

10-7-2020

## Fuzzy Open Mapping and Fuzzy Closed Graph Theorems in Fuzzy Length Space

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### Recommended Citation

Sabri, Raghad I. (2020) "Fuzzy Open Mapping and Fuzzy Closed Graph Theorems in Fuzzy Length Space," *Al-Qadisiyah Journal of Pure Science*: Vol. 25: No. 4, Article 5.

DOI: 10.29350/qjps.2020.25.4.1189

Available at: <https://qjps.researchcommons.org/home/vol25/iss4/5>

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Al-Qadisiyah Journal of Pure Science

# Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490

ISSN(Online): 2411-3514

DOI: /10.29350/jops.

<http://qu.edu.iq/journalsc/index.php/IOPS>

## Fuzzy Open Mapping and Fuzzy Closed Graph Theorems in Fuzzy Length Space

<b>Authors Names</b>	<b>ABSTRACT</b>
<p><b>a .Raghad I.Sabri</b></p> <p><b>Article History</b>            Received on:26 / 8/2020            Revised on: 17 /9 /2020            Accepted on:19 /9 /2020</p> <p><b>Keywords:</b>            Fuzzy Point, Fuzzy Length Space,            Fuzzy Cauchy Sequence, linear            operator.</p> <p><b>DOI:</b> <a href="https://doi.org/10.29350/jops.2020.25.4.1189">https://doi.org/10.29350/jops.2020.25.4.1189</a></p>	<p>The theory of fuzzy set includes many aspects that regard important and significant in different fields of science and engineering in addition to their applications. Fuzzy metric and fuzzy normed spaces are essential structures in the fuzzy set theory. The concept of fuzzy length space has been given analogously and the properties of this space are studied few years ago. In this work, the definition of a fuzzy open linear operator is presented for the first time and the fuzzy Barise theorem is established to prove the fuzzy open mapping theorem in a fuzzy length space. Finally, the definition of a fuzzy closed linear operator on fuzzy length space is introduced to prove the fuzzy closed graph theorem.</p>

### 1. Introduction

The fuzzy set theory has wide use in mathematics and other sciences especially in new problems that need making decisions. After the pioneering work of Zadeh [1], many authors have been developed this concept in diverse branches. In functional analysis, the fuzzy sets are utilized broadly and many authors reinforced this matter like Kramosil [2], George and Veermani [3] by introducing the notion of fuzzy metric spaces, in addition to Katras [4] who introduced the idea of the fuzzy norm and Felbin [5] who defined a fuzzy norm on a linear space. In 2011, Bag [6] modified the concept of Felbin's type of fuzzy normed linear space presented in [5] and established in a fuzzy framework the main theorems such as open mapping theorem, closed graph theorem, and uniform boundedness principle theorem. Authors in [7] studied properties of the fuzzy norm of linear operators and investigated the bounded inverse theorem on fuzzy normed

linear spaces and proved Hahn Banach theorem, closed graph theorem, and uniform boundedness theorem on fuzzy normed linear spaces. Several authors have been published a large number of papers related to the notion of the fuzzy norm and its applications for reference see [8-11].

In 2016, Kider and Mousa [12] introduced the definition of fuzzy length space on a fuzzy set and studied many properties of this space. Furthermore, several concepts like fuzzy convergence sequence, fuzzy bounded set, fuzzy dense set, and fuzzy continuous operator are given. Gozutok and Sagiroglu in 2017 [13] introduced another notion of fuzzy length space and studied its basic properties.

The essential aim of this paper is to introduce the definition of fuzzy open linear operators and fuzzy closed linear operators to proved main theorems such as fuzzy Baire's theorem, fuzzy open mapping theorem, and fuzzy closed graph theorem in the fuzzy length space given in [12].

Structurally, the paper comprises the following: some properties and basic notions of the fuzzy length space are presented in Section 2. In Section 3, the definition of a fuzzy open linear operator in a fuzzy length space is introduced and fuzzy Baire's theorem is proved in order to prove the main theorem namely fuzzy open mapping theorem. Furthermore, the concept of a fuzzy closed linear operator in a fuzzy length space is introduced and a fuzzy closed graph theorem is established. The paper finished with a conclusion section.

## 2. Preliminaries

In this section, some main notions of fuzzy length space are presented.

### Definition 2.1:[3]

A binary operation  $\odot: [0,1]^2 \rightarrow [0,1]$  is called t-norm

(or continuous triangular norm ) if  $\forall f, g, t, s \in [0,1]$

the conditions are satisfied :

$$(i) f \odot g = g \odot f$$

$$(ii) f \odot 1 = f$$

$$(iii) (f \odot g) \odot t = f \odot (g \odot t),$$

$$(iv) \text{ If } f \leq g \text{ and } t \leq s \text{ then } f \odot t \leq g \odot s.$$

### Remark 2.2:[3]

For each  $f > g$ , there is  $t$  with  $f \odot t \geq g$  and for all  $s$ , there is  $\sigma$  with  $s \odot s \geq \sigma$  where  $f, g, t, \sigma, s \in [0,1]$ .

