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ABSTRACT

In this paper, we will give some results about Frobenius number, gaps, and determine number of arf closure of telescopic numerical semigroup S_k such that

$$S_k = \langle 8, 8k+2, x \rangle$$

where $k \geq 1$, $k \in \mathbb{Z}$, $j \neq 0, 2, 4, ..., 2(k-1)$ and $x = 8k+2+(2j+1)$ is odd integer number.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of nonnegative integers and integers, respectively. The subset S of $\mathbb{N}$ is a numerical semigroup if $0 \in S$, $x+y \in S$, *for all* $x, y \in S$, and $Card(\mathbb{N} \backslash S) < \infty$ (this condition is equivalent to $\gcd(S) = 1$, $\gcd(S) =$ greatest common divisor the element of S) . Let S be a numerical semigroup, then $F(S) = \max(\mathbb{Z} \backslash S)$ and $m(S) = \min\{s \in S : s > 0\}$ are called Frobenius number and multiplicity of S, respectively. Also, $n(S) = Card(\{0,1,2,...,F(S)\} \cap S)$ is called the number determine of S. If $F(S) - x \in S$ then is called symmetric numerical semigroup, for all $x \in \mathbb{Z} \backslash S$. It is known that $S = < a, b >$ is symmetric numerical semigroup, and if S is a symmetric numerical semigroup then $n(S) = G(S) = \frac{F(S)+1}{2}$ (for details see [1], [6]).

If S is a numerical semigroup such that $S = < a_1, a_2, ..., a_n >$, then we observe that

$$S = < a_1, a_2, ..., a_n > = \{s_0 = 0, s_, , s_2, ..., s_{n-1}, s_n = F(S)+1, \rightarrow ...\}$$

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where $s_i < s_{i+1},\ n = n(S)$, and the arrow means that every integer greater than $F(S)+1$ belongs to S, for $i = 1,2,...,n = n(S)$.

If $x \in \mathbb{N}$ and $x \notin S$, then x is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S) = \mathbb{N} \backslash S$ and, the $G(S) = Card(H(S))$ is called the genus of S. Also, It is know that $G(S) = F(S) + 1 - n(S)$. Let $S = < s_1, s_2, s_3 >$ is a triply-generated telescopic numerical semigroup if $s_3 \in < \frac{s_1}{d}, \frac{s_2}{d} >$ where $d = \gcd(s_1 s_2)$ (see [3],[5],[7]). If S is a numerical semigroup such that $S = < a_1, a_2, ..., a_n >$, then $L(S) = \langle a_1, a_2 - a_1, a_3 - a_1, ..., a_n - a_1 \rangle$ is called Lipman numerical semigroup of S, and it is known that

$$L_0(S) = S \subseteq L_1(S) = L(L_0(S)) \subseteq L_2 = L(L_1(S)) \subseteq ... \subseteq L_m = L(L_{m-1}(S)) \subseteq ... \subseteq \mathbb{N}.$$

A numerical semigroup S is Arf if $a + b - c \in S$, for all $a, b, c \in S$ such that $a \geq b \geq c$. The intersection of any family of Arf numerical semigroups is again an Arf numerical semigroup. Thus, since $\mathbb{N}$ is an Arf numerical semigroup, one can consider the smallest Arf numerical semigroup containing a given numerical semigroup. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and it is denoted by $Arf(S)$ (see [4],[6]).

In this paper, we will give some results about Frobenius number, gaps, and determine number of Arf closure of telescopic numerical semigroup S_k such that $S_k = \langle 8, 8k+2, x \rangle$ where $k \geq 1, k \in \mathbb{Z}$, $j \neq 0, 2, 4, ..., 2(k-1)$ and $x = 8k + 2 + (2j+1)$ is odd integer number. We note that any telescopic numerical semigroup is not symmetric. For example, $S = \langle 6, 9, 23 \rangle$ is telescopic numerical semigroup but it is not symmetric since $F(S) = 40$ and for $x = 3$ $F(S) - x = 37 \notin S$. But, here $S_k = \langle 8, 8k+2, x \rangle$ is symmetric numerical semigroup where $k \geq 1, k \in \mathbb{Z}$.

2. Main Results

Proposition 1. ([8]) $S_k = \langle 8, 8k+2, x \rangle$ is a telescopic numerical semigroups where $k \geq 1,\ k \in \mathbb{Z},\ j \neq 0, 2, 4, ..., 2(k-1)$ and $x = 8k + 2 + (2j+1)$ is odd integer number.

