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ABSTRACT

Our aim in this article is to introduce the notion of algebra fuzzy absolute value in order to introduce new type of fuzzy normed space that is called algebra fuzzy normed space. After that some examples are introduced to illustrate these notions. Then some properties of algebra fuzzy absolute value space are stated and proved after that basic properties of algebra fuzzy normed space are also proved.

MSC: 30C45, 30C50

1. Introduction

In 2009 [14] the fuzzy topological structure of a fuzzy normed space was studied by Sadeqi and Kia. In 2011 [2] Kider introduced a fuzzy normed space and also he proved such fuzzy normed space has a completion see [3]. Again in 2012 [4] Kider introduced a new type of fuzzy normed space. In 2015 [13] the properties of fuzzy continuous mapping which is defined on a fuzzy normed spaces was studied by Nadaban.

In 2017 [5] Kider and Kadhum introduce the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties was proved by Kadhum in 2017 [12]. In 2018 [1] Ali proved basic properties of complete fuzzy normed algebra. Again in 2018 [6] Kider and Ali introduce the notion of fuzzy absolute value and study properties of finite dimensional fuzzy normed space.

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The concept of general fuzzy normed space were presented by Kider and Gheeab in 2019 [7] [8] and they proved basic properties of this space and the general fuzzy normed space GFB(V, U). In 2019 [9] Kider and Kadhum introduce the notion fuzzy compact linear operator and proved its basic properties. In 2020 Kider [10] introduce the notion fuzzy soft metric space after that he investigated and proved some basic properties of this space again Kider in 2020 [11] introduced new type of fuzzy metric space called algebra fuzzy metric space after that the basic properties of this space are proved.

In this paper first we introduce the notion of algebra fuzzy absolute value space and we see that there are more than one definition of fuzzy absolute value for a real number after that some properties of this space is proved because it needed in the next section. The aim of introducing algebra fuzzy absolute value is to introduce a new type of fuzzy normed space which is called algebra fuzzy normed space at this point basic properties of this space is proved.

2. Algebra fuzzy absolute value with its basic properties:

In this section we shall introduced basic properties of algebra fuzzy absolute value space.

Definition 2.1:[11]

Assume that S be a nonempty set, a fuzzy set $\widetilde{D}$ in S is represented by $\widetilde{D} = \{(S, \mu_{\widetilde{D}}(s)): s \in S,$

$0 \leq \mu_{\widetilde{D}}(s) \leq 1\}$ where $\mu_{\widetilde{D}}(s): S \to I$ is a membership function where I=[0, 1].

Definition 2.2:[11]

Let $\odot$:I × I →I be a binary operation then $\odot$ is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions

(i)p $\odot$ q= q $\odot$ p;

(ii) p $\odot$ [q $\odot$ w] = [p $\odot$ q] $\odot$ w;

(iii) $\odot$ is continuous function;

(iv)p $\odot$ 0 = 0, (v) (p $\odot$ z) ≤ (q $\odot$ w) whenever p ≤ q and z ≤ w. For all p, q, z, w ∈ I

Lemma 2.3:[11]

If $\odot$ is a continuous t-conorm on [0, 1] then

(i)1 $\odot$ 1=1, (ii)0 $\odot$ 1 = 1 $\odot$ 0 =1. (iii)0 $\odot$ 0 = 0, (iv)p $\odot$ p ≥ p for all p ∈ [0, 1].

Remark 2.4:[11]

If $\odot$ is a continuous t-conrm then

(i)For any p, q ∈ (0, 1) with p > q we have w ∈ (0, 1) whenever p > q $\odot$ w. In general for any p, q ∈ (0, 1) with p > q we can find $w_1, w_2, \ldots, w_k$ ∈ (0, 1) whenever p > q $\odot$ $w_1 \odot w_2 \odot \ldots \odot w_k$ where k ∈ ℕ.

(ii) For any p ∈ (0, 1) there exists q ∈ (0, 1) such that q $\odot$ q ≤ p. In general for any p ∈ (0, 1) there exists $w_1, w_2, \ldots, w_k$ ∈ (0, 1) such that $w_1 \odot w_2 \odot \ldots \odot w_k$ ≤ p where k ∈ ℕ.

Example 2.5:[11]

The algebra product p $\odot$ q = p + q – pq is a continuous t-conorm for all p, q ∈ [0, 1].

Definition 2.6:

Let $\odot$ be a continuous t-conorm and a:ℝ →I be a fuzzy set then a is called **algebra fuzzy absolute value on** ℝ if

(1)0< a(α) ≤1;

(2)a(α)=0 if and only if α=0;

(3)a($\alpha\beta$) ≤ a(α) . a(β);

(4)a($\alpha + \beta$) ≤ a(α) $\odot$ a(β);

For all $\alpha, \beta \in \mathbb{R}$. Then $(\mathbb{R}, a, \circledcirc)$ is called **algebra fuzzy absolute value space.**

Definition 2.7:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space then define $\mathbf{a(\alpha)= a(-\alpha)}$ for all $\alpha \in \mathbb{R}$.

Example 2.8:

Let $|.|$ be absolute value on $\mathbb{R}$ and $\alpha \circledcirc \beta= \alpha+ \beta - \alpha\, \beta$ for all $\alpha, \beta \in$ I. Define

$$a_{|.|}(\alpha) = \begin{cases} \frac{|\alpha|}{1+|\alpha|} & if\ \alpha \neq 1 \\ 1 & if\ \alpha = 1 \end{cases}$$

for all $\alpha \in \mathbb{R}$. Then $(\mathbb{R}, a_{|.|}, \circledcirc)$ is algebra fuzzy absolute value space. Also $a_{|.|}$ is called the **standard algebra fuzzy absolute value on $\mathbb{R}$.**

Proof:

We shall show that all conditions of Definition 2.6 are satisfied

(1)It is clear that $0\leq a_{|.|} \leq 1$.

(2) $a_{|.|}(\alpha)=0$ if and only if $\frac{|\alpha|}{1+|\alpha|} = 0$ if and only if $|\alpha|=0$ if and only if $\alpha=0$.

