Al-Qadisiyah Journal of Pure Science

Volume 25 | Number 4 Article 1

10-7-2020

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Recommended Citation

Murugusundraramoorthy, Gangadharan and Porwal, Saurabh (2020) "Univalent Functions with PositiveCoefficients Involving Touchard Polynomials," *Al-Qadisiyah Journal of Pure Science*: Vol. 25: No. 4, Article 1.
DOI: 10.29350/qjps.2020.25.4.1176
Available at: https://qjps.researchcommons.org/home/vol25/iss4/1

Al-Qadisiyah Journal of Pure Science Vol.(25) Issue (4) (2020) pp. Math. 1–8

Al-Qadisiyah Journal of Pure Science

QJPS

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: /10.29350/jops.

http://qu.edu.iq/journalsc/index.php/JOPS

Al-Qadisiyah Journal of Pure Science

Univalent Functions with Positive Coefficients Involving Touchard Polynomials

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Article History

Received on: 19/07/2020

Revised on: 16/08/2020

Accepted on: 20/08/2020

Keywords:

Univalent, Starlike functions, Convex functions, Hadamard product, Touchard polynomials.

DOI:https://doi.org/10.29350/qjps2020.25.4.1176

ABSTRACT

The purpose of the present paper is to establish connections between various subclasses of analytic univalent functions by applying certain convolution operator involving Touchard polynomials. To be more precise, we investigate such connections with the classes of analytic univalent functions with positive coefficients in the open unit disk $\mathbb{U}$.

MSC: 30C45.

1. Introduction

The application of special function on Geometric Function Theory is a current and interesting topic of research. It is frequently applied in various branches Mathematics, Physics, Sciences, Engineering and Technology. The surprising use of generalized hypergeometric function by L. de Branges [6] in the solution of the famous Bieberbach conjecture. There is an extensive literature dealing with analytical and geometric properties of various types of special functions, especially for the generalized, Gaussian hypergeometric functions [5,10,11,16,18].

The Touchard polynomials, studied by Jacques Touchard [19], also called the exponential generating polynomials (see [3], [15], [17]) or Bell polynomials (see [1]) comprise a polynomial sequence of binomial type that for X is a random variable with a Poisson distribution with expected value ℓ, then its nth moment is $E(X_\kappa) = \mathcal{T}(\kappa, \ell)$, leading to the form:

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$$\mathcal{T}(\kappa,\ell) = e^{\kappa}\sum_{n=0}^{\infty}\frac{\kappa^n n^{\ell}}{n!}z^n \tag{1.1}$$

Touchard polynomials coefficients after the second force as following

Lately, introduce Touchard polynomials coefficients after the second force as following

$$\Phi_{\kappa}^{\ell}(z) = z + \sum_{n=2}^{\infty}\frac{(n-1)^{\ell}\kappa^{n-1}}{(n-1)!}e^{-\kappa}z^n, \qquad z \in \mathbb{D}, \tag{1.2}$$

where $\ell \geq 0; \kappa > 0$ and we note that, by ratio test the radius of convergence of above series is infinity. Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ z \in \mathbb{U}. \tag{1.3}$$

We also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$.

Denote by $\mathcal{V}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.4}$$

For functions $f \in \mathcal{A}$ given by (1.3) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.5}$$

Now, we define the linear operator

$$\mathcal{I}(l, m, z): \mathcal{A} \to \mathcal{A}$$

defined by the convolution or hadamard product

$$\mathcal{I}(l, m, z)f = \Phi_m^{\ell}(z) * f(z) = z + \sum_{n=2}^{\infty}\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}a_n z^n, \tag{1.6}$$

where $\Phi_m^{\ell}(z)$ is the series given by (1.2).