In this study, we will take $j = 1$ in $S_k = \langle 8, 8k+2, x \rangle$, i.e., $S_k = \langle 8, 8k+2, 8k+5 \rangle$.

Proposition 2. ([2]) Let $S = < u_1, u_2, ..., u_n >$ be a numerical semigroup and $d = \gcd\{u_1, u_2, ..., u_{n-1}\}$. If $T = < \frac{u_1}{d}, \frac{u_2}{d}, ..., \frac{u_{n-1}}{d} >$ numerical semigroup then

(a) $F(S) = d.F(T) + (d-1).u_n$

(b) $G(S) = d.G(T) + \frac{(d-1)(u_n - 1)}{2}$.

Proposition 3. Let $S_k = \langle 8, 8k+2, 8k+5 \rangle$ be a telescopic numerical semigroup, where $k \geq 1,\ k \in \mathbb{Z}$. Then, we have

(a) $F(S_k) = 32k+3$

(b) $n(S_k) = 16k+2$

(c) $G(S_k) = 16k+2$.

Proof. (a) We find that $F(T) = 16k+4-4k-4-1 = 12k-1$ since $d = \gcd\{8, 8k+2\} = 2$ and $T = <\frac{8}{2}, \frac{8k+2}{2}> = <4, 4k+1>$, where $k \geq 1,\ k \in \mathbb{Z}$. In this case, we obtain that $F(S) = 2(12k-1)+(2-1).(8k+5) = 32k+3$ from Proposition 2/(1).

(b)-(c) It is trivial $n(S) = G(S) = \frac{F(S)+1}{2} = \frac{32k+4}{2} = 16k+2$ from S_k is symmetric numerical semigroup.

Theorem 1. Let $S_k = \langle 8, 8k+2, x \rangle$ be a telescopic numerical semigroup, where $k \geq 1,\ k \in \mathbb{Z}$. Then, $Arf(S) = \{0, 8, 16, 24, ..., 8k, 8k+2, x-1, \rightarrow ...\}$.

Proof. It is trivial $m_0 = 8$ since $L_0(S) = S$. Thus, we write $L_1(S) = <8, 8k-6, x-8>$. In this case,

(1) If $8k-6 < 8$ (if $k=1$) then we obtain $L_1(S) = <8, 8k-6, x-8> = <2, x-8>,\ m_1 = 2$ and we have $L_2(S) = <2, x-10>$.

In here, $x-10 > 2$ and $m_2 = 2$. So, we have $L_3(S) = <2, x-12>$.

In here, if $x-12 < 2$ (if $x=13$) then $L_3(S) = <2,1> = <1> = \mathbb{N},\ m_3 = 1$.

If $x-12 > 2$ then we find that $m_3 = 2$ since $L_3(S) = <2, x-12>$ .If we are continued , we have that $L_i(S) = <2, x-2(i+3)>$ and $m_i = 2$ or $m_i = 1$, for $i \geq 1$. Thus, we obtain

$$Arf(S) = \{0, 8, 16, 24, ..., 8k, 8k+2, x-1, \rightarrow ...\}.$$

(2) If $8k-6 > 8$ then $m_1 = 8$, and we have $L_2(S) = <8, 8k-14, x-16>$. In this case, if $8k-14 < 8$ (if $k=2$) then $L_2(S) = <8, 8k-14, x-16> = <8, 2, x-16> = <2, x-16>$ and $m_2 = 2$ from $x-16 > 2$. Thus, we have $L_3(S) = <2, x-18>$. In here,

if $x-18 < 2$ (if $x=13$) then $L_3(S) = <2,1> = <1>,\ m_3 = 1$.

if $x-18 > 2$ then we write that $m_3 = 2$ since $L_3(S) = <2, x-18>$.

If we are continued , we have that $L_i(S) = <2, x-2(i+6)>$, and $m_i = 2$ or $m_i = 1$, for $i \geq 2$. So, we obtain $Arf(S) = \{0, 8, 16, 24, ..., 8k, 8k+2, x-1, \rightarrow ...\}$.

Corollary 1 . Let $S_k = \langle 8, 8k+2, 8k+5 \rangle$ be a telescopic numerical semigroup, where $k \geq 1$, $k \in \mathbb{Z}$. Then, we have

(a) $F(Arf(S_k)) = x - 2 = 8k + 3$

(b) $n(Arf(S_k)) = k + 2$

(c) $G(Arf(S_k)) = 7k + 2$.

Proof. (a) It is clear.