(3) $a(\alpha\beta) = \frac{|\alpha\beta|}{1+|\alpha\beta|} \leq [\frac{|\alpha|}{1+|\alpha|}].[\frac{|\beta|}{1+|\beta|}] = a(\alpha)\, a(\beta)$

(4) $a(\alpha + \beta) = \frac{|\alpha+\beta|}{1+|\alpha+\beta|}$ and $a(\alpha) \circledcirc a(\beta) = a(\alpha) + a(\beta) - a(\alpha)\, a(\beta)$. Now

$a(\alpha) \circledcirc a(\beta) = [\frac{|\alpha|}{1+|\alpha|}]+[\frac{|\beta|}{1+|\beta|}] - [\frac{|\alpha|}{1+|\alpha|}].[\frac{|\beta|}{1+|\beta|}] = \frac{|\alpha|[[1+|\beta|]+|\beta|[[1+|\alpha|]}{[1+|\alpha|][1+|\beta|]} - \frac{|\alpha||\beta|}{[1+|\alpha|][1+|\beta|]}$

$= \frac{|\alpha|+|\alpha||\beta|+|\beta|+|\beta||\alpha|}{[1+|\alpha|][1+|\beta|]} - \frac{|\alpha||\beta|}{[1+|\alpha|][1+|\beta|]} = \frac{|\alpha|+|\alpha||\beta|+|\beta|}{[1+|\alpha|][1+|\beta|]} \geq \frac{|\alpha+\beta|}{1+|\alpha+\beta|} = a(\alpha + \beta)$

Hence $(\mathbb{R}, a_{|.|}, \circledcirc)$ is algebra fuzzy absolute value space.

Example 2.9:

If $a\circledcirc b=a + b \rightarrow ab$ for all a, b $\in$I define a: $\mathbb{R} \rightarrow$I by

$$a^{|.|}(r) = \begin{cases} \frac{1}{|r|} & if\ r \neq 0 \\ 0 & if\ r = 0 \end{cases}$$

for all r $\in$ $\mathbb{R}$. Then $(\mathbb{R}, a^{|.|}, \circledcirc)$ is algebra fuzzy absolute value space. This space is called the **algebra fuzzy absolute value space induced by $|.|$.**

Proof:

We shall show that all conditions of Definition 2.6 are satisfied.

(1)it is clear that $a^{|.|}(r) \in$I.

(2) $a^{|.|}(r)=0$ if and only if r=0 follows immediately from definition of $a^{|.|}$.

(3) $a^{|.|}(r) \cdot a^{|.|}(t) = \frac{1}{|r|}\frac{1}{|t|} = \frac{1}{|rt|} = a^{|.|}(r.\,t)$

(4) $a^{|.|}(t) \circledcirc a^{|.|}(r) = \frac{1}{|t|} \circledcirc \frac{1}{|r|} = \frac{1}{|t|} + \frac{1}{|r|} - \frac{1}{|tr|}$

$= \frac{|r|+|t|}{|t||r|} - \frac{1}{|tr|} \geq \frac{1}{|t+r|} = a^{|.|}(t + r)$

Hence $(\mathbb{R}, a^{|.|}, \circledcirc)$ is algebra fuzzy absolute value space.

Definition 2.10:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, we say that $\{p_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit p as n approaches to ∞ if for every s $\in$ (0,1) there exists N $\in \mathbb{N}$ such that $a(p_n - p) < s$, for all $n\geq N$. If p_n is fuzzy approaches to the limit p we write $\lim_{n\to\infty} p_n = p$ or $p_n \rightarrow p$ or $\lim_{n\to\infty} a(p_n - p)=0$.

Definition 2.11:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, we say that $\{p_n\}_{n=1}^{\infty}$ is fuzzy Cauchy sequence in $\mathbb{R}$ if for every $s \in (0,1)$ there exists $N \in \mathbb{N}$ such that $a(p_n - p_m) < s$, for all $n, m \geq N$.

Theorem 2.12:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space if $\{p_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} p_n = q$ then $p=q$.

Proof:

Since $p_n \to p$ and $p_n \to q$ so $\lim_{n\to\infty} a[p_n - p]=0$ and $\lim_{n\to\infty} a[p_n - q]=0$. Now
$a[p-q] = a[p-p_n + p_n - q] < a[p-p_n] \odot a[p_n - q]$
Taking limit to both sides as n approaches to ∞ we obtain
$a[p-q] < \lim_{n\to\infty} a[p_n - p] \odot \lim_{n\to\infty} a[p_n - q] < 0 \odot 0 = 0$
Hence $a[p-q] = 0$ so $p-q = 0$ that is $p=q$.

Theorem 2.13:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space and if the sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is fuzzy approaches to the limit p then any subsequence of it is also fuzzy approaches to p.

Proof:

Since $p_n \to p$ then $\lim_{n\to\infty} a[p_n - p] = 0$. Also (p_n) is a Cauchy sequence then $\lim_{n\to\infty} a[p_n - p_m] = 0$ when $n \to \infty$ and $m \to \infty$. Now

$$a[p_{n_k} - p] = a[p_{n_k} - p_n + p_n - p] \leq a[p_{n_k} - p_n] \odot a[p_n - p]$$

Thus $\lim_{n\to\infty} a[p_{n_k} - p] \leq 0 \odot 0 = 0$. Hence (p_{n_k}) fuzzy approaches to p

Definition 2.14:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space. The sequence $\{q_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to be fuzzy bounded if there exists $t \in (0,1)$ such that $a(q_n) < t$ for all $n \in \mathbb{N}$.

Theorem 2.15:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space and if the sequence $\{q_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is fuzzy approaches to the limit q then it is fuzzy bounded.

Proof:

Suppose that $\{q_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is fuzzy approaches to the limit q then for every $s \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $a(q_n - q) < s$ for all $n \geq N$. This implies $a(q_n) = a(q + q_n - q) \leq a(q) \odot a(q_n - q) \leq a(q) \odot s$. Now put $t = a(q)$ for some $t \in (0,1)$ then $a(q_n) \leq t \odot s$.
Choose $p \in (0, 1)$ with $t \odot s < p$. Hence $a(q_n) < p$ for each $n \in \mathbb{N}$. Thus $\{q_n\}_{n=1}^{\infty}$ is fuzzy bounded.