The class $\mathcal{M}(\alpha)$ of starlike functions of order $1 < \alpha \leq \frac{4}{3}$

$$\mathcal{M}(\alpha) := \left\{f \in \mathcal{A}: \Re\frac{zf'(z)}{f(z)} < \alpha, z \in \mathbb{U}\right\}$$

and the class $\mathcal{N}(\alpha)$ of convex functions of order $1 < \alpha \leq \frac{4}{3}$

$$\mathcal{N}(\alpha):=\left\{f\in\mathcal{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\alpha, z\in\mathbb{U}\right\}=\{f\in\mathcal{A}:zf'\in\mathcal{M}(\alpha)\}$$

were introduced by Uralegaddi et al. [20] (see[7,9]). Also let $\mathcal{M}^*(\alpha)\equiv\mathcal{M}(\alpha)\cap\mathcal{V}$ and $\mathcal{N}^*(\alpha)\equiv\mathcal{N}(\alpha)\cap\mathcal{V}$.

In this paper we introduce two new subclasses of $\mathcal{S}$ namely $\mathcal{M}(\lambda,\alpha)$ and $\mathcal{N}(\lambda,\alpha)$ to discuss some inclusion properties.

For some $\alpha\ (1 < \alpha \leq \frac{4}{3})$ and $\lambda(0\leq\lambda<1)$, we let $\mathcal{M}(\lambda,\alpha)$ and $\mathcal{N}(\lambda,\alpha)$ be two new subclass of $\mathcal{S}$ consisting of functions of the form (1.3) satisfying the analytic criteria

$$\mathcal{M}(\lambda,\alpha):=\left\{f\in\mathcal{S}:\Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)<\alpha,\ z\in\mathbb{U}\right\}. \tag{1.7}$$

$$\mathcal{N}(\lambda,\alpha):=\left\{f\in\mathcal{S}:\Re\left(\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right)<\alpha,\ z\in\mathbb{U}\right\}. \tag{1.8}$$

We also let $\mathcal{M}^*(\lambda,\alpha)\equiv\mathcal{M}(\lambda,\alpha)\cap\mathcal{V}$ and $\mathcal{N}^*(\lambda,\alpha)\equiv\mathcal{N}(\lambda,\alpha)\cap\mathcal{V}$.

Note that $\mathcal{M}(0,\alpha)=\mathcal{M}(\alpha)$, $\mathcal{N}(0,\alpha)=\mathcal{N}(\alpha)$; $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ the subclasses of studied by Uralegaddi et al. [20].

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [5,10,11,16,18]), we obtain necessary and sufficient condition for function $\Phi(l,m,z)$ to be in the classes $\mathcal{M}(\lambda,\alpha)$, $\mathcal{N}(\lambda,\alpha)$ and connections between $\mathcal{R}^\tau(A,B)$ by applying convolution operator.

2. Preliminary results

To prove our main results we shall require the following definitions and lemmas.

Definition 2.1. The l^{th} moment of the Poisson distribution is defined as

$$\mu_l' = \sum_{n=0}^{\infty}\frac{n^l m^n}{n!}e^{-m}.$$

Lemma 2.1. For some $\alpha\ (1 < \alpha \leq \frac{4}{3})$ and $\lambda(0\leq\lambda<1)$, and if $f\in\mathcal{V}$ then $f\in\mathcal{M}^*(\lambda,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\alpha]|a_n|\leq\alpha-1. \tag{2.9}$$

Lemma 2.2. For some α $(1 < \alpha \leq \frac{4}{3})$ and $\lambda(0 \leq \lambda < 1)$, and if $f \in \mathcal{V}$ then $f \in \mathcal{N}^*(\lambda, \alpha)$ if and only if

$$\sum_{n=2}^{\infty} n\,[n - (1 + n\lambda - \lambda)\alpha]a_n \leq \alpha - 1. \tag{2.10}$$

3. Main Results

For convenience throughout in the sequel, we use the following notations:

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} = e^m - 1 \tag{3.11}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} = me^m \tag{3.12}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} = m^2e^m \tag{3.13}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-4)!} = m^3e^m \tag{3.14}$$