### Definition 2.3:[2]

Suppose that  $U$  be a universal set,  $u \in U$ , and  $\alpha \in [0,1]$ . The fuzzy point with value  $\alpha$  and support  $u$  is defined as :

$$u_\alpha(v) = \begin{cases} \alpha & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}$$

for each  $v \in U$ .

**Definition 2.4:[12]**

Let  $\tilde{U}$  be a fuzzy set in a linear space  $\mathcal{L}$  over field  $\mathbb{K}$ . Suppose that  $\tilde{F}$  be a fuzzy set from  $\tilde{U}$  to  $[0,1]$  and  $\odot$  is a t-norm such that:

$$(\tilde{F}1) \tilde{F}(u, \alpha) > 0 \text{ for all } (u, \alpha) \in \tilde{U}$$

$$(\tilde{F}2) \tilde{F}(u, \alpha) = 0 \text{ if and only if for } (u, \alpha) = 0$$

$$(\tilde{F}3) \tilde{F}(cu, \alpha) = \tilde{F}\left(u, \frac{\alpha}{|c|}\right) \text{ where } c \in \mathbb{K}, c \neq 0$$

$$(\tilde{F}4) \tilde{F}((u, \alpha) + (v, \beta)) \geq \tilde{F}((u, \alpha)) \odot \tilde{F}((v, \beta)) \text{ for all } (u, \alpha), (v, \beta) \in \tilde{U}$$

$$(\tilde{F}5) \tilde{F} \text{ is a continuous fuzzy set for all } (u, \alpha), (v, \beta) \in \tilde{U} \text{ and } \alpha, \beta \in [0,1].$$

Then  $(\tilde{U}, \tilde{F}, \odot)$  is called a fuzzy length space (briefly  $\mathcal{FL}$  – space) on the fuzzy set  $\tilde{U}$ .

**Definition 2.5:[12]**

Let  $(\tilde{U}, \tilde{F}, \odot)$  be an  $\mathcal{FL}$  – space, and let  $(u, \alpha) \in \tilde{U}$ , where  $\alpha \in [0,1]$ . Given  $0 < \varepsilon < 1$  then:

$$\tilde{B}((u, \alpha), \varepsilon) = \{(v, \beta) \in \tilde{U} : \tilde{F}((v, \beta) - (u, \alpha)) > 1 - \varepsilon\}$$

$$\tilde{B}[(u, \alpha), \varepsilon] = \{(v, \beta) \in \tilde{U} : \tilde{F}((v, \beta) - (u, \alpha)) \geq 1 - \varepsilon\}$$

is called the fuzzy open ball and fuzzy closed ball respectively of center  $(u, \alpha) \in \tilde{U}$  and radius  $\varepsilon$ .

**Definition 2.6:[12]**

Let  $(\tilde{U}, \tilde{F}, \odot)$  be an  $\mathcal{FL}$  – space and  $\tilde{C} \subseteq \tilde{U}$ . Then  $\tilde{C}$  is called fuzzy open if for every  $(v, \beta) \in \tilde{C}$  there is  $\tilde{B}((v, \beta), \varepsilon) \subseteq \tilde{C}$ . A subset  $\tilde{X} \subseteq \tilde{U}$  is called fuzzy closed if  $\tilde{X}^c = \tilde{U} - \tilde{X}$  is fuzzy open

**Definition 2.7:[12]**

Let  $(\tilde{U}, \tilde{F}, \odot)$  be an  $\mathcal{FL}$  – space and  $\tilde{C} \subseteq \tilde{U}$ . The fuzzy closure of  $\tilde{C}$  is written by  $\tilde{C}^c$  which is the smallest fuzzy closed set which includes  $\tilde{C}$ .

**3. Fuzzy Open Mapping and Fuzzy Closed Graph Theorems**

In this section, the definitions of the fuzzy open operator and fuzzy close operator are introduced. Then fuzzy open mapping theorem and fuzzy closed graph theorem will be proved.

**Definition 3.1:**

Let  $(\tilde{U}, \tilde{F}_{\tilde{U}}, \odot)$  and  $(\tilde{V}, \tilde{F}_{\tilde{V}}, \odot)$  be two  $\mathcal{FL}$  – spaces. The operator  $T: D(T) \rightarrow \tilde{V}$  where  $D(T) \subseteq \tilde{U}$  is called fuzzy open if for any fuzzy open set  $\tilde{C}$  in  $D(T)$ ,  $T(\tilde{C})$  is a fuzzy open set in  $\tilde{U}$ .