Theorem 2.16:

Let $(\mathbb{R}, a, \odot)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$ if $\{p_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit p and $\{q_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit q then $\{(p_n + q_n)\}_{n=1}^{\infty}$ is fuzzy approaches to the limit $(p + q)$.

Proof:

Since $\{p_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit p then for every $r \in (0, 1)$ there exists $N_1 \in \mathbb{N}$ such that $a(p_n - p) < r$ for all $n \geq N_1$. Also Since $\{q_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit q then for every $p \in (0, 1)$ there exists $N_2 \in \mathbb{N}$ such that $a(q_n - q) < p$ for all $n \geq N_2$. Now choose $N = \min\{N_1, N_2\}$ and for each $n \geq N$ we have $a[(p_n + q_n) - (p + q)] \leq a(p_n - p) \odot a(q_n - q) < r \odot p$. Put $r \odot p = t$ for some $t \in (0, 1)$ then

$a[(p_n + q_n) - (p + q)] < t$ for all $n \geq N$.

Theorem 2.17:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and $0 \neq c \in \mathbb{R}$. If $\{a_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit p then $\{cp_n\}_{n=1}^{\infty}$ is fuzzy approaches the limit cp.

Proof:

Since $\{p_n\}_{n=1}^{\infty}$ approaches the limit a then for every $r \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $a(a_n - a) < r$ for all $n \geq N$. Now

$a(cp_n - cp) = a[c(p_n - p)] \leq a(c) . a(p_n - p) < a(c).r$

Put $a(c) = \sigma$ where $\sigma \in (0, 1)$. Let α where $0 < \alpha < 1$ such that $a(c).r = \alpha$ so $a(cp_n - cp) < \alpha$. Hence $\{cp_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit cp.

Theorem 2.18:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space and let $\{p_n\}_{n=1}^{\infty}$ and $\{q_m\}_{m=1}^{\infty}$ be two sequences in $\mathbb{R}$. If $\{p_n\}_{n=1}^{\infty}$ fuzzy approaches the limit p and $\{q_m\}_{m=1}^{\infty}$ fuzzy approaches the limit q then $\{(p_n . q_n)\}_{n=1}^{\infty}$ fuzzy approaches the limit (p.q).

Proof:

Since $\{p_n\}_{n=1}^{\infty}$ is fuzzy approaches to the limit p then for every $r \in (0,1)$ there exists $N_1 \in \mathbb{N}$ such that $a(p_n - p) < r$ for all $n \geq .N_1$ Also Since $\{q_m\}_{m=1}^{\infty}$ is fuzzy approaches to the limit p then for every $s \in (0,1)$ there exists $N_2 \in \mathbb{N}$ such that $a(q_m - q) < s$ for all $m \geq .N_2$ Choose $N = \min\{N_1, N_2\}$ now

$a[p_n q_n - pq) = a[p_n q_n - pq_n + pq_n - pq \leq a[p_n q_n - pq_n] \circledcirc a[pq_n - pq]$
$\leq a[p_n - p] . a(q_n) \circledcirc a(p) . a[b_n - b] \leq r . a(q_n) \circledcirc a(p) . s$

Put $a(q_n) = \alpha$ and $a(p) = \delta$ for some $0 < \alpha, \delta < 1$. Now let t for some $t \in [0, 1]$ be chosen so that $[r . \alpha \circledcirc \delta . s] < t$. Hence $a[p_n q_n - pq) < t$ for all $n \geq N$.

Theorem 2.19:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space. If $\{q_n\}_{n=1}^{\infty}$ is a fuzzy Cauchy sequence in $\mathbb{R}$ then $\{q_n\}_{n=1}^{\infty}$ is fuzzy bounded.

Proof:

Since $\{q_n\}_{n=1}^{\infty}$ is a fuzzy Cauchy sequence in $\mathbb{R}$ then for every $r \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $a(q_m - q_n) < r$ for all $n, m \geq N$. Then $a(q_m - q_N) < r$ for all $m \geq N$. Hence if $m \geq N$ we have $a(q_m) = a(q_m - q_N + q_N) \leq a(q_m - q_N) \odot a(q_N)$ and so $a(q_m) \leq r \circledcirc a(q_N)$. Now put $t = \max\{a(q_1), a(q_2), \ldots, a(q_{N-1})\}$. Then $a(q_m) \leq r \circledcirc t$. Hence $\{q_n\}_{n=1}^{\infty}$ is fuzzy bounded.

Definition 2.20;

The algebra fuzzy absolute value $(\mathbb{R}, a, \circledcirc)$ is called fuzzy complete if every fuzzy Cauchy sequence in $\mathbb{R}$ fuzzy approaches to a point in $\mathbb{R}$.

The following is the main result in this section

Theorem 2.21:

Let $(\mathbb{R}, a, \circledcirc)$ be algebra fuzzy absolute value space then every fuzzy Cauchy $\{p_n\}_{n=1}^{\infty}$ sequence in $\mathbb{R}$ is fuzzy approaches to the limit $p \in \mathbb{R}$. That is $(\mathbb{R}, a, \circledcirc)$ is fuzzy complete.

