-closure of \(A\) and shortened by \(\overline{A^n}\).

Proposition (2.29): Let \(X\) be a tp-s \& \(A, B\) are subset of \(X\) with \(B\) \(n\)-open set. If \(x \in B\) and \(B \cap A = \emptyset,\) thus \(x \notin \overline{A^n}.\)

Proof: Assume that \(G\) is \(n\)-open set, \(x \in G\) and \(G \cap A = \emptyset.\) Then \(A \subseteq G^c, G^c\) is \(n\)-closed and \(x \notin G^c.\)

Therefore \(x \notin \overline{A^n}.\)

Remark (2.30): Assume that \((X, t)\) is a tp-s \& \(A, B\) subsets of \(X.\) If \(A\) is \(n\)-closed, then \(A = \overline{A^n}.\)
Example (2.31): In Example (2.25), \( \{ z, q \}^n = \{ z, q \} \), but \( \{ z, q \} \) is not n-closed set.

Definition (2.32): Let \( X \) and \( Z \) be two tp-s. Then a function \( f : X \to Z \) is called \( n \)-irresolute function if for all \( n \)-open set \( A \) in \( Z \). Therefore \( f^{-1}(A) \) is an \( n \)-open set in \( X \).

Theorem (2.33): Let \( X \) and \( Z \) be two tp-s. If a function \( f : X \to Z \) is \( n \)-irresolute function, then for each \( x \in X \) and each \( n \)-neighborhood \( W \) of \( f(x) \) in \( Z \), there is a \( n \)-neighborhood \( G \) of \( x \) in \( X \) such that \( f(G) \subseteq W \).

Proof: Let \( f : X \to Z \) be a \( n \)-irresolute function and \( W \) be a \( n \)-neighborhood of \( f(x) \) in \( Z \). To prove that there is a \( n \)-neighborhood \( G \) of \( x \) in \( X \) such that \( f(G) \subseteq W \). Since \( f \) is a \( n \)-irresolute then \( f^{-1}(W) \) is a \( n \)-neighborhood of \( x \) in \( X \). Let \( G = f^{-1}(W) \), then \( f(G) = f(f^{-1}(W)) \subseteq W \) and hence \( f(G) \subseteq W \).

3. On \( n \)-Convergence of Filters

This section introduces an explanation and some theorems about a modern class of convergence, which is \( n \)-convergence of filters.

Definition (3.1): [15]. Let \( X \neq \emptyset \), \( \Sigma \neq \emptyset \subseteq X \), that \( \Sigma \) is a filter on \( X \) if:

1. \( F_1 \cap F_2 \in \Sigma \) for all \( F_1, F_2 \in \Sigma \).
2. When \( F_1 \in \Sigma \), \( F_1 \subseteq F_2 \) then \( F_2 \in \Sigma \).

Definition (3.2): [10]. Let \( X \neq \emptyset \), \( \Sigma \) be a filter on \( X \) and \( \Sigma^o \) be sub family of a filter \( \Sigma \), then \( \Sigma^o \) is named a filter base shortened by \( (F_b) \) if and only if every element of \( \Sigma \) includes some element of \( \Sigma^o \). This means that for each \( F \in \Sigma \) there is \( F^o \in \Sigma^o \) such that \( F^o \subseteq F \).

Proposition (3.3): [10]. Let \( X \neq \emptyset \) and \( \Sigma \) be a filter on \( X \). If \( \Sigma^o \) is \( F_b \) for a filter \( \Sigma \), then \( \Sigma = \{ F \subseteq X : F^o \subseteq F, for some \ F^o \in \Sigma^o \} \) is named filter generated \( (F_b) \) by \( \Sigma^o \).

Proposition (3.4): [10]. Assume that \( X \neq \emptyset \), let \( \Sigma^o \neq \emptyset \) be a family of \( X \) and \( \Sigma = \{ A \subseteq X : B \subseteq A \ for \ some \ B \in \Sigma^o \} \), then \( \Sigma \) is a filter on \( X \) if and only if for all \( V, W \in \Sigma^o \), \( \exists F \in \Sigma \) such that \( F \subseteq V \cap W \).