In an  $\mathcal{FL}$  – space  $(\tilde{U}, \tilde{F}, \odot)$ , a sequence  $\{(u_n, \alpha_n)\}$  is called fuzzy converges to a fuzzy point  $(u, \alpha) \in \tilde{U}$  if for each  $0 < \gamma < 1$ ,  $\exists N$  with  $\tilde{F}((u_n, \alpha_n) - (u, \alpha)) > 1 - \gamma$ , for all  $n \geq N$ . Also a sequence  $\{(u_n, \alpha_n)\}$  is called fuzzy Cauchy if for any  $0 < \gamma < 1$ ,  $\exists N$  with  $\tilde{F}((u_n, \alpha_n) - (u_m, \alpha_m)) > 1 - \gamma$ , for all  $n, m \geq N$ . An  $\mathcal{FL}$  – space  $(\tilde{U}, \tilde{F}_{\tilde{U}}, \odot)$  is called complete if each fuzzy Cauchy sequence in  $\tilde{U}$  converges to a fuzzy point in  $\tilde{U}$ .

The following theorem is the key to proving the main results

**Theorem (Fuzzy Baire's Theorem) 3.2:**

Let  $(\tilde{U}, \tilde{F}_{\tilde{U}}, \odot)$  be a complete  $\mathcal{FL}$  – space and let  $\{\tilde{G}_n\}$  be a countable collection of dense fuzzy open sets of  $\tilde{U}$ . Then the intersection  $\bigcap_n \tilde{G}_n$  is dense in  $\tilde{U}$ .

**Proof:** Assume that  $\tilde{U}$  a complete  $\mathcal{FL}$  – space. Let  $\tilde{B}_0$  be a fuzzy open set and assume that  $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \dots$  be dense fuzzy open sets in  $\tilde{U}$ . Since  $\tilde{G}_1$  dense in  $\tilde{U}$ ,  $\tilde{B}_0 \cap \tilde{G}_1 \neq \emptyset$ . Let  $(u_1, \alpha_1) \in \tilde{B}_0 \cap \tilde{G}_1$ . Since  $\tilde{B}_0 \cap \tilde{G}_1$  is fuzzy open, there exist  $0 < \varepsilon_1 < 1$ , such that  $\tilde{B}((u_1, \alpha_1), \varepsilon_1) \subseteq \tilde{B}_0 \cap \tilde{G}_1$ . Now choose  $\varepsilon_1 < \varepsilon_1$  such that  $\tilde{B}[(u_1, \alpha_1), \varepsilon_1] \subseteq \tilde{B}_0 \cap \tilde{G}_1$ . Let  $\tilde{B}_1 = \tilde{B}[(u_1, \alpha_1), \varepsilon_1]$ . Since  $\tilde{G}_2$  dense in  $\tilde{U}$ ,  $\tilde{B}_1 \cap \tilde{G}_2 \neq \emptyset$ . Let  $(u_2, \alpha_2) \in$

$\overline{\mathcal{B}}_1 \cap \overline{\mathcal{G}}_2$ . Since  $\overline{\mathcal{B}}_1 \cap \overline{\mathcal{G}}_2$  is fuzzy open, there exist  $0 < \varepsilon_2 < \frac{1}{2}$ , such that  $\check{\mathcal{B}}((u_2, \alpha_2), \varepsilon_2) \subset \overline{\mathcal{B}}_1 \cap \overline{\mathcal{G}}_2$ . Choose  $\varepsilon_2 < \varepsilon_2$  such that  $\check{\mathcal{B}}[(u_2, \alpha_2), \varepsilon_2] \subset \overline{\mathcal{B}}_1 \cap \overline{\mathcal{G}}_2$ . Continuation in the same way it will be found a  $(u_n, \alpha_n) \in \overline{\mathcal{B}}_{n-1} \cap \overline{\mathcal{G}}_n$ . Since  $\overline{\mathcal{B}}_{n-1} \cap \overline{\mathcal{G}}_n$  is fuzzy open, there exist  $0 < \varepsilon_n < \frac{1}{n}$ , such that  $\check{\mathcal{B}}((u_n, \alpha_n), \varepsilon_n) \subset \overline{\mathcal{B}}_{n-1} \cap \overline{\mathcal{G}}_n$ . Choose  $\varepsilon_n < \varepsilon_n$  such that  $\check{\mathcal{B}}[(u_n, \alpha_n), \varepsilon_n] \subset \overline{\mathcal{B}}_{n-1} \cap \overline{\mathcal{G}}_n$ . Let  $\check{\mathcal{B}}_n = \check{\mathcal{B}}((u_n, \alpha_n), \varepsilon_n)$ . Now claim that  $\{(u_n, \alpha_n)\}$  is a fuzzy Cauchy sequence. For any  $\gamma > 0$ , choose  $n_0$  with  $\gamma > \frac{1}{n_0}$ . Then for  $m \geq n$  and  $n \geq n_0$ ,  $\check{\mathcal{F}}((u_n, \alpha_n) - (u_m, \alpha_m)) \geq 1 - \frac{1}{n_0} \geq 1 - \gamma$ . Therefore  $\{(u_n, \alpha_n)\}$  is a fuzzy Cauchy sequence. Since  $\check{\mathcal{U}}$  is complete,  $\{(u_n, \alpha_n)\}$  converges to a fuzzy point  $(u, \alpha)$ . But  $(u_k, \alpha_k) \in \check{\mathcal{B}}[(u_n, \alpha_n), \varepsilon_n]$  for all  $k \geq n$  and by ([12], Lemma 4.10)  $\check{\mathcal{B}}[(u_n, \alpha_n), \varepsilon_n]$  is a fuzzy closed set. Hence  $(u, \alpha) \in \check{\mathcal{B}}[(u_n, \alpha_n), \varepsilon_n] \subset \overline{\mathcal{B}}_{n-1} \cap \overline{\mathcal{G}}_n$ . Hence  $\overline{\mathcal{B}}_0 \cap (\bigcap_{n=1}^{\infty} \overline{\mathcal{G}}_n) \neq \emptyset$ . Therefore  $\bigcap_{n=1}^{\infty} \overline{\mathcal{G}}_n$  is dense in  $\check{\mathcal{U}}$ .