Proof:

Let $\{p_n\}_{n=1}^{\infty}$ be a fuzzy Cauchy sequence in $\mathbb{R}$ then $\{p_n\}_{n=1}^{\infty}$ has a monotonic subsequence $\{p_{n_j}\}_{n=1}^{\infty}$ but $\{p_n\}_{n=1}^{\infty}$ is fuzzy bounded [by Theorem 2.19] hence $\{p_{n_j}\}_{n=1}^{\infty}$ is fuzzy bounded. Thus $\{a_{n_j}\}_{n=1}^{\infty}$ fuzzy approaches the limit $p \in \mathbb{R}$. Then for every $r \in (0, 1)$ there exists $N \in \mathbb{N}$ such that $a(p_{n_j} - p) < r$ for all $n \geq N$. Since $\{p_n\}_{n=1}^{\infty}$ is a fuzzy Cauchy sequence in $\mathbb{R}$ then for every $t \in (0,1)$ there exists $M \in \mathbb{N}$ such that $a(p_m - p_n) < t$ for all

$n, m \geq M$. We may choose $M \geq N$. Now for each $k \geq M$ then $k \geq N$, we have

$a(p_{n_k} - p_k) < t$. Hence

$a(p_k - p) = a(p_k - p_{n_k} + p_{n_k} - p) \leq a(p_k - p_{n_k}) \circledcirc a(p_{n_k} - p) \leq t \circledcirc r < \varepsilon$

For some $\varepsilon \in (0, 1)$. Thus $\{p_k\}_{k=1}^{\infty}$ is fuzzy approaches to the limit p ∈ ℝ.

3. Algebra fuzzy normed space and its basic properties

In this section we shall introduce the notion of algebra fuzzy normed space and proved its basic properties.

Definition 3.1:

Let U be a vector space over ℝ and let ⊚ be a continuous t-conorm. Let (ℝ, a, ⊚) be algebra fuzzy absolute value space and n: U→I be a fuzzy set then n is called **algebra fuzzy norm on U** if

(1)0< n(u) ≤ 1;

(2)n(u) =0 if and only if u=0;

(3)n(αu) ≤ a(α) n(u) for all 0≠ $\alpha \in$ ℝ.

(4)n(u + v) ≤ n(u) ⊚ n(v).

For all u, v ∈ U. Then (U, n, ⊚) is called **algebra fuzzy normed space**.

Example 3.2:

Let U=C[p, b], t⊚s=t + s −ts for all t, s ∈I and (ℝ, a, ⊚) is algebra fuzzy absolute space.

Define n(r) = $max_{s\in[p,b]}$ a[r(s)] for all r ∈ U. Then (U, n, ⊚) is algebra fuzzy normed space

Proof:

We will show that all conditions of Definition 3.1 are satisfied

(1)it is clear that n(r) ∈I for all r ∈ U.

(2)n(r) = 0 if and only if $max_{s\in[p,b]}$ a[x(s)] = 0 if and only if a[r(s)] = 0 for all s ∈ [p, b] if and only if r(s) =0 for all s ∈ [p, b] if and only if r=0.

(3)n(αr) = $max_{s\in[p,b]}$ a[αr(s)] ≤ a(α) . $max_{s\in[p,b]}$ a[r(s)] = a(α) .n(r), for all 0≠ $\alpha \in$ ℝ.

(4)n(r + z) = $max_{s\in[p,b]}$ a[r(s) + z(s)] ≤ $max_{s\in[p,b]}$ a[r(s)] + $max_{s\in[p,b]}$ a[z(s)]

= n(r) + n(z)

Hence (U, n, ⊚) is algebra fuzzy normed space.

Theorem 3.3:

If (ℝ, a, ⊚) is algebra fuzzy absolute space and p⊚q=p + q −pq for all p, q ∈I define n: ℝ →I by n(r)= a(r) for all r ∈ ℝ.Then (ℝ, n, ⊚) is algebra fuzzy normed space.

Proof:

We shall show that all conditions of Definition 3.1 are satisfied

(1)It is clear that n(r) ∈I for all r ∈ ℝ.

(2) n(r)=0 if and only if a(r)=0 if and only if r=0.

(3)n(cr) = a(cr) ≤ a(c) . a(r) =a(c). n(r) for all 0≠c ∈ ℝ.

(4)n(t + r)=a(t + r) ≤ a(t) ⊚ a(r) = n(t) + n(r)

Hence (ℝ, n, ⊚) is algebra fuzzy normed space

Example 3.4:

Let (U, ‖. ‖) be a normed space and $\alpha \circledcirc \beta = \alpha + \beta - \alpha.\beta$ for all $\alpha, \beta \in$ I. Define $n_{\|.\|}(u) = \frac{\|u\|}{1+\|u\|}$ for all u ∈U. Then (U, $n_{\|.\|}$, ⊚) is algebra fuzzy normed space with the algebra fuzzy absolute value space (ℝ, $a_{|.|}$, ⊚) where $a_{|.|}(\alpha) = \frac{|\alpha|}{1+|\alpha|}$ for all $\alpha \in$ ℝ. Then $n_{\|.\|}$ is called the **standard algebra fuzzy norm on U.**

Proof:

We will show that all conditions of Definition 3.1 are satisfied.

(1)It is clear that $0 \leq n_{\|.\|}(u) \leq 1$.

(2) $n_{\|.\|}(u)=0$ if and only if $\frac{\|u\|}{1+\|u\|} = 0$ if and only if $\|u\|=0$ if and only if u=0.

(3) $n_{\|.\|}(\alpha u)=\frac{\|\alpha u\|}{1+\|\alpha u\|} = \frac{|\alpha|\|u\|}{1+|\alpha|\|u\|} \leq [\frac{|\alpha|}{1+|\alpha|}].[\frac{\|u\|}{1+\|u\|}] = a_{|.|}(\alpha)\, n_{\|.\|}(u)$, for all $0 \neq \alpha \in \mathbb{R}$.

(4) $n_{\|.\|}(u+v) = \frac{\|u+v\|}{1+\|u+v\|}$ and $n_{\|.\|}(u) \odot n_{\|.\|}(v) = n_{\|.\|}(u) + n_{\|.\|}(v) - n_{\|.\|}(u).\, n_{\|.\|}(v)$

$$n_{\|.\|}(u) \odot n_{\|.\|}(v) = = [\frac{\|u\|}{1+\|u\|}]+[\frac{\|v\|}{1+\|v\|}] - [\frac{\|u\|}{1+\|u\|}].[\frac{\|v\|}{1+\|v\|}]$$
$$= \frac{\|u\|[[1+\|v\|]+\|v\|[[1+\|u\|]}{[1+\|u\|][1+\|v\|]} - \frac{\|u\|\|v\|}{[1+\|u\|][1+\|v\|]} = \frac{\|u\|+\|u\|\|v\|+\|v\|+\|v\|\|u\|}{[1+\|u\|][1+\|v\|]} - \frac{\|u\|\|v\|}{[1+\|u\|][1+\|v\|]}$$
$$= \frac{\|u\|+\|u\|\|v\|+\|v\|}{[1+\|u\|][1+\|v\|]} \geq \frac{\|u+v\|}{1+\|u+v\|} = n_{\|.\|}(u+v)$$

Hence (U, $n_{\|.\|}$, ⊙) is algebra fuzzy normed space.