Theorem (3.5): [15]. Assume that \( X \neq \emptyset \) and \( \emptyset \neq X^* \subseteq X \), if \( \Sigma^o \) is \( F_b \) on \( X^* \), then \( \Sigma^o \) is \( F^o \) of a filter on \( X \).

Definition (3.6): [10]. A filter \( \Sigma \) on a tp-s \( (X, x) \) is named convergent to a point \( y \in X \) a abbreviated by \( (\Sigma \to y) \) if and only if \( N(y) \subseteq \Sigma \). The point \( y \in X \) is named a limit point of \( \Sigma \).

It is said that \( y \in X \) a (cluster point) of \( \Sigma \) and this one is shortened by \( (\Sigma \to y) \) if and only if \( F \cap V \neq \emptyset \), for all \( F \in \Sigma \) and \( V = N(y) \).

Proposition (3.7): [10]. Let \( X_1, X_2 \) be two \( tp-s \) and \( f : X_1 \to X_2 \) be a function from \( X_1 \) in to \( X_2 \), then:

1. If \( \Sigma \) is a filter on \( X_1 \). Then \( f(\Sigma) \) is a filter on \( X_2 \) which has for a base the sets \( f(F) \), \( F \in \Sigma \) for a base.
2. If \( \Sigma^o \) is \( F_b \) on \( X_1 \). Then \( f(\Sigma^o) \) is \( F_b \) on \( X_2 \).

Proposition 3.8: [15]. Let \( X_1, X_2 \) be two \( tp-s \) and \( f : X_1 \to X_2 \) be a function from \( X_1 \) in to \( X_2 \), \( y \in X_1 \). Then \( f \) is a continuous function if and only if for \( \Sigma \to y \) in \( X_1 \), then \( f(\Sigma) \to f(y) \) in \( X_2 \).

Definition (3.9): A filter \( \Sigma \) on a \( tp-s \) \( (X, x) \) is named \( n \)-convergent to a point \( y \in X \) (written \( \Sigma \to^n y \)) if \( \Sigma_n(y) \subseteq \Sigma \); a filter \( \Sigma \) on a \( tp-s \) \( (X, x) \) has \( y \in X \) as \( n \)-cluster point abbreviated by \( \left( \Sigma \to^n y \right) \) if \( F \in \Sigma \) meets each \( G \in \Sigma_n(y) \).

Theorem (3.10): A filter \( \Sigma \) located on a \( tp-s \) \( (X, x) \) has \( y \in X \) as an \( n \)-cluster point if \( y \in \cap \bar{F}^n \), for all \( F \in \Sigma \).

Proof: Let \( \mathcal{F} \in \Sigma \) and \( y \in \cap \bar{F}^n \). Then for all \( G \in \Sigma_n(y) \), for all \( F \in \Sigma \), \( F \cap G \neq \emptyset \). Therefore \( \Sigma \to^n y \).
Example (3.11): (i) Let $X = \{a, b, c, d, e\}$ with $T = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$. Then

$fo(X) = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$, and

$so(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

Hence $N_0(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $no(X) = \{\emptyset, X, \{a\}\}$. Since $N_n(a) = \{X\}, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $no(X) = \{\emptyset, X\}$. Let $\Sigma = \{X\}$ be a filter on $X$. Since $N(c) = \{X\}$, then $N(c) \subseteq \Sigma$, thus $\Sigma$ converges to $c$. Since $N_n(a) = \{X\}, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$, therefore $\Sigma$ is not $n-$converges to $c$.

(ii) Let $X = \{a, b\}$ with $T = \{\emptyset, X, \{a\}\}$. Then $fo(X) = \{\emptyset, X, \{a\}\}$, and $so(X) = \{\emptyset, X, \{a\}\}$. Let $\Sigma = \{X\}$ be a filter on $X$. Since $N_n(a) = \{X\}$, then $N_n(a) \subseteq \Sigma$, thus $\Sigma$ is $n-$converges to $c$. Since $N(a) = \{X\}$, then $N(a) \not\subseteq \Sigma$, therefore $\Sigma$ is not converges to $a$.