According to the previous theorem, the fuzzy open mapping theorem is established in the fuzzy length space as follows.

**Theorem [Fuzzy Open Mapping Theorem] 3.3:**

If  $(\check{\mathcal{U}}, \check{\mathcal{F}}_{\check{\mathcal{U}}}, \odot)$  and  $(\check{\mathcal{V}}, \check{\mathcal{F}}_{\check{\mathcal{V}}}, \odot)$  are two complete  $\mathcal{FL}$  – spaces and  $\mathcal{T}: \check{\mathcal{U}} \rightarrow \check{\mathcal{V}}$  be a fuzzy bounded linear surjective operator then  $\mathcal{T}$  is a fuzzy open mapping. Thus, if  $\mathcal{T}$  is bijective operator, then  $\mathcal{T}^{-1}$  is fuzzy continuous which implies that  $\mathcal{T}^{-1}$  fuzzy bounded.

**Proof:** Proving a theory requires several steps

**First step.** Consider  $\check{E}$  a neighborhood of 0 in  $\check{\mathcal{U}}$ . It will be shown that  $0 \in \text{int}[\overline{\mathcal{T}(\check{E})}]$ . Let  $\check{E}^*$  be a balanced neighborhood of 0 with  $\check{E}^* + \check{E}^* \subset \check{E}$ . Since  $\mathcal{T}$  is surjective,  $\check{\mathcal{V}} = \mathcal{T}(\check{\mathcal{U}})$  it follows that  $\check{\mathcal{V}} = \bigcap_n \mathcal{T}(n\check{E}^*)$ . So by Theorem 3.4, there exist  $n_0 \in N$  in which  $\overline{\mathcal{T}(n_0\check{E}^*)}$  has a nonempty interior. Thus  $0 \in \text{int}[\overline{\mathcal{T}(\check{E}^*)}] - \text{int}[\overline{\mathcal{T}(\check{E}^*)}]$ . On the other hand,  $\text{int}[\overline{\mathcal{T}(\check{E}^*)}] - \text{int}[\overline{\mathcal{T}(\check{E}^*)}] \subset \overline{\mathcal{T}(\check{E}^*)} - \overline{\mathcal{T}(\check{E}^*)} = \overline{\mathcal{T}(\check{E}^*)} - \mathcal{T}(\check{E}^*) \subset \overline{\mathcal{T}(\check{E})}$ . Therefore, the set  $\overline{\mathcal{T}(\check{E})}$  contains the neighborhood  $\text{int}[\overline{\mathcal{T}(\check{E}^*)}] - \text{int}[\overline{\mathcal{T}(\check{E}^*)}]$  of 0.

**Second step.** It is shown  $0 \in \text{int}[\overline{\mathcal{T}(\check{E})}]$ . Since  $0 \in \check{E}$  and  $\check{E}$  is a fuzzy open set, then there exists  $0 < \varepsilon < 1$  such that  $\check{\mathcal{B}}(0, \varepsilon) \subset \check{E}$ . On the other side,  $0 \in \overline{\mathcal{T}(\check{\mathcal{B}}(0, \varepsilon_n))}$  and by Step 1, there exist  $0 < \delta_n < 1$  such that  $\check{\mathcal{B}}(0, \delta_n) \subset \overline{\mathcal{T}(\check{\mathcal{B}}(0, \varepsilon_n))}$ . Now it is shown that  $\check{\mathcal{B}}(0, \delta_1) \subset \text{int}(\mathcal{T}(\check{E}))$ . Suppose that  $(u_0, \alpha_0) \in \check{\mathcal{B}}(0, \delta_1)$ . Then  $(u_0, \alpha_0) \in \check{\mathcal{B}}(0, \varepsilon_1)$  and so for  $\delta_2 > 0$  the fuzzy open ball  $\check{\mathcal{B}}(u_0, \delta_2)$  intersects  $\mathcal{T}(\check{\mathcal{B}}(0, \varepsilon_1))$ . Therefore there exists  $(v_1, \beta_1) \in \check{\mathcal{B}}(0, \varepsilon_1)$  such that  $\mathcal{T}(v_1, \beta_1) \in \check{\mathcal{B}}(u_0, \delta_2)$ , this mean  $\check{\mathcal{F}}_{\check{\mathcal{V}}}((u_0, \alpha_0) - \mathcal{T}(v_1, \beta_1)) > 1 - \delta_2$  or equivalently  $(u_0, \alpha_0) - \mathcal{T}(v_1, \beta_1) \in \check{\mathcal{B}}(0, \delta_2) \subset \overline{\mathcal{T}(\check{\mathcal{B}}(0, \varepsilon_1))}$  and by a similar case there exist  $(v_2, \beta_2)$  in  $\check{\mathcal{B}}(0, \varepsilon_2)$  such that  $\check{\mathcal{F}}_{\check{\mathcal{V}}}((u_0, \alpha_0) - (\mathcal{T}(v_1, \beta_1) + \mathcal{T}(v_2, \beta_2))) = \check{\mathcal{F}}_{\check{\mathcal{V}}}(((u_0, \alpha_0) - (\mathcal{T}(v_1, \beta_1)) - \mathcal{T}(v_2, \beta_2))) > 1 - \delta_3$ .

