Some Fixed Point Theorems in n−normed spaces

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ABSTRACT

In this article, we recall the definition of a real n-normed space and some basic properties. fixed point theorems for types of Kannan, Chatterge, Zamfirescu, n-Weak contraction and n- (α, k)-Weak contraction mappings in $n-$ Banach spaces..

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1. Introduction

The term metric fixed point theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and (or) mappings play a crucial role.' The basic idea of the metric fixed point principle firstly appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of the solution to an integral equation"[18]. This principle (Banach's contraction mapping principle) is remarkable in its simplicity contraction it is perhaps the most widely applied fixed point theorem in all of the analysis. "This is because the contraction condition on the mappings is simple and easy to test, it requires only complete metric space for its

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setting, it provides a contraction algorithm (iterative method), it finds almost canonical applications in the theory of differential and integral equations especially existence and uniqueness solution" [9]. Later, the researchers presented several studies that included different generalizations of the Banach principle in various spaces such as [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17]. This paper deals with three independent principles for existence a unique fixed point, namely: Let (A, ρ) mapping on A. For all u, v in A

   i.   Banach's principle, $\exists \lambda, 0 \leq \lambda < 1$ such that $\rho(Tu, Tv) \leq \lambda \rho(u, v)$.

   ii.   Kanann's   principle, $\exists \ \beta, \ 0 \leq \ \beta \ < 1/2$ such that $\rho(Tu,Tv) \leq \ \beta [\rho(u,Tu)+\rho(v,Tv)]$.

   iii.   Chatterge's principle, $\exists \ \alpha, \ 0 \leq \alpha < 1/2$ such that $\rho(Tu,Tv) \leq \alpha [\rho(u,Tv)+ \rho(u,Tv)]$.

  The conditions in (i), (ii) and (iii) are independent that is if the mapping T satisfies (i) will be continuous but, if it satisfies (ii) or (iii) may be discontinuous. The following examples illustrate these facts [11]:

T satisfies (i) is continuous: Let T:[0,1] $\rightarrow$[0,1] , T$u = u/3$ but does not satisfy (ii) when $u=0$ and $y=1/3$.

T satisfies (ii) and T is discontinuous: Let T: [0, 1] $\rightarrow$ [0,1] such that T$u =u/4$ if $u \in$[0, 1/2) and T$u =u/5$ if $u \in$[1/2 ,1].

T satisfies (ii) but does not satisfy (iii): Let T$:R \rightarrow R$, T$u = -u/2$ take $u=2$, $v = -2$.

T satisfies (iii) but does not satisfy (ii): Let T:[0,1] $\rightarrow$[0,1], T$u =u/2$ if $u \in$[0,1) and T$u =0$ if $u =1$ take $u=1/2, v=0$.

  Throughout this paper, we prove the above principles in n-normed spaces.

Definition(1.1): [7]. Let A be a real linear space with dim A $= n$,$n \in N$ and $\|., \dots, . \|$: $A^n \rightarrow [0, \infty)$ be a function. Then (A ,$\|., \dots, . \|$) is called a linear n-normed space, if

for all $w_1, \dots, w_n \in$ A, and $\propto \ \in$ R.

(N₁) $\|w_1, \dots, w_n\| = 0 \Leftrightarrow w_1, \dots, w_n$ are linearly dependent.

(N₂) $\|w_1, \dots, w_n\|$ is invariant under permutation.

(N₃) $\|w_1, \dots, w_{n-1}, \propto w_n\| = |\propto| \; \|w_1, \dots, w_{n-1}, w_n\|, \; \forall \propto \in R$

(N₄) $\|w_1, \dots, w_{n-1}, u + v\| \leq \|w_1, \dots, w_{n-1}, u\| + \|w_1, \dots, w_{n-1}, v\|, \forall \, u, v \in R.$

A usual example of an $n-$normed space is the following

Example(1.2): [6]. As an example of an n-normed space, we may take A= R^n, equipped with the Euclidean n-norm

$$\|w_1, w_2, \dots, w_n\|_E = |det(w_{ij})| = abs \left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix} \right)$$

where $w_i = (w_{i1}, \dots, w_{in}) \in R^n$ for each $i = 1,2, \dots, n.$

$\|w_1, w_2, \dots, w_n\|_E = $ the volume of the n-dimensional parallelepiped spanned by the vectors $w_1, w_2, \dots, w_n$, in A. In special case $n = 1$, (If A $= R$, R is the set of real numbers) $\|w\|_E$=the absolute value of w, $|w|$.