(iii) Let $X = \{a, b, c\}$ with $T = \{\emptyset, X, \{a\}\}$. Then $so(X) = \{\emptyset, X, \{a\}\}$. Let $\Sigma = \{X\}$ be a filter on $X$. Then $\Sigma \not\alpha a$, but $a$ is not cluster point to $\Sigma$.

(iv) Let $X = \{a, b, c, d\}$ with $T = \{\emptyset, X, \{a\}\}$. Then $so(X) = \{\emptyset, X, \{a\}\}$. Let $\Sigma = \{X\}$ be a filter on $X$. Then $\Sigma \not\alpha c$, but $c$ is not $n-$cluster point of $\Sigma$.

Theorem (3.12): Assume that $\Sigma$ be a filter on a $tp-s (X, t)$ and $y \in X$. If $\Sigma \Rightarrow y$, then $\Sigma \Rightarrow y$.

Proof: Let $\Sigma \Rightarrow y$, then $N_n(y) \subseteq \Sigma$, consequently $F \cap G \in \Sigma$, for all $G \in N_n(y)$ and for all $F$ belonging to $\Sigma$. Since $\Sigma$ is a filter on $X$, therefore $F \cap G \neq \emptyset$ for all $G \in N_n(y)$ and for all $F \in \Sigma$, Therefore $\Sigma \Rightarrow y$.

However, in general the opposite of Theorem (3.12) is not valid as the next example:

Example(3.13): In Example (3.11(i)), where $\Sigma = \{X, \{b, c, d, e\}\}$. Then $\Sigma \not\alpha e$ but $\Sigma$ is not $n-$converges to $e$.

Theorem (3.14): Assume that $\Sigma$ be a filter on a $tp-s (X, t)$ and $y \in X$. If $\Sigma \Rightarrow y$, then every filter smaller than $\Sigma$ is also $n-$converges to $y$.

Proof: (Clear).

However, in general the opposite of Theorem (3.14) is not valid as the next example:

Example(3.15): In Example (3.11(i)), let $\Sigma' = \{X, \{b, c, d, e\}\}$ and $\Sigma = \{X\}$. Thus $\Sigma'$ is $n-$converges to $b$. But $\Sigma \subseteq \Sigma'$ and $\Sigma$ is not $n-$converges to $b$.

Definition (3.16): Let $(X, t)$ be a $tp-s$ and $n$ be a filter on $(X, t)$, then $n$ is said to be $n-$convergence to $y \in X$ (written $\Sigma \Rightarrow y$) if and only if the filter generated by $\Sigma$ is $n-$convergence to $y$.

Furthermore, suppose the $F \in \Sigma$ has $y \in X$ as an $n-$cluster point (written $\Sigma \Rightarrow y$) if and only if each $F \in \Sigma$ matches $G \in N_n(y)$. 

**Definition (3.17):** Let $\Sigma_*$ be a $F_b$ on a $tp\text{-}s \ (X, t)$ and $y \in X$. Then:

1. If a point $k \in \Sigma_*^n$, for every $F_\ast \in \Sigma_*$, then $k$ is defined as an $n$-adherent point of $\Sigma_*$.
2. If a point $k \in \cap \Sigma_*^n$, for every $F_\ast \in \Sigma_*$, then $k$ is said to be $n$-accumulation point of $\Sigma_*$.

**Theorem (3.18):** A $F_b \Sigma_*^n$ on a $tp\text{-}s \ (X, t)$ is $n$-convergence to a point $y$ belongs to $X$ if and only if for each $G$ belongs to $N_n(y)$, there exists $F_\ast \in \Sigma_*$ such as $F_\ast \subseteq G$.