By continuing this process a sequence  $\{(v_n, \beta_n)\}$  is obtained such that  $(v_n, \beta_n) \in \check{\mathcal{B}}(0, \varepsilon_n)$  and

$$\check{\mathcal{F}}_{\check{\mathcal{V}}}\left((u_0, \alpha_0) - \sum_{j=1}^{n-1} \mathcal{T}(v_j, \beta_j)\right) > (1 - \delta_{n+1}), n = 1, 2, 3, \dots$$

Now if  $n \in N$  and  $\{(z_n, \beta_n)\}$  is increasing sequence, then

$$\begin{aligned} \check{\mathcal{F}}_{\check{\mathcal{U}}}\left(\sum_{j=1}^n (v_j, \beta_j) - \sum_{j=1}^{n+(z_n, \beta_n)} (v_j, \beta_j)\right) &= \check{\mathcal{F}}_{\check{\mathcal{U}}}\left(\sum_{j=n+1}^{n+(z_n, \beta_n)} (v_j, \beta_j)\right) \\ &\geq \check{\mathcal{F}}_{\check{\mathcal{U}}}((v_{n+1}, \beta_{n+1})) \odot \check{\mathcal{F}}_{\check{\mathcal{U}}}((v_{n+2}, \beta_{n+2})) \odot \dots \odot \check{\mathcal{F}}_{\check{\mathcal{U}}}((v_{n+(z_n, \beta_n)}, \beta_{(z_n, \beta_n)})) \end{aligned}$$

$$> (1 - \varepsilon_{n+1}) \circledast (1 - \varepsilon_{n+2}) \circledast \dots \circledast (1 - \varepsilon_{n+(z_n, \beta_n)})$$

Put  $(1 - \varepsilon_{n+1}) \circledast (1 - \varepsilon_{n+2}) \circledast \dots \circledast (1 - \varepsilon_{n+(z_n, \beta_n)}) = (1 - \mu)$  for some  $0 < \mu < 1$ , therefore  $\check{F}_{\check{U}}\left(\sum_{j=n+1}^{n+(z_n, \beta_n)}(\nu_j, \beta_j)\right) > (1 - \mu)$ . So the sequence  $\{\sum_{j=1}^n(\nu_j, \beta_j)\}$  is a fuzzy Cauchy sequence.  $\{\sum_{j=1}^n(\nu_j, \beta_j)\}$  converges to some fuzzy point  $(\nu_1, \beta_1) \in \check{U}$  because by the assumption  $\check{U}$  is complete. Now for  $n, n_0 \in N$  and  $n > n_0$ ,  $\check{F}_{\check{V}}\left((u_0, \alpha_0) - \sum_{j=1}^{n-1} \mathcal{T}(\nu_j, \beta_j)\right) \geq (1 - \delta_n)$  and thus  $\check{F}_{\check{V}}\left((u_0, \alpha_0) - \sum_{j=1}^{n-1} \mathcal{T}(\nu_j, \beta_j)\right) \rightarrow 1$ . Therefore,  $(u_0, \alpha_0) = \lim_n \sum_{j=1}^{n-1} \mathcal{T}(\nu_j, \beta_j) = \mathcal{T}(\nu_0, \beta_0)$ .

But

$$\begin{aligned} \check{F}_{\check{U}}(\nu_0, \beta_0) &\geq \limsup_n \check{F}_{\check{U}}\left(\sum_{j=1}^n(\nu_j, \beta_j)\right) \geq \limsup_n [\check{F}_{\check{U}}((\nu_1, \beta_1)) \circledast \dots \circledast \check{F}_{\check{U}}((\nu_n, \beta_n))] \\ &\geq \limsup_n [(1 - \varepsilon_1) \circledast \dots \circledast (1 - \varepsilon_n)] \geq (1 - \delta) \text{ or some } 0 < \delta < 1. \end{aligned}$$

Hence  $(\nu_0, \beta_0) \in \check{B}(0, \delta)$ .