Example 3.5:

Let (U, ‖. ‖) be a normed space. If s⊙t=s + t −st for all s, t ∈I define $n^{\|\cdot\|}$: U →I by $n^{\|\cdot\|}(p) =$

$$\begin{cases} \frac{1}{\|p\|} & if\ p \neq 0 \\ 0 & if\ \ p = 0 \end{cases}$$

for all p ∈ **U.** Then (U**,** $n^{\|\cdot\|}$, ⊙) is algebra fuzzy normed space with

$$a^{|\cdot|}(r) = \begin{cases} \frac{1}{|r|} & if\ r \neq 0 \\ 0 & if\ \ r = 0 \end{cases}$$

for all r ∈ ℝ where (ℝ**,** $a^{|\cdot|}$, ⊙) is algebra fuzzy absolute value space. This space is called the **algebra fuzzy normed space induced by**‖. ‖**.**

Proof:

We will show that all conditions of definition 3.1 are satisfied.

(1)It is clear that $n^{\|\cdot\|}(p) \in$I.

(2) $n^{\|\cdot\|}(p)=0$ if and only if p=0 follows immediately from definition of n.

(3)$a^{|\cdot|}(r) \cdot n^{\|\cdot\|}(p) = \frac{1}{|r|}\frac{1}{\|p\|} = \frac{1}{\|rp\|} = n^{\|\cdot\|}(r.\, p)$

(4) $n^{\|\cdot\|}(p) \odot n^{\|\cdot\|}(q) = \frac{1}{\|p\|} \odot \frac{1}{\|q\|} = \frac{1}{\|p\|} + \frac{1}{\|q\|} - \frac{1}{\|pq\|}$
$$=\frac{\|q\|+\|p\|}{\|p\|\|q\|} - \frac{1}{\|pq\|} \geq \frac{1}{\|p+q\|} = n^{\|\cdot\|}(p+q)$$

Hence (U**,** $n^{\|\cdot\|}$, ⊙) is algebra fuzzy normed space.

Theorem 3.6:

If $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are two algebra fuzzy normed spaces then (U, n, ⊙) is algebra fuzzy normed space where U= $U_1 \times U_2$ and n[(u_1, u_2)]=$n_1(u_1) \odot n_2(u_2)$ for all $(u_1, u_2) \in$ U.

Proof:

We will show that all conditions of Definition 3.1 is satisfied

(1)Since $0 \leq n_1(u_1) \leq 1$ and $0 \leq n_2(u_2) \leq 1$ so $0 \leq n[(u_1, u_2)] \leq 1$

(2) $n[(u_1, u_2)]=0$ if and only if $n_1(u_1) \odot n_2(u_2)=0$ if and only if $n_1(u_1)=0$ and $n_2(u_2)=0$ if and only if $u_1=0$ and $u_2=0$ if and only if $(u_1, u_2)=(0, 0)$.

(3) $n[c(u_1, u_2)] = n[(cu_1, cu_2)] = n_1(cu_1) \odot n_2(cu_2) \leq a(c).\, n_1(u_1) \odot a(c).\, n_2(u_2) \leq a(c)\, [n_1(u_1) \odot n_2(u_2)]=a(c).\, n[(u_1, u_2)]$

(4)$n[(u_1, u_2) + (v_1, v_2)] = n[(u_1+ v_1) + (u_2 + v_2)] = n_1(u_1 + v_1) \odot n_2(u_2 + v_2)$
$$\leq n_1(u_1) \odot n_1(v_1) \odot n_2(u_2) \odot n_2(v_2)$$
$$\leq [n_1(u_1) \odot n_2(u_2)] \odot [n_1(v_1) \odot n_2(v_2)]$$
$$= n[(u_1, u_2)] \odot n[(v_1, v_2)]$$

Hence (U, n, ⊚) is algebra fuzzy normed space.
The proof of the next result is similar to the proof of Theorem 3.5 and hence is omitted.

Corollary 3.7:

If $(U_1, n_1, ⊚)$, $(U_2, n_2, ⊚)$, ..., $(U_k, n_k, ⊚)$ are algebra fuzzy normed spaces then (U, n, ⊚) is algebra fuzzy normed space where U= $U_1 \times U_2 \times ... \times U_k$ and n[$(u_1, u_2, ..., u_k)$]= $n_1(u_1)$ ⊚ $n_2(u_2)$ ⊚, ..., ⊚ $n_k(u_k)$ for all $(u_1, u_2, ...,u_k)$ ∈ U.

Definition 3.8:

Let (U, n, ⊚) be algebra fuzzy normed space and let (u_k) be a sequence in U, we say that (u_k) is fuzzy converges to the limit u as k approaches to ∞ if for every s ∈ (0,1) there exists N ∈ ℕ such that $n(u_k - u) < s$, for all k≥N. If (u_k) is fuzzy approaches to the limit u we write $\lim_{k\to\infty} u_k = u$ or $u_k \to u$ or $\lim_{n\to\infty} n(u_k - u) = 0$.

Definition 3.9:

Suppose that (U, n, ⊚) is algebra fuzzy normed space . Put fb(u, r)= {v ∈U: n(u−v) < t} and fb[u, r]= {v ∈ U: n(u−v) ≤ t}.Then fb(u, r) and fb[u,r] is called **open and closed fuzzy ball** with the center u ∈U and radius t, with t ∈ (0, 1).