Remark (1.3): [8]. In an n-normed space (A,$\|., \dots, .\|$), we have

1-$\|w_1, \dots, w_n\| \geq 0,$

2-$\|w_1, \dots, w_{n-1}, w_n\| = \|w_1, \dots, w_{n-1}, w_n + \alpha_1 w_1 + \cdots + \alpha_n w_n\|, \forall \, w_1, \dots, w_n \in A$ and

$\alpha_1, \dots, \alpha_n \in R.$

Definition(1.4): [6]. A sequence $\{u_n\}$ in a n-normed space (A,$\|., \dots, .\|$) is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} \|w_1, \dots, w_{n-1}, u_n - u_m\| = 0, \forall \, v \in A.$

Definition(1.5): [6]. A sequence $\{u_n\}$ in a n-normed space (A,$\|., \dots, .\|$) is said to be convergent if there is a point u in A such that $\lim\limits_{n\to\infty}\|w_1, \dots, w_{n-1}, u_n - u\| = 0, \forall \, v \in$ A if $\{u_n\}$ converges to u we write $u_n \to u$ as $n \to \infty$, for every $w_1, \dots, w_{n-1}$ in A.

Definition(1.6): [8]. A n-normed space is said to be complete if every Cauchy sequence is convergent to an element of A. A complete n-normed space A is called n-Banach space.

   We give some topological concepts in n-normed space.

Definition(1.7): [6]. Let (A, $\|., \ldots, .\|$) be a linear n-normed space, G be a subset of A then the closure of G is $\bar{G} = \{\, u \in A;$ there is a sequence u_n of G : $u_n \to u\}$. We say G is sequentially closed if $G = \bar{G}$.

Theorem(1.8): [15]. Let (A,$\|., \ldots, .\|$) be a linear n-normed space, B be a nonempty subset of A and $u \in B$, then B is said to be u-bounded if there exist some $M > 0$, $\|w_1, \ldots, w_{n-1}. u\| \leq$ M, $\forall u \in B$. If for all $u \in B$, B is u-bounded then B is called a bounded set.

Definition(1.9): [15] and [10]. Let (A,$\|., \ldots, .\|$) be a linear n-normed space. Then the mapping T: A $\to$ A is said to be a $n-$contraction if there exists h $\in [0,1)$,

$$\|w_1, \ldots, w_{n-1}, Tu - Tv\| \leq \text{h} \|w_1, \ldots, w_{n-1}, u - v\|, \forall\, w_1, \ldots, w_{n-1}, u, v \in A.$$

a $n-$nonexpansive mapping with respect to C, if $\|w_1, \ldots, w_{n-1}, Tu - Tv\| \leq \|w_1, \ldots, w_{n-1}, u - v\|, \forall\, w_1, \ldots, w_{n-1}, u, v \in A.$

  The following lemma used by Sukaesih and at et. [15] to show when a vector is zero, which is essential in proving theorems.

Lemma(1.10): [15]. Let $l \in A$. If $\|l, w_{i2}, \ldots, w_{in}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1,2, \ldots, n\}$ then $l = 0$.

   In the following, we recall the fixed point theorem for a contractive mappings on a closed and bounded subset concerning$W = \{w_1, \ldots, w_n\}$.

Theorem(1.11): [15]. Let (A,$\|., \ldots, .\|$) be a n-Banach space and B $\subset$ A be a nonempty closed and bounded with respect to W. If T : B $\to$ B is contraction mapping with respect to W, then T has a unique fixed point in B.

2. n −Zamfirescu mappings

In this section, we give Kannan fixed point theorem and Chatterjea fixed point theorem

arising from n −normed spaces.

Definition(2.1): Let $(A,\|.,\dots,.\|)$ be a n-normed space. A mapping T : A →A is called Picard

mapping if $\exists u^* \in A \ni F_T = \{u^*\}$ and $T^n(u_0) \to u^*, \forall u_0 \in A$.

Definition(2.2): Let $(A,\|.,\dots,.\|)$ be a n-normed space. Then the mapping T : A →A is said to be a

n-contractive we get $\|w_1,\dots,w_{n-1},Tu - Tv\| < \|w_1,\dots,w_{n-1},u - v\|, \forall u,v \in A$.