**Third Step.** Let  $\check{C}$  be a fuzzy open subset of  $\check{U}$  and  $(\nu, \beta) \in \check{C}$ . Then,  $\mathcal{T}(\check{C}) = \mathcal{T}(\nu, \beta) + \mathcal{T}[\check{C} - (\nu, \beta)] \supset \mathcal{T}(\nu, \beta) + \text{int}[\mathcal{T}[\check{C} - (\nu, \beta)]]$ .

Hence  $\mathcal{T}(\check{C})$  will be a fuzzy open because it contains a neighborhood of all its fuzzy points. Now if  $\mathcal{T}^{-1}: \check{V} \rightarrow \check{U}$  exists it is continuous since  $\mathcal{T}$  is fuzzy open. Because  $\mathcal{T}^{-1}$  is linear hence by ([14], Theorem 3.4) it is fuzzy bounded.

#### Definition 3.4:

Let  $(\check{U}, \check{F}_{\check{U}}, \circledast)$  and  $(\check{V}, \check{F}_{\check{V}}, \circledast)$  are  $\mathcal{FL}$  – spaces and  $\mathcal{T}: \mathfrak{D}(\mathcal{T}) \rightarrow \check{V}$  a linear operator with  $\mathfrak{D}(\mathcal{T})$  is a fuzzy subset of  $\check{U}$ . The operator  $\mathcal{T}$  is said to be fuzzy closed linear operator if the graph of  $\mathcal{T}$

$\Gamma(\mathcal{T}) = \{((u, \alpha), (\nu, \beta)) : (u, \alpha) \in \mathfrak{D}(\mathcal{T}), (\nu, \beta) = \mathcal{T}(\nu, \beta)\}$  is closed in an  $\mathcal{FL}$ -space  $(\check{U} \times \check{V}, \check{F}, \circledast)$  where  $\check{F}((u, \alpha), (\nu, \beta)) = \check{F}_{\check{U}}((u, \alpha)) \circledast \check{F}_{\check{V}}((\nu, \beta))$ .

Let  $(\check{U}, \check{F}_{\check{U}}, \circledast)$  and  $(\check{V}, \check{F}_{\check{V}}, \circledast)$  be two  $\mathcal{FL}$  – space and  $\mathcal{T}: D(\mathcal{T}) \rightarrow \check{V}$  is a linear operator where  $D(\mathcal{T}) \subset \check{U}$ . The operator is called fuzzy bounded if there is  $0 < r < 1$  ( $r$  real number) such that for all  $(u, \alpha) \in D(\mathcal{T})$ ,  $\check{F}_{\check{V}}(\mathcal{T}((u, \alpha))) > (1 - r)$  where  $D(\mathcal{T})$  is the domain of  $\mathcal{T}$  [14].

Now fuzzy closed graph theorem is proved in the fuzzy length space.

#### Theorem (Fuzzy Closed Graph Theorem) 3.5:

Let  $(\check{U}, \check{F}_{\check{U}}, \circledast)$  and  $(\check{V}, \check{F}_{\check{V}}, \circledast)$  be complete  $\mathcal{FL}$  – spaces and  $\mathcal{T}: \mathfrak{D}(\mathcal{T}) \rightarrow \check{V}$  be a fuzzy closed linear operator, where  $\mathfrak{D}(\mathcal{T}) \subset \check{U}$ . If  $\mathfrak{D}(\mathcal{T})$  fuzzy closed in  $\check{U}$ , then the operator  $\mathcal{T}$  is fuzzy bounded.

**Proof:** Proof of theorem requires first prove that  $(\check{U} \times \check{V}, \check{F}, \circledast)$  is a complete  $\mathcal{FL}$ -space. Let  $\{(u_n, \alpha_n), (\nu_n, \beta_n)\}$  be a fuzzy Cauchy sequence in  $\check{U} \times \check{V}$ , that is for each

$$((u, \alpha), (\nu, \beta)) \in \check{U} \times \check{V}, \lim_{n \rightarrow \infty} \check{F}[(u_n, \alpha_n), (\nu_n, \beta_n)] - ((u_m, \alpha_m), (\nu_m, \beta_m))] = \lim_{n \rightarrow \infty} \check{F}_{\check{U}}[(u_n, \alpha_n) - (u_m, \alpha_m)] \circledast \lim_{n \rightarrow \infty} \check{F}_{\check{V}}[(\nu_n, \beta_n) - (\nu_m, \beta_m)].$$

Hence

$$\lim_{n \rightarrow \infty} \check{F}_{\check{U}}[(u_n, \alpha_n) - (u_m, \alpha_m)] = 1 \text{ and } \lim_{n \rightarrow \infty} \check{F}_{\check{V}}[(v_n, \beta_n) - (v_m, \beta_m)] = 1.$$

Therefore  $\{(u_n, \alpha_n)\}$  is a fuzzy Cauchy sequence in  $(\check{U}, \check{F}_{\check{U}}, \odot)$  and  $\{(v_n, \beta_n)\}$  is a fuzzy Cauchy sequence in  $(\check{V}, \check{F}_{\check{V}}, \odot)$ . But  $(\check{U}, \check{F}_{\check{U}}, \odot)$  and  $(\check{V}, \check{F}_{\check{V}}, \odot)$  are complete  $\mathcal{FL}$ -space, then there is a fuzzy point  $(u, \alpha) \in \check{U}$  and  $(v, \beta) \in \check{V}$  such that  $\lim_{n \rightarrow \infty} \check{F}_{\check{U}}[(u_n, \alpha_n) - (u, \alpha)] = 1$  and  $\lim_{n \rightarrow \infty} \check{F}_{\check{V}}[(v_n, \beta_n) - (v, \beta)] = 1$ .