Definition 3.10:

Let (U, n, ⊚) be algebra fuzzy normed space and let (u_k) be a sequence in U, we say that (u_k) is fuzzy Cauchy sequence in U if for every s ∈ (0, 1) there exists N ∈ ℕ such that $n(u_k - u_m) < s$, for all k, m ≥N.

The proof of the next result is clear and hence is omitted.

Lemma 3.11:

If (U, n, ⊚) is algebra fuzzy normed space then a[n(u)−n(v)] ≤ n(u−v) for all u, v ∈ U.

Lemma 3.12:

If (U, n, ⊚) is algebra fuzzy normed space then the function u ⟼ n(u) is a fuzzy continuous function from (U, n, ⊚) → (ℝ, n, ⊚).

Proof:

Let (u_k) be a sequence in U with $u_k \to u$ so $\lim_{k\to\infty} n(u_k - u) = 0$. Now a[n$(u_k)$−n(u)] ≤ n$(u_k$ −u) taking limit to both sides as k→ ∞ we have $\lim_{k\to\infty}$ a[n(u_k) − n(u)] ≤ $\lim_{k\to\infty}$ n$(u_k$ − u) =0. So n(u_k) → n(u).
Hence u ⟼ n(u) is a fuzzy continuous function from U → ℝ.

Lemma 3.13:

If (U, n, ⊚) is algebra fuzzy normed space then n(u−v) = n(v−u) for all u, v ∈ U.

Proof:

n(u−v) = n[(−1)(v−u)] ≤ a(−1). n(v−u) =a(1). n(v−u)= n(v−u). Similarly n(v−u) ≤ n(u−v) from these two inequalities we conclude that n(u−v) = n(v−u) for all u, v ∈ U.

Definition 3.14:[14]

A triple (S, m, ⊚) is said to be the algebra fuzzy metric space if S≠ ∅, ⊚ is a continuous t-conorm and m: S×S→[0, 1] satisfying the following conditions:

(A_1) 0 ≤ m(s, r) ≤ 1, (A_2) m(s, r) = 0 if and only if s = r, (A_3) m(s, r) = m(r, s) ;
(A_4) m(s, t) ≤ m(s, r) ⊚ m(r, t). For all s, r, t ∈ S then the triple (S, m, ⊚) is said to be the **algebra fuzzy metric space .**

Theorem 3.15:

If (U, n, ⊚) is algebra fuzzy normed space then (U, m_n, ⊚) is algebra fuzzy metric space where m_n(u, v) = n(u−v) for all u, v ∈ U. (U, m_n, ⊚) is called the **algebra fuzzy metric space induced by n.**

Proof:

We will show that all conditions of Definition 3.10 are satisfied.

(A1)It is clear that m_n(u, v) ∈ I.

(A2) m_n(u, v)= 0 if and only if n(u−v)=0 if and only if u−v=0 if and only if u=v.

(A3) m_n(u, v) = n(u−v) =n(v−u)= m_n(v, u) by Lemma 3.10.

(A4) m_n(u, v) = n(u−v)=n(u−z + z −v) ≤ n(u−z) ⊚ n(z −v) ≤ m_n(u, z) ⊚ m_n(z, v)

Hence (U, m_n, ⊚) is algebra fuzzy metric space.

Lemma 3.16:

The algebra fuzzy metric m_n induced by the algebra fuzzy norm n satisfies

(1)m_n(u + r, v + r) = m_n(u, v), (2) m_n(αu, αv) ≤ a(α). m_n(u, v)

For all u, v, r ∈ U and $\alpha \neq 0 \in \mathbb{R}$.

Proof:

(1)m_n(u + r, v + r)=n[(u + r) – (v + r)]=n(u−v)= m_n(u, v)

(2) m_n(αu, αv)=n[αu – αv] = n[α(u−v)] ≤ a(α). n(u−v)= a(α). m_n(u, v)

Definitions 3.17:

If (U, n, ⊚) is algebra fuzzy normed space and W⊆U is known as **fuzzy open** if fb(w, j) ⊆W for any arbitrary w∈W and for some j ∈(0, 1). Also D⊆U is known as **fuzzy closed** if D^C is fuzzy open. Moreover the **fuzzy closure** of D, $\overline{\text{D}}$ is defined to be the smallest fuzzy closed set contains D.

Definition 3.18:

If (U, n, ⊚) is algebra fuzzy normed space then D⊆S is known as **fuzzy dense** in U if whenever $\overline{\text{D}}$ = U.

Theorem 3.19:

If fb(s, j) is open fuzzy ball in algebra fuzzy normed space (U, n, ⊚) then it is a fuzzy open set.

Proof:

Let fb(v, j) be open fuzzy ball open where v∈U and j ∈ (0, 1) . Let u∈fb(v, j) so n(v−u) < j , let t = n(v−u) so t< j , then there is i ∈ (0, 1) such that t ⊚ i < j by Remark 2.3 (ii). Now assume the open fuzzy ball fb(u, i), we will show that fb(u, i) ⊆ fb(v, j). Let z∈fb(u, i) so n(u – z) <i. Hence n(v− z) ≤ n(v− u) ⊚ n(u− z) or n(v− z) ≤ t ⊚ i < j. so z∈ fb(v, j) that is fb(u, i) ⊆ fb(v, j).Therefore fb(v, j) is a fuzzy open set.

Definition 3.20:[1]

Let U is a nonempty set a collection T of subset of U is said to be a **fuzzy topology** on U if

(i)U and φ belongs to T, (ii)If $\mathrm{A_1}$, $\mathrm{A_2}$,...., $\mathrm{A_n}$ ∈ T then $\cap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{A_i}$ ∈ T.

(iii)If {$\mathrm{A_j}$: j∈J}∈ T then $\cup_{\mathrm{j\in J}} \mathrm{A_j}$ ∈ T.

Theorem 3.21:

Every algebra fuzzy normed space is a fuzzy topological space.

Proof:

If (U, n, ⊚) is algebra fuzzy normed space then put T_n= {W⊂U: w∈W if and only if we have j∈ (0, 1) with fb(w, j) ⊂W }. Now we will prove T_n is a fuzzy topology on U.