Definition(2.3): Let $(A,\|.,\dots,.\|)$ be a n-normed space. Then the mapping T : A →A is said to be a

n-quasi nonexpansive where p is fixed point of T, we have

$$\|w_1,\dots,w_{n-1},Tu - Tp\| \leq \|w_1,\dots,w_{n-1},u - p\|.$$

Theorem(2.4): Let $(A,\|.,\dots,.\|)$ be a n-Banach space and H be a nonempty closed and bounded
subset of A, if T: H → H satisfying :

$$\|w_1,\dots,w_{n-1},Tu - Tv\| \leq \propto [\|w_1,\dots,w_{n-1},u - Tu\| + \|w_1,\dots,w_{n-1},v - Tv\|] \tag{2.1}$$

where $\propto \in [0,\frac{1}{2})$ then T is Picard operator

Proof: Let $u_0 \in A$ and $\{u_n\}_{n=1}^{\infty}$ be a sequence in H defined by Picard iteration Scheme i.e.
$u_n = Tu_{n-1}$, we have :

$$\|w_1,\dots,w_{n-1},u_n - u_{n+1}\| = \|w_1,\dots,w_{n-1},Tu_{n-1} - Tu_n\| \leq$$

$$\propto \|w_1,\dots,w_{n-1},u_{n-1} - u_n\| + \propto \|w_1,\dots,w_{n-1},u_n - u_{n+1}\| \tag{2.2}$$

$$\|w_1,\dots,w_{n-1},u_n - u_{n+1}\| \leq \frac{\propto}{1-\propto} \|w_1,\dots,w_{n-1},u_{n-1} - u_n\| \tag{2.3}$$

Note, that $\propto \in [0,\frac{1}{2})$, then $h = \frac{\propto}{1-\propto} \in [0,1)$

Thus, T is a contraction mapping, also, from (2.3) we have :

$$\|w_1,\dots,w_{n-1},u_n - u_{n+1}\| \leq \left(\frac{\propto}{1-\propto}\right)^n \|w_1,\dots,w_{n-1},u_0 - u_1\|.$$

Now, we show that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Let $m, n > 0$ with m > n, Taking $m = n+p$

$$\|w_1,\dots,w_{n-1},u_n - u_m\| = \|w_1,\dots,w_{n-1},u_n - u_{n+p}\| \leq$$

$$\|w_1,\dots,w_{n-1},u_n - u_{n+1}\| + \|w_1,\dots,w_{n-1},u_{n+2} - u_{n+1}\| + \dots+$$

$$\|w_1,\dots,w_{n-1},u_{n+p-1} - u_{n+p}\|. \tag{2.4}$$

From (2.4) we have :

$$\|w_1, \dots, w_{n-1}, u_n - u_m\| \leq h^n \|w_1, \dots, w_{n-1}, u_0 - u_1\| +$$

$$h^{n+1} \|w_1, \dots, w_{n-1}, u_0 - u_1\| + \dots + h^{n+p-1} \|w_1, \dots, w_{n-1}, u_0 - u_1\|. \tag{2.5}$$

Letting $p, n \to \infty$ in (2.5) we have :

$$\lim_{n\to\infty} \|w_1, \dots, w_{n-1}, u_n - u_m\| = \lim_{n\to\infty} \|w_1, \dots, w_{n-1}, u_n - u_{n+p}\| \leq 0. \tag{2.6}$$

Thus, $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in H.

Hence, there exists $v \in H$ such that $\lim_{n\to\infty} u_n = v$ \hfill (2.7)

Now, we show that $v \in$ H is a fixed point of T. Using condition (iv) in Definition (1.1)

$$\|w_1, \dots, w_{n-1}, Tv - v\| \leq \|w_1, \dots, w_{n-1}, Tv - u_n\| + \|w_1, \dots, w_{n-1}, u_n - v\| \tag{2.8}$$

Letting $n \to \infty$ in (2.8) we have

$$\|w_1, \dots, w_{n-1}, Tv - v\| \leq \|w_1, \dots, w_{n-1}, Tv - v\| + \|w_1, \dots, w_{n-1}, v - v\|$$

$$\lim_{n\to 0} \|w_1, \dots, w_{n-1}, Tv - v\| = 0.$$

Therefore, $v = Tv$ and implies that v is a fixed point of T.

Now, assume that $\acute{v}$ is another fixed point of T. Thus, we have $Tv = \acute{v}$ and

$$\|w_1, \dots, w_{n-1}, v - \acute{v}\| = \|w_1, \dots, w_{n-1}, Tv - T\acute{v}\|$$

$$\leq \propto [\|w_1, \dots, w_{n-1}, v - Tv\| + \|w_1, \dots, w_{n-1}, \acute{v} - T\acute{v}\| = 0 \tag{2.9}$$

The inequality (2.9) implies that $v = \acute{v}$. Hence, we obtain that the fixed point is unique,

This completes the proof.