Thus

$$\lim_{n \rightarrow \infty} \check{F}[(u_n, \alpha_n), (v_n, \beta_n) - ((u, \alpha), (v, \beta))] = \lim_{n \rightarrow \infty} \check{F}_{\check{U}}[(u_n, \alpha_n) - (u, \alpha)] \odot \lim_{n \rightarrow \infty} \check{F}_{\check{V}}[(v_n, \beta_n) - (v, \beta)] = 1.$$

Hence  $\{(u_n, \alpha_n), (v_n, \beta_n)\}$  fuzzy converges to  $((u, \alpha), (v, \beta))$  in  $\check{U} \times \check{V}$ . This shows  $(\check{U} \times \check{V}, \check{F}, \odot)$  is a complete an  $\mathcal{FL}$ -space.

By assumption,  $\Gamma(\mathcal{T})$  is closed in  $\check{U} \times \check{V}$  and  $\mathfrak{D}(\mathcal{T})$  fuzzy closed in  $\check{U}$ . Hence  $\Gamma(\mathcal{T})$  and  $\mathfrak{D}(\mathcal{T})$  are complete[15]. Now define  $S: \Gamma(\mathcal{T}) \rightarrow \mathfrak{D}(\mathcal{T})$  by  $S[(u, \alpha), \mathcal{T}(u, \alpha)] = (u, \alpha)$  then  $S$  is linear and  $S$  is a fuzzy bounded since

$$\begin{aligned} \check{F}(S[(u, \alpha), \mathcal{T}(u, \alpha)]) &= \check{F}_{\check{U}}((u, \alpha)) \geq \check{F}_{\check{U}}((u, \alpha)) \odot \check{F}_{\check{V}}(\mathcal{T}(u, \alpha)) \\ &= \check{F}((u, \alpha), \mathcal{T}(u, \alpha)) \end{aligned}$$

and  $S$  is bijective so  $S^{-1}$  exists where  $S^{-1}: \mathfrak{D}(\mathcal{T}) \rightarrow \Gamma(\mathcal{T})$  defined by  $S^{-1}((u, \alpha))$  since  $\mathfrak{D}(\mathcal{T})$  and  $\Gamma(\mathcal{T})$  are complete, it is possible to apply Theorem 3.3 and see  $S^{-1}$  is fuzzy bounded, say  $\check{F}((u, \alpha), \mathcal{T}(u, \alpha)) \geq (1 - r) \odot \check{F}_{\check{U}}((u, \alpha))$  where  $0 < r < 1$  and for all  $(u, \alpha) \in \mathfrak{D}(\mathcal{T})$ . Hence  $\mathcal{T}$  is fuzzy bounded because

$$\begin{aligned} \check{F}_{\check{V}}(\mathcal{T}(u, \alpha)) &\geq \check{F}_{\check{V}}(\mathcal{T}(u, \alpha)) \odot \check{F}_{\check{U}}((u, \alpha)) = \check{F}((u, \alpha), \mathcal{T}(u, \alpha)) \\ &\geq (1 - r) \odot \check{F}_{\check{U}}((u, \alpha)) \text{ for all } (u, \alpha) \in \mathfrak{D}(\mathcal{T}). \end{aligned}$$

Let  $(\check{U}, \check{F}, \odot)$  be a  $\mathcal{FL}$ -space and  $\check{C} \subseteq \check{U}$ . Then  $(v, \beta) \in \check{C}$  if and only if there exists a sequence  $\{(v_n, \beta_n)\}$  in  $\check{C}$  with  $\{(v_n, \beta_n)\}$  converges to  $(v, \beta)$ .

### Lemma 3.6:

Let  $(\check{U}, \check{F}_{\check{U}}, \odot)$  and  $(\check{V}, \check{F}_{\check{V}}, \odot)$  be two  $\mathcal{FL}$  - spaces and let  $\mathcal{T}: \mathfrak{D}(\mathcal{T}) \rightarrow \check{V}$  be a linear operator with  $\mathfrak{D}(\mathcal{T}) \subset \check{U}$  then  $\mathcal{T}$  is fuzzy closed if and only if when a sequence  $\{(u_n, \alpha_n)\}$  converges to  $(u, \alpha)$  where  $(u_n, \alpha_n) \in \mathfrak{D}(\mathcal{T})$  and  $\{\mathcal{T}(u_n, \alpha_n)\}$  converges to  $(v, \beta)$  then  $(u, \alpha) \in \mathfrak{D}(\mathcal{T})$  and  $\mathcal{T}(u, \alpha) = (v, \beta)$  where  $(v, \beta) \in \check{V}$ .