(i)Clearly ϕ and U belong to T_n since ϕ and S are fuzzy open.

(ii)Let $\mathrm{W_1}$, $\mathrm{W_2}$,...., $\mathrm{W_n}$ ∈ T_n and put V = $\cap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{W_i}$.We shall show that V∈ T_n. Let v∈V then v$\in$W$_i$ for each $1 \leq i \leq n$. Hence there exists $0 \leq r_i \leq 1$ such that fb(v, r_i) $\subset$ W$_i$ since W$_i$ is fuzzy open for each i=1, 2, ...,n. Put r = min{ r_i : $1 \leq i \leq n$ } thus $r \leq r_i$ for all $1 \leq i \leq n$. So fb(v, r) $\subseteq$ W$_i$ for all $1 \leq i \leq n$. Therefore fb(v, r) $\subseteq \cap_{i=1}^{n}$ W$_i$ = V, thus V $\in \mathcal{T}_n$.

(iii) Suppose that {W$_j$: j $\in$J}$\in \mathcal{T}_n$ and put W = $\cup_{j \in J}$ W$_j$. We shall show that W$\in \mathcal{T}_n$. Let w$\in$W then w$\in\cup_{j \in J}$ W$_j$ so w$\in$ W$_j$ for some j $\in$J since W$_j$ $\in \mathcal{T}_n$ there exists 0< r < 1 such that fb(w, r) $\subset$ W$_j$.Thus fb(w, r) $\subset$ W$_j$ $\subseteq \cup_{j \in J}$ W$_j$ = W that is W$\in \mathcal{T}_n$. Hence (U, $\mathcal{T}_n$) is a fuzzy topological space. $\mathcal{T}_n$ is known as the **fuzzy topology induced by n**.

Definition 3.22:

An algebra fuzzy normed space (U, n, ⊙) is known as fuzzy complete if (s_k) is fuzzy Cauchy sequence in U then $s_k \to$s $\in$ U.

Theorem 3.23:
In algebra fuzzy normed space (U, n, ⊙) if $u_k \to$u $\in$ U then (u_k) is fuzzy Cauchy.
Proof:
Suppose that (u_k) in U and $u_k \to$u $\in$ U then for any r $\in$(0, 1) we can find N with n(u_k − u) < r , for all k≥N . Using Remark 2.3 we can find t $\in$(0, 1) with r ⊙ r < t. Now n($u_k - u_m$) ≤ n(u_k − u) ⊙ n(u− u_m))< r ⊙ r < t for each m, k ≥ N. Hence (u_k) is fuzzy Cauchy sequence.
Theorem 3.24:
In algebra fuzzy normed space (U, n, ⊙) if (u_k) $\in$ U with $u_k \to$ u and (d_k) $\in$ U with n($u_k - d_k$) → 0 as k→ ∞. Then $d_k \to$ u.
Proof:
Since $u_k \to$ u so n(u_k − u) → 0 as k→ ∞. Now
n(d_k − u) ≤ n($d_k - u_k$) ⊙ n(u_k − u) → 0 as k→ ∞. Hence $d_k \to$ u.
Theorem 3.25:

In algebra fuzzy normed space (U, n, ⊙) when D$\subset$U then d$\in\overline{D}$ if and only if there is (d$_k$) $\in$ D with d$_k \to$ d.

Proof:

Suppose that d $\in \overline{D}$, if d$\in$D then choose the sequence of that type is (d, d,...,d,...) .
If d $\notin$ D, it is a limit point of D. Hence we construct the sequence (d$_k$) $\in$ D by

n(d$_k$ − d) < $\frac{1}{k}$ for each n = 1, 2, 3,.....

The fuzzy ball fb(d, $\frac{1}{k}$) contains d$_k$ $\in$ D and d$_k \to$ d because $\lim_{k\to\infty}$n(d$_k$ − d) = 0.

Conversely if (d$_k$) in D and d$_k \to$ d then d $\in$D or every fuzzy ball of a contain points d$_k \neq$ d, so that d is a point of accumulation of D, hence d $\in \overline{D}$ by the definition of the closure.

Theorem 3.26:

In algebra fuzzy normed space (U, n, ⊙) when D$\subset$U then $\overline{D}$=U if and only if for any u $\in$U there is d∈D such that n(u – d) < r for some r∈(0, 1).

Proof:

Suppose that D is fuzzy dense in U and u ∈U so u ∈ $\overline{D}$ and by Theorem 2.25 there is a sequence (d_n) ∈D such that $d_n \to u$ that is for any r∈(0, 1) we can find N with $n(d_n - u) < r$ for all n ≥ N. Take d = d_N, so n(u – d) < r.

Conversely to prove D is fuzzy dense in U we have to show that D ⊆ $\overline{D}$. Let u∈U then there is d_k ∈D such that $n(d_k - u) < \frac{1}{k}$. Now take 0 < r < 1 such that $\frac{1}{k} < r$ for each k ≥ N for N∈ ℕ.

Hence we have a sequence (d_k) ∈D such that $n(d_k - u) < \frac{1}{k} < r$ for all k ≥ N that is $d_k \to u$ so u ∈ $\overline{D}$.

Theorem 3.27:

Let D be a dense subset of algebra fuzzy normed space (U, n, ⊙). If every fuzzy Cauchy sequence of points of D converges in U then U is fuzzy complete.

Proof:

Suppose that (u_k) is a fuzzy Cauchy sequence in U, since D is fuzzy dense then for every u_k ∈U there is d_k ∈D such that $n(u_k - d_k) < s$ for some 0 < s < 1 by Theorem 3.26. Now by Remark 2.3 we can find t ∈(0, 1) with s ⊙ s < t. But (u_k) is fuzzy Cauchy so (d_k) is fuzzy Cauchy thus $d_k \to u$∈U by our assumption. Now $n(u_k - u) \le n(u_k - d_k) \odot n(d_k - u) \le s \odot s < t$. Hence $u_k \to u$.