(b) Let A_1 and A_2 be the cardinalities of the subsets $\{8, 16, 24, ..., 8k\}$ and $\{4k+2, x-1\}$ of $Arf(S) = \{0, 8, 16, 24, ..., 8k, 8k+2, x-1, \rightarrow ...\}$, respectively. In this case, we have $A_1 = \frac{8k-8}{8} + 1 = k$ and $A_2 = 2$. Thus, we obtain $n(Arf(S_k)) = A_1 + A_1 = k + 2$.

(c) $G(Arf(S_k)) = F(Arf(S_k)) + 1 - n(Arf(S_k)) = 8k + 3 + 1 - (k+2) = 7k + 2$.

Corollary 2. Let $S_k = \langle 8, 8k+2, 8k+5 \rangle$ be a telescopic numerical semigroup, where $k \geq 1$, $k \in \mathbb{Z}$. Then, we have

(a) $F(S_k) = F(Arf(S_k)) + 24k$

(b) $n(S_k) = n(Arf(S_k)) + 15k$

(c) $G(S_k) = G(Arf(S_k)) + 9k$

Proof. It is trivial from Proposition 3 and Corollary 1.

The following corollaries are satisfied from Propositions 3 and Corollary 1:

Corollary 3. Let $S_k = \langle 8, 8k+2, 8k+5 \rangle$ be a telescopic numerical semigroup where $k \geq 1$, $k \in \mathbb{Z}$. Then, it satisfies following equalities:

(a) $F(S_{k+1}) = F(S_k) + 32$

(b) $n(S_{k+1}) = n(S_k) + 16$

(c) $G(S_{k+1}) = G(S_k) + 16$.

Corollary 4. Let $S_k = \langle 8, 8k+2, 8k+5 \rangle$ be a telescopic numerical semigroup, where $k \geq 1$, $k \in \mathbb{Z}$. Then, we have :

(a) $F(Arf(S_{k+1})) = F(Arf(S_k)) + 8$

(b) $n(Arf(S_{k+1})) = n(Arf(S_k)) + 1$

(c) $G(Arf(S_{k+1})) = G(Arf(S_k)) + 7$.

Example 7. We put $k = 1$ in $S_k = \langle 8, 8k+2, 8k+5 \rangle$ triply-generated telescopic numerical semigroups. Then we have

$$S_1 = <8,10,13> = \{0,8,10,13,16,18,20,21,23,24,26,28,29,30,31,32,33,34,36,\rightarrow ...\}.$$

In this case, we obtain

$$F(S_1) = 35,\ n(S_1) = 18,\ H(S_1) = \{1,2,3,4,5,6,7,9,11,12,14,15,17,19,22,25,27,35\},$$

$$G(S_1) = 18,\ Arf(S_1) = \{0,8,10,12,\rightarrow ...\},$$

$$F(Arf(S_1)) = 11,\ H(Arf(S_1)) = \{1,2,3,4,5,6,7,9,11\}$$

$G(Arf(S_1)) = 9$ and $n(Arf(S_1)) = 3$.

If $k = 2$ then we write in $S_k = \langle 8, 8k+2, 8k+5 \rangle$ triply-generated telescopic numerical semigroups. Then we write

$$S_2 = <8,18,21> = \{0,8,16,18,21,24,26,29,32,34,36,37,39,40,42,44,45,47,48,50,52,...,60,61,...,66,68 \rightarrow ...\}.$$

Thus, we have

$$F(S_2) = 67,\ n(S_2) = 34,\ G(S_2) = 34,\ Arf(S_2) = \{0,8,16,18,20,\rightarrow ...\},$$

$F(Arf(S_2)) = 19$, $n(Arf(S_2)) = 4$ and $G(Arf(S_2)) = 16$.

So, we obtain

$$G(Arf(S_1)) + 9 = 9 + 9 = 18 = G(S_1),$$

$$F(Arf(S_1)) + 24 = 11 + 24 = 35 = F(S_1),$$

$$n(Arf(S_1)) + 15 = 3 + 15 = 18 = n(S_1),$$

$$F(S_1) + 32 = 35 + 32 = 67 = F(S_2),$$

$$n(S_1) + 16 = 18 + 16 = 34 = n(S_2),$$

$G(S_1) + 16 = 18 + 16 = 34 = G(S_2)$ and

$$F(Arf(S_1)) + 8 = 11 + 8 = 19 = F(Arf(S_2)),$$

$$n(Arf(S_1)) + 1 = 3 + 1 = 4 = n(Arf(S_2)),$$

$$G(Arf(S_1)) + 7 = 9 + 7 = 16 = G(Arf(S_2)).$$

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