Theorem 3.1. If $m > 0(m \neq 0, -1, -2, \ldots)$, $l \in N_0$ then $\Phi(l, m, z) \in \mathcal{M}^*(\lambda, \alpha)$ if and only if

$$\begin{cases} (1 - \alpha\lambda)\mu'_{l+1} + (1 - \alpha)\mu'_l, & l \geq 1 \\ (1 - \alpha\lambda)m + (1 - \alpha)(1 - e^{-m}), & l = 0 \end{cases} \leq \alpha - 1. \tag{3.15}$$

Proof. To prove that $\Phi(l, m, z) \in \mathcal{M}^*(\lambda, \alpha)$, then by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} [n - (1 + n\lambda - \lambda)\alpha] \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} \leq \alpha - 1. \tag{3.16}$$

Now

$$e^{-m} \sum_{n=2}^{\infty} [n(1 - \lambda\alpha) - \alpha(1 - \lambda)] \frac{(n-1)^l m^{n-1}}{(n-1)!}$$

$$= e^{-m} \sum_{n=2}^{\infty} [(n-1)(1 - \lambda\alpha) + 1 - \alpha] \frac{(n-1)^l m^{n-1}}{(n-1)!}$$

$$= e^{-m}\sum_{n=1}^{\infty}[n(1-\lambda\alpha)+1-\alpha]\frac{n^l m^n}{n!}$$

$$= e^{-m}\sum_{n=1}^{\infty}[(1-\lambda\alpha)\frac{n^{l+1}m^n}{n!}+(1-\alpha)\frac{n^l m^n}{n!}]$$

$$= \begin{cases} (1-\alpha\lambda)\mu'_{l+1}+(1-\alpha)\mu'_l, & l\ge 1 \\ (1-\alpha\lambda)m+(1-\alpha)(1-e^{-m}), & l=0 \end{cases}$$

But this expression is bounded above by $\alpha - 1$ if and only if (3.15) holds. Thus the proof is complete. □

Theorem 3.2. If $m > 0 (m \neq 0,-1,-2,\dots)$, $l \in N_0$ then $\Phi(l,m,z) \in \mathcal{N}^*(\lambda,\alpha)$ if and only if

$$\begin{cases} (1-\alpha\lambda)\mu'_{l+2}+(2-\alpha\lambda-\alpha)\mu'_{l+1}+(1-\alpha)\mu'_l, & l\ge 1 \\ (1-\alpha\lambda)(m^2+m)+(2-\alpha\lambda-\alpha)m+(1-\alpha)(1-e^{-m}), & l=0 \end{cases} \le \alpha-1. \tag{3.17}$$

Proof. To prove that $\Phi(l,m,z) \in \mathcal{N}^*(\lambda,\alpha)$, then by virtue of Lemma 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n\,[n-(1+n\lambda-\lambda)\alpha]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m} \le \alpha-1. \tag{3.18}$$

Now

$$e^{-m}\sum_{n=2}^{\infty} n\,[n(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(n-1)^l m^{n-1}}{(n-1)!}$$

$$= e^{-m}\sum_{n=2}^{\infty}[(1-\lambda\alpha)(n-1)^2+(2-\alpha\lambda-\alpha)(n-1)+1-\alpha]\frac{(n-1)^l m^{n-1}}{(n-1)!}$$

$$= e^{-m}\sum_{n=2}^{\infty}[(1-\lambda\alpha)\frac{(n-1)^{l+2}m^{n-1}}{(n-1)!}+(2-\alpha\lambda-\alpha)\frac{(n-1)^{l+1}m^{n-1}}{(n-1)!}+(1-\alpha)\frac{(n-1)^l m^{n-1}}{(n-1)!}]$$

$$= \begin{cases} (1-\alpha\lambda)\mu'_{l+2}+(2-\alpha\lambda-\alpha)\mu'_{l+1}+(1-\alpha)\mu'_l, & l\ge 1 \\ (1-\alpha\lambda)(m^2+m)+(2-\alpha\lambda-\alpha)m+(1-\alpha)(1-e^{-m}), & l=0 \end{cases}$$

But this expression is bounded above by $\alpha - 1$ if and only if (3.17) holds. Thus the proof is complete.