**Proof:** Let $\Sigma_*^n \to y$, then a filter $\Sigma_*$ generated by $\Sigma_*$ and $\Sigma_*^n \to y$. Then $N_n(y) \subseteq \Sigma_*$, therefore for every $G \in N_n(y)$, $V \in \Sigma_*$. Therefore there is $F_\ast \in \Sigma_*$ such that $F_\ast \subseteq G$.

On the other hand, to prove that $\Sigma_*^n \to y$ means, $\Sigma_*$ to be a filter on $X$ generated by $\Sigma_*$ with $\Sigma \to y$. Let $G \in N_n(y)$ then, by hypothesis, there is $F_\ast \in \Sigma_*$ such that $F_\ast \subseteq G$. Because $\Sigma_*$ is a filter on $X$, then $G \in \Sigma_*$. Then $G \in \Sigma$ and $N_n(y) \subseteq \Sigma$. Then $\Sigma_*^n \to y$.

**Proposition (3.19):** A filter $\Sigma_*$ on a $tp\text{-}s \ (X, t)$ has $y \in X$ as an $n$-cluster point if and only if there exists a filter $\Sigma^*$ smaller than $\Sigma$ which $n$-convergence to $y$.

**Proof:** Assume the $\Sigma_*^n \to y$. So by Definition (3.16) every $F \in \Sigma$ meets every $G \in N_n(y)$. So $\Sigma_*=\{F \cap G:F \in \Sigma, G \in N_n(y)\}$ is a filter base for some filter $\Sigma^*$ which is finer , means $\Sigma$ and $n$-convergence to $y$.

On the other hand, give $\Sigma \subseteq \Sigma^*$ and $\Sigma_*^n \to y$, then $\Sigma^* \to y$ and $N_n(y) \subseteq \Sigma^*$. Hence each $F \in \Sigma$ and each $G \in N_n(y)$ belongs to $\Sigma^*$. Because $\Sigma^*$ is a filter, then $F \cap G \neq \emptyset$.

**Theorem (3.20):** Assume that $(X, t)$ be a $tp\text{-}s$ and $A \subseteq X$, $y \in X$. If $y \in \overline{A^n}$, then we have a filter $\Sigma_*$ on $X$ so that $A \in \Sigma_*$ and $\Sigma_*^n \to y$.

**Proof:** If $y \in \overline{A^n}$, then $A \cap G \neq \emptyset$ is necessary for all $G \in N_n(y)$. Therefore $\Sigma_* = \{A \cap G: G \in N_n(y)\}$ is a $F_b$ for some filter $\Sigma$. The result filter includes $A$ and $\Sigma_*^n \to y$.

**Corollary (3.21):** Let $(X, t)$ be a $tp\text{-}s$ and $A \subseteq X$, $y \in X$. If $y \in \overline{A^n}$, then there is $F_b \Sigma_*^n$ on $X$ such that $A \in \Sigma_*$ and $\Sigma_*^n \to y$.

**Theorem (3.22):** Let $f:X_1 \rightarrow X_2$ be a function and $\Sigma_*$ is a filter on $X_1$, $y \in X_1$. If $f$ is $n$-irresolute, then whenever $\Sigma_*^n \rightarrow y$ in $X_1$, then $f(\Sigma)_*^n \rightarrow f(y)$ in $X_2$.

**Proof:** Assume that $f$ is $n$-irresolute function and $\Sigma_*^n \rightarrow y$. To prove $f(\Sigma)_*^n \rightarrow f(y)$ in $X_2$. Let $W \in N_n(f(y))$, as $f$ is $n$-irresolute , then there is $G \in N_n(y)$ such that $f(G) \subseteq W$. Since $\Sigma_*^n \rightarrow y$, then $G \in \Sigma_*$. But $W \in f(\Sigma)_*$, hence $f(\Sigma)_*^n \rightarrow f(y)$.