**Proof:**  $\Gamma(\mathcal{T})$  is fuzzy closed if and only if  $(c, \gamma) = ((u, \alpha), (v, \beta)) \in \overline{\Gamma(\mathcal{T})}$  implies  $(c, \gamma) \in \Gamma(\mathcal{T})$ . Now by Theorem 3.5,  $(c, \gamma) \in \overline{\Gamma(\mathcal{T})}$  if and only if there exist  $(c_n, \gamma_n) = ((u_n, \alpha_n), \mathcal{T}(u_n, \alpha_n)) \in \Gamma(\mathcal{T})$  such that the sequence  $\{(c_n, \gamma_n)\}$  converges to  $(c, \gamma)$  hence  $\{(u_n, \alpha_n)\}$  converges to  $(u, \alpha)$ ,  $\{\mathcal{T}(u_n, \alpha_n)\}$  converges to  $(v, \beta)$  and  $(c, \gamma) = ((u, \alpha), (v, \beta)) \in \Gamma(\mathcal{T})$  if and only if  $(u, \alpha) \in \mathfrak{D}(\mathcal{T})$  and  $(v, \beta) = \mathcal{T}(u, \alpha)$ .

The following result according to the previous lemma is proved.

### Theorem 3.7:

Let  $(\check{U}, \check{F}_{\check{U}}, \odot)$  and  $(\check{V}, \check{F}_{\check{V}}, \odot)$  be two  $\mathcal{FL}$  - spaces and  $\mathcal{T}: \mathfrak{D}(\mathcal{T}) \rightarrow \check{V}$  be a linear operator with  $\mathfrak{D}(\mathcal{T}) \subset \check{U}$  then:

1-If  $\mathfrak{D}(\mathcal{T})$  is fuzzy closed in  $\check{U}$  then  $\mathcal{T}$  is fuzzy closed

2-If  $\mathcal{T}$  is fuzzy closed and  $\check{V}$  is complete  $\mathcal{FL}$  - space then  $\mathfrak{D}(\mathcal{T})$  is fuzzy closed in  $\check{U}$ .

**Proof:**

1-Suppose that  $\{(u_n, \alpha_n)\} \in \mathfrak{D}(\mathcal{T})$  and  $\{(u_n, \alpha_n)\}$  converges to  $(u, \alpha)$  such that  $\{\mathcal{T}(u_n, \alpha_n)\}$  converges to  $\mathcal{T}(u, \alpha)$  then  $(u, \alpha) \in \overline{\mathfrak{D}(\mathcal{T})} = \mathfrak{D}(\mathcal{T})$  since  $\mathfrak{D}(\mathcal{T})$  is fuzzy closed and sequence  $\{\mathcal{T}(u_n, \alpha_n)\}$  converges to  $\mathcal{T}(u, \alpha)$  because  $\mathcal{T}$  is fuzzy continuous. Hence  $\mathcal{T}$  is fuzzy closed by Lemma 3.6.

2-Let  $(u, \alpha) \in \overline{\mathfrak{D}(\mathcal{T})}$  there is a sequence  $\{(u_n, \alpha_n)\}$  in  $\mathfrak{D}(\mathcal{T})$  with  $\{(u_n, \alpha_n)\}$  converges to  $(u, \alpha)$  since  $\mathcal{T}$  is fuzzy bounded

$$\begin{aligned} \check{\mathcal{F}}_{\check{\nu}} [\mathcal{T}(u_n, \alpha_n) - \mathcal{T}(u_m, \alpha_m)] &= \check{\mathcal{F}}_{\check{\nu}} [\mathcal{T}((u_n, \alpha_n) - (u_m, \alpha_m))] \\ &\geq \check{\mathcal{F}}[\mathcal{T}] \circledast \check{\mathcal{F}}_{\check{u}} [(u_n, \alpha_n) - (u_m, \alpha_m)]. \end{aligned}$$

This show that  $\{\mathcal{T}(u_n, \alpha_n)\}$  is fuzzy Cauchy sequence so  $\{\mathcal{T}(u_n, \alpha_n)\}$  converges to  $(\nu, \beta) \in \check{\mathcal{V}}$  since  $\check{\mathcal{V}}$  is complete. Also since  $\mathcal{T}$  is fuzzy closed  $(u, \alpha) \in \mathfrak{D}(\mathcal{T})$  and  $\mathcal{T}(u, \alpha) = (\nu, \beta)$ . Hence  $\mathfrak{D}(\mathcal{T})$  is fuzzy closed because  $(u, \alpha) \in \overline{\mathfrak{D}(\mathcal{T})}$  was arbitrary.

**4. Conclusion**

This paper the definition of open linear operators and closed linear operators in the fuzzy length space is introduced, and fuzzy open mapping theorem and fuzzy closed graph theorem are proved. Moreover, some basic properties of this type of operators are investigated.

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