Theorem(2.5): Let $(A, \|. , \dots, . \|)$ be a n-Banach space, H be a nonempty n-closed and n-bounded subset of A. If T: H $\to$ H satisfying :

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \beta [\|w_1, \dots, w_{n-1}, u - Tv\| + \|w_1, \dots, w_{n-1}, v - Tu\|] \tag{2.10}$$

where $\beta \in [0, \frac{1}{2})$ then T is Picard operator.

Proof: Let $u_0 \in A$ and $\{u_n\}_{n=1}^\infty$ be a sequence in A defined as $u_n = Tu_{n-1} = T_{u_0}^n$, $n = 1,2, \dots$.

We have $\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| = \|w_1, \dots, w_{n-1}, Tu_{n-1} - Tu_n\|$

$$\leq \beta [\|w_1, \dots, w_{n-1}, u_{n-1} - Tu_n\| + \|w_1, \dots, w_{n-1}, u_n - Tu_{n-1}\|]$$

$$= \beta [\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\| + \|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|] \tag{2.11}$$

From (2.11) we obtain that

$$\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| \leq \frac{\beta}{1-\beta} \|w_1, \dots, w_{n-1}, u_{n-1} - u_n\| \tag{2.12}$$

Note that $\beta \in [0, \frac{1}{2})$, then $h = \frac{\beta}{1-\beta} \in [0, 1)$. Also, from (2.12) we have

$$\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| \leq h^n \|w_1, \dots, w_{n-1}, u_0 - u_1\| \tag{2.13}$$

Using condition (N_4) of n-norm, we have

$$\|w_1, \dots, w_{n-1}, u_n - u_m\| \leq \|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| + \|w_1, \dots, w_{n-1}, u_{n+1} - u_{n+2}\| + \dots +$$

$$\|w_1, \dots, w_{n-1}, u_{m-2} - u_{m-1}\| \leq h^n \|w_1, \dots, w_{n-1}, u_0 - u_1\| [1 + h + h^n + \dots] \tag{2.14}$$

Letting m, n $\rightarrow \infty$ in (2.14) we have

$$\lim_{m,n\to\infty} \|w_1, \dots, w_{n-1}, u_n - u_m\| = 0 \tag{2.15}$$

Thus, we obtain that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H, hence there exists $u^* \in H$ such that

$(u_n) \rightarrow u^*$ as $n \rightarrow \infty$.

Now, we will show that $u^* \in H$ is a fixed point of T.

$$\|w_1, \dots, w_{n-1}, u^* - Tu^*\| \leq \|w_1, \dots, w_{n-1}, u^* - u_n\| + \|w_1, \dots, w_{n-1}, u_n - Tu^*\| \leq$$

$$\|w_1, \dots, w_{n-1}, u^* - u_n\| + \|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| + \|w_1, \dots, w_{n-1}, u_{n+1} - Tu^*\|$$

$$\leq \|w_1, \dots, w_{n-1}, u^* - u_n\| + h^n \|w_1, \dots, w_{n-1}, u_0 - u_1\| + \|w_1, \dots, w_{n-1}, Tu_n - Tu^*\|$$

$$\leq \|w_1, \dots, w_{n-1}, u^* - u_n\| + h^n \|w_1, \dots, w_{n-1}, u_0 - u_1\| +$$

$$\beta [\|w_1, \dots, w_{n-1}, u_n - Tu^*\| + \|w_1, \dots, w_{n-1}, u^* - Tu_n\|] \tag{2.16}$$

Letting $n \rightarrow \infty$ in (2.16) we obtain that

$$\|w_1, \dots, w_{n-1}, u^* - Tu^*\| \leq 2\beta \|w_1, \dots, w_{n-1}, u^* - Tu^*\| \tag{2.17}$$

Inequality (2.17) is a contradiction unless $\|w_1, \dots, w_{n-1}, u^* - Tu^*\| = 0$. This implies that $u^* = Tu^*$ and we obtain that $u^* \in H$ is a fixed point of T. To complete the proof, we show that the fixed point is unique. Assume that $ú \in H$ is another fixed point of T. Hence, we have $Tú = ú$ and $\|w_1, \dots, w_{n-1}, u^* - ú\| = \|w_1, \dots, w_{n-1}, Tu^* - Tú\|$

$$\leq \beta [\|w_1, \dots, w_{n-1}, u^* - Tú\| + \|w_1, \dots, w_{n-1}, ú - Tu^*\|]$$

$$\leq 2\beta \|w_1, \dots, w_{n-1}, u^* - Tú\| \tag{2.18}$$

The inequality (2.18) is a contradiction with $\beta \in [0, \frac{1}{2})$ unless $\|w_1, \dots, w_{n-1}, u^* - ú\| = 0$.