**Theorem (3.23):** Suppose $(X, t)$ is a $tp\text{-}s$ and $A \subseteq X$. A point $y \in X$ is $n$-limit point of $A$ if and only if $A - \{y\}$ is a subset of some filter $\Sigma$ which is $n$-convergence to $y$.

**Proof:** Assume $y$ is $n$-limit point of $A$, then $A - \{y\} \cap G \neq \emptyset$ for all $G \in N_n(y)$. $\Sigma_* = \{A - \{y\} \cap G: G \in N_n(y)\}$is $F_b$ for some filter $\Sigma$. The outcome filter includes $A - \{y\}$ and $\Sigma_*^n \rightarrow y$. In the other hand, If $A - \{y\} \subseteq \Sigma$ with $\Sigma_*^n \rightarrow y$, then $A - \{y\} \subseteq \Sigma$ and $N_n(y) \subseteq \Sigma$. Since $\Sigma$ is a filter, then $A - \{y\} \cap G \neq \emptyset$ for all $G \in N_n(y)$. Consequently $y$ is the $n$-limit point of a set $A$.

**Definition (3.24):** Suppose $\{h_a: a \in D\}$ is a net in a $tp\text{-}s \ (X, t)$, $\Sigma_*$ is a filter generated by a $F_b \Sigma_*$ made up of the sets $A_a = \{h_a: a \geq a, a \in D\}$ is named a filter generated by $\{h_a: a \in D\}$ meaning, $\Sigma_* = \{A_a \subseteq X: h_a \text{ is eventually in } A_a\}$ is a $F_b$, $\Sigma_*$ is a filter on $X$ and its named a filter associated with the net $\{h_a: a \in D\}$.
Proposition (3.25): A net \( \{h_a: a \in D\} \) in a \( tp-s \) (\( X, t \)) is \( n \)-convergence to \( y \in X \) if and only if a filter \( \Sigma \) generated by \( \{h_a: a \in D\} \) is \( n \)-convergent to \( y \).

Proof: A net \( \{h_a: a \in D\} \) is an \( n \)-convergent to \( y \in X \) if and only if each \( V \in N_n(y) \) will have a tail of \( \{h_a: a \in D\} \) is a base for a filter generated by \( \{h_a: a \in D\} \).

Definition (3.26): [15]. Let \( \Sigma \) be a \( F_b \) on a \( tp-s \) (\( X, t \)), for all \( B_1, B_2 \in \Sigma \), we put \( B_1 \sqsupseteq B_2 \) if and only if \( B_1 \subseteq B_2 \), then \( (\Sigma, \sqsupseteq) \) is a directed set. For all \( F \in \Sigma \), defines \( h: \Sigma \to \mathcal{F} \), \( F \in \Sigma \) such that for any \( F \in \Sigma \) takes (fixed) \( h_F \in \mathcal{F} \) such that \( h(F) = h_\Sigma \). Therefore \( \{h_B: B \in \Sigma\} \) is a net in \( X \) and is named a net associated with \( F_b \) \( \Sigma \).

Theorem (3.27): Assume that \( \{h_B: B \in \Sigma\} \) is a net associated with a \( F_b \Sigma \) on a \( tp-s \) (\( X, t \)) and \( y \in X \). If \( \Sigma \) \( \to y \), then \( h_B \rightarrow y \).

Proof: Assume that \( \Sigma \rightarrow y \) and \( G \in N_n(y) \). Thus there is \( B_\ast \in \Sigma \) such that \( B_\ast \subseteq G \), then \( h_{B_\ast} \in G \), so \( h_B \in G \) for all \( B \supseteq B_\ast \). This emplaces \( h_B \rightarrow y \).

Definition (3.28): [15]. Let \( \Sigma \) be \( F_b \) on a \( tp-s \) (\( X, t \)). Take \( D = \{(k, B): k \in B, B \in \Sigma\} \), \( (D, \geq) \) is directed set by the relation, \( (k_1, B_1) \geq (k_2, B_2) \) if and only if \( B_1 \subseteq B_2 \), then set the function \( h: D \rightarrow X \) by \( h(a) = h_a \in X \), where \( a = (k, B) \). Then \( \{h_a: a \in D\} \) is named the canonical net (net based) of \( \Sigma \).