4.Inclusion Properties

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^{\tau}(A,B)$ was introduced earlier by Dixit and Pal [8].

It is of interest to note that if

$$\tau = 1,\ A = \beta \text{ and } B = -\beta \ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in \mathbb{U})$$

which was studied by (among others) Padmanabhan [13] and Caplinger and Causey [4].

Lemma 4.3. [8] If $f \in \mathcal{R}^{\tau}(A,B)$ is of form (1.3), then

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{4.1}$$

The result is sharp.

Making use of the Lemma 4.3 we will study the action of the Poissons distribution series on the class $\mathcal{M}(\lambda,\alpha)$.

Theorem 4.3. Let $> 0(m \neq 0,-1,-2,\dots)$, $l \in N_0$. If $f \in \mathcal{R}^{\tau}(A,B)$,then $\mathcal{I}(l,m,z)f \in \mathcal{N}^*(\lambda,\alpha)$ if and only if

$$(A-B)|\tau|\begin{cases}(1-\alpha\lambda)\mu'_{l+1} + (1-\alpha)\mu'_l, & l \geq 1\\(1-\alpha\lambda)m + (1-\alpha)(1-e^{-m}), & l = 0\end{cases} \leq \alpha - 1. \tag{4.2}$$

Proof. Let f be of the form (1.3) belong to the class $\mathcal{R}^{\tau}(A,B)$. By virtue of Lemma 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n\,[n-(1+n\lambda-\lambda)\alpha]\frac{(n-1)^l m^{n-1}}{(n-1)!}|a_n| \leq \alpha - 1.$$

Let Now

$$e^{-m}\sum_{n=2}^{\infty}[\,n(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(n-1)^l m^{n-1}}{(n-1)!}|a_n|$$
$$\leq (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[\,(n-1)(1-\lambda\alpha)+1-\alpha]\frac{(n-1)^l m^{n-1}}{(n-1)!}$$

$$= (A-B)|\tau|e^{-m}\sum_{n=1}^{\infty}[n(1-\lambda\alpha)+1-\alpha]\frac{n^l m^n}{n!}$$
$$= (A-B)|\tau|e^{-m}\sum_{n=1}^{\infty}[(1-\lambda\alpha)\frac{n^{l+1}m^n}{n!}+(1-\alpha)\frac{n^l m^n}{n!}]$$
$$= (A-B)|\tau|\begin{cases}(1-\alpha\lambda)\mu'_{l+1}+(1-\alpha)\mu'_l, & l\geq 1\\ (1-\alpha\lambda)m+(1-\alpha)(1-e^{-m}), & l=0\end{cases}$$
$$\leq \alpha-1.$$

Theorem 4.4. Let $m>0(m\neq 0,-1,-2,\dots)$, $l\in N_0$ then $\mathcal{L}(l,m,z)=\int_0^z\frac{\mathcal{I}(l,m,t)}{t}dt$ is in $\mathcal{N}^*(\lambda,\alpha)$ if and only if inequality (3.15) is satisfied.
Proof. Since

$$\mathcal{L}(l,m,z)=z+\sum_{n=2}^{\infty}\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}\frac{z^n}{n}.$$

By virtue of Lemma 3.1, it suffices to show that

$$\sum_{n=2}^{\infty}n\,[n-(1+n\lambda-\lambda)\alpha]\frac{(n-1)^l m^{n-1}}{n(n-1)!}e^{-m}\leq \alpha-1.$$

Now,

$$\sum_{n=2}^{\infty}n\,[n-(1+n\lambda-\lambda)\alpha]\frac{(n-1)^l m^{n-1}}{n(n-1)!}e^{-m}=\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\alpha]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}.$$

Proceeding as in Theorem 3.1 we obtain the required result. □

4. Open Problems

t is interesting to find the result of Theorems 3.1-4.4, when l is a real number.

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