Theorem 3.28:

If $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are two algebra fuzzy normed spaces then(U, n, ⊙) is fuzzy complete algebra fuzzy normed space if and only if $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are fuzzy complete where U= $U_1 \times U_2$ and $n[(u_1, u_2)] = n_1(u_1) \odot n_2(u_2)$ for all $(u_1, u_2) \in$ U.

Proof:

Let $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are two fuzzy complete algebra fuzzy normed spaces. Let (u_k) be fuzzy Cauchy sequence in U then $(u_k) = (u_{1k}, u_{2k})$ where $(u_{1k}) \in U_1$ and $(u_{2k}) \in U_2$. Hence $n(u_k - u_m)$ fuzzy converges to zero as k→ ∞ and m→ ∞ this implies that $[n_1(u_{1k} - u_{1m}) \odot n_2(u_{2k} - u_{2m})]$ fuzzy converges to zero as k→ ∞ and m→ ∞.
Hence $n_1(u_{1k} - u_{1m})$ fuzzy converges to zero in $(U_1, n_1, \odot)$ as k→ ∞ and m→ ∞ and $n_2(u_{2k} - u_{2m})$ fuzzy converges to zero in $(U_2, n_2, \odot)$ as k→ ∞ and m→ ∞. Therefore (u_{1k}) is fuzzy Cauchy sequence in $(U_1, n_1, \odot)$ and (u_{2k}) is fuzzy Cauchy sequence in $(U_2, n_2, \odot)$ but $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are fuzzy complete so there is $u_1 \in U_1$ and $u_2 \in U_2$ such that (u_{1k}) fuzzy converges to $u_1 \in U_1$and (u_{2k}) fuzzy converges to $u_2 \in U_2$. Put u=(u_1, u_2) then u ∈U and (u_k) fuzzy converges to u ∈U since $n(u_k - u) = n[(u_{1k}, u_{2k}) - (u_1, u_2)] = [(u_{1k} - u_1) + (u_{2k} - u_2)] = n_1[(u_{1k} - u_1)] \odot n_2[(u_{2k} - u_2)]$. By taking limit to both sides as k→ ∞ we have $n(u_k - u) \to 0$.
Conversely assume that (U, n, ⊙) is fuzzy complete we will prove that $(U_1, n_1, \odot)$ and $(U_2, n_2, \odot)$ are fuzzy complete. Let (u_{1k}) is fuzzy Cauchy sequence in $(U_1, n_1, \odot)$ and (u_{2k}) is fuzzy Cauchy sequence in $(U_2, n_2, \odot)$.Then $n_1(u_{1k} - u_{1m})$ fuzzy converges to zero in $(U_1, n_1, \odot)$ as k→ ∞ and m→ ∞ and $n_2(u_{2k} - u_{2m})$ fuzzy converges to zero in $(U_2, n_2, \odot)$ as k→ ∞ and m→ ∞. Put

(u_k) = (u_{1k}, u_{2k}) where $(u_{1k}) \in U_1$ and $(u_{2k}) \in U_2$. Since

n$(u_k - u_m)$ =$[n_1(u_{1k} - u_{1m}) \circledcirc n_2(u_{2k} - u_{2m})]$ fuzzy converges to zero as k→ ∞ and m→ ∞. Hence (u_k) is fuzzy Cauchy sequence in U but U is fuzzy complete so there is u=(u_1, u_2) ∈U such that n$(u_k$ –u) fuzzy converges to zero but n$(u_k$ –u)=n$[(u_{1k}, u_{2k})$ –$(u_1, u_2)]$= $[(u_{1k} - u_1)$+ $(u_{2k} - u_2)]$=$n_1[(u_{1k} - u_1)] \circledcirc n_2[(u_{2k} - u_2)]$ so $n_1[(u_{1k} - u_1)]$ fuzzy converges to zero in $(U_1, n_1, \circledcirc)$ as k→ ∞ and $n_2[(u_{2k} - u_2)]$ fuzzy converges to zero in $(U_2, n_2, \circledcirc)$ as k→ ∞. Hence (u_{1k}) fuzzy converges to $u_1 \in U_1$ as k→ ∞ and (u_{2k}) fuzzy converges to $u_2 \in U_2$ as k→ ∞. It follows that $(U_1$,n_1, ⊚) and $(U_2$,n_2, ⊚) are fuzzy complete.

In similar way we can prove the next result

Corollary 3.29:

If $(U_1, n_1, \circledcirc)$, $(U_2, n_2, \circledcirc)$, ..., $(U_k, n_k, \circledcirc)$ are algebra fuzzy normed spaces then (U, n, ⊚) is fuzzy complete algebra fuzzy normed space if and only if $(U_1, n_1, \circledcirc)$, $(U_2, n_2, \circledcirc)$, ..., $(U_k, n_k, \circledcirc)$ are fuzzy complete where U= $U_1 \times U_2 \times \ldots \times U_k$ and n$[(u_1, u_2, \ldots, u_k)]$=$n_1(u_1) \circledcirc n_2(u_2) \circledcirc, \ldots, \circledcirc n_k(u_k)$ for all $(u_1, u_2, \ldots, u_k)$ ∈ U.

The proof of the next result is similar to Theorem 3.28 and hence is deleted.

Theorem 3.30:

If (U, n, ⊚) is algebra fuzzy normed spaces then $(U^2, n_2, \circledcirc)$ is fuzzy complete algebra fuzzy normed space if and only if (U, n, ⊚) is fuzzy complete where U^2= U × U and $n_2[(u_1, u_2)]$=n(u_1) ⊚ n(u_2) for all $(u_1, u_2) \in U^2$.

In similar way we can prove the next result

Corollary 3.31:

If (U, n, ⊚) is algebra fuzzy normed spaces then $(U^k, n_k , \circledcirc)$ is fuzzy complete algebra fuzzy normed space if and only if (U, n, ⊚) are fuzzy complete where U^k = U × U × ... × U [k-times] where k ∈ ℕ and $n_k[(u_1, u_2, \ldots, u_k)]$=n$(u_1)$ ⊚ n(u_2) ⊚, ..., ⊚n(u_k) For all $(u_1, u_2, \ldots, u_k) \in U^k$.

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