Proposition (3.29): Assume that \( (X, t) \) \( tp-s \) \( \Sigma \) be \( F_b \) on a \( (X, t) \) then \( \sum \) is \( n \)-convergence to a point \( k \in X \) if and only if the canonical net of \( \Sigma \) is \( n \)-convergence to \( k \).

Proof: Assume that \( \sum \rightarrow k \) with \( G \in N_n(k) \), then there is \( F_\star \in \Sigma \) so that \( F_\star \subseteq G \). Since \( F_\star \neq \emptyset \), there is \( k_\ast \in F_\star \). Use \( a_\ast = (k_\ast, F_\star) \), then \( h_{a_\ast} \in G \) for all \( a \geq a_\ast \). Hence \( h_{a_\ast} \rightarrow k \) and \( G \in N_n(k) \), there is \( a_\ast \in D \) such that \( h_{a_\ast} \in G \) for all \( a \geq a_\ast \). Therefore there are \( F_\star \in \Sigma \) and \( k_\ast \in F_\star \) so that \( a_\ast = (k_\ast, F_\star) \). To prove \( F_\star \subseteq G \). Assume that this is \( y_\ast \in F_\star \). So \( a = (k, F) \geq (k_\ast, F_\star) = a_\ast \), thus \( h_a \in G \). Hence \( F_\star \subseteq G \), this emplaces \( \sum \rightarrow k \).

Corollary (3.30): A \( F_b \Sigma \) on a \( tp-s \) (\( X, t \)) has \( k \in X \) as an \( n \)-cluster point if and only if the canonical net on \( \Sigma \) has \( k \) as an \( n \)-cluster point.

Definition (3.31): A \( tp-s \) (\( X, t \)) is \( n-T_2 \)-space if for each \( a, b \in X \) with \( a \neq b \), there are \( V \in N_n(a) \) and \( W \in N_n(b) \) such that \( V \cap W = \emptyset \).

Theorem (3.32): Let \( (X, t) \) \( tp-s \) then the next two point are equivalent

1. \( (X, t) \) is \( n-T_2 \)-space
2. Each \( n \)-convergent filter in \( X \) has only one \( n \)-limit point.

Proof: (1 \( \Rightarrow \) 2) Suppose that \( (X, t) \) is an \( n-T_2 \)-space with \( \Sigma \) be a filter on \( X \) so that \( \Sigma \rightarrow a \) and \( \Sigma \rightarrow b \) with \( a \neq b \). Since \( X \) is an \( n-T_2 \)-space, then there are \( V \in N_n(a) \) and \( W \in N_n(b) \) such that \( V \cap W = \emptyset \). Since \( \Sigma \rightarrow a \) then \( N_n(a) \subseteq \Sigma \) and \( \Sigma \rightarrow b \) then \( N_n(b) \subseteq \Sigma \). Since \( \Sigma \) is a filter, then \( V \cap W \neq \emptyset \). That is the opposite, and then comes the conclusion.

(2 \( \Rightarrow \) 1) On the other hand when we wont to prove that \( X \) is an \( n-T_2 \)-space. Assume that not \( n-T_2 \)-space, then there are \( a, b \in X \) with \( a \neq b \) such that for all \( V \in N_n(a) \) and for all \( W \in N_n(b) \), \( V \cap W \neq \emptyset \). So \( \Sigma = \{V \cap W: V \in N_n(a) \text{ and } W \in N_n(b)\} \) is a filter base for some filter \( \Sigma \). The resulting filter \( n \)-convergence at a and b. It is an inconsistency, because \( X \) is an \( n-T_2 \)-space.
Conclusion

In this paper, we studied some of the properties of the n-open sets through their behavior on convergence and filters, which led to obtaining some new definitions and theories that are important in topological spaces. We also found several examples of reverse instances, which are usually incorrect in this study.

References