Thus, $u^* = ú$. This completes the proof.

To confirm the independence of condition (2.1) from n-contraction and n-nonexpansive, we give the following example:

Example(2.6): In the case $n = 1$ we recall a simple example, let A=R and T: H→H, $T(u) = 0$, if $u \in (-\infty, 2]$ and $Tu = -\frac{1}{2}$, if $u > 2$. then :

i)T is not continuous (in the sense of usual metric)

ii)T achieves (with$\propto = \frac{1}{5}$) and hence, by Theorem (2.4), T is a Picard operator.

iii)T is not 1-nonexpansive (to show this, take $u = 2$, and $v = \frac{9}{4}$).

Corollary(2.7): i-Let the assumptions in Theorem (2.5) be satisfied. Then the error estimates of the Picard iteration are given by

$$\|w_1, \dots, w_{n-1}, u_n - u^*\| \leq \frac{\alpha^n}{1-\alpha} \|w_1, \dots, w_{n-1}, u_0 - u_1\|, \;\; n = 0,1,2, \dots \tag{2.19}$$

$$\|w_1, \dots, w_{n-1}, u_n - u^*\| \leq \frac{\alpha}{1-\alpha} \|w_1, \dots, w_{n-1}, u_n - u_{n-1}\|, \;\; n = 0,1,2, \dots \tag{2.20}$$

where $\alpha = \frac{\propto}{1-\propto}$

ii-If T any mapping on A and there exists $k \in N \ni T^k$ is a Picard operator, then $F_T = \{u^*\}$.

Proof: i-From (2.12) and (2.13) in Theorem (2.2) we have (2.19) and (2.20).

ii- Let $T^k = S$. By Theorem (1.9), T^k has a unique fixed point u^*. Hence, $T^{k+1}u^* = T(T^k u^*) = T(u^*)$. So $T(u^*)$ is also fixed point of T^k. Since the fixed point of T^k is unique, it must be the case that $Tu^* = u^*$.

Now, we give a generalization of Theorems (2.4)-(2.5) by generalizing (2.1) and (2.10).

Theorem(2.8): Let $(A, \|., \dots, . \|)$ be a n-Banach space and   T : H→H be a mapping (we call it $n -$Zamfirescu mappings) for which there exist the real numbers α, β and γ satisfying $0 \leq \alpha < 1, 0 \leq \beta < 0.5, \ni u, v \in$ H, at least one of the

following is true :

$z_1)$ $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \alpha \|w_1, \dots, w_{n-1}, u - v\|$

$z_2)$ $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \beta [\|w_1, \dots, w_{n-1}, u - Tu\| + \|w_1, \dots, w_{n-1}, v - Tv\|]$

$z_3)$ $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \gamma [\|w_1, \dots, w_{n-1}, u - Tv\| + \|w_1, \dots, w_{n-1}, v - Tu\|]$.

Then T is a Picard operator.

Proof: We first fix $u, v \in$ H. At least one of (z_1), (z_2) or (z_3) is true. If (z_2) holds, then we have

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \beta [\|w_1, \dots, w_{n-1}, u - Tu\| + \|w_1, \dots, w_{n-1}, v - Tv\|]$$

$$\leq \beta\{\|w_1, \dots, w_{n-1}, u - Tu\| + [\|w_1, \dots, w_{n-1}, v - u\|$$

$$+ \|w_1, \dots, w_{n-1}, u - Tu\| + \|w_1, \dots, w_{n-1}, Tu - Tv\|]\}.$$

So

$$(1 - \beta)\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq 2\beta\|w_1, \dots, w_{n-1}, u - Tu\| + \beta\|w_1, \dots, w_{n-1}, u - v\|,$$

Which yields

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \frac{2\beta}{1-\beta}\|w_1, \dots, w_{n-1}, u - Tu\| + \frac{\beta}{1-\beta}\|w_1, \dots, w_{n-1}, u - v\|. \tag{2.21}$$

If (iii) holds, then similarly we get

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \frac{2\gamma}{1-\gamma}\|w_1, \dots, w_{n-1}, u - Tu\| + \frac{\gamma}{1-\gamma}\|w_1, \dots, w_{n-1}, u - v\| \tag{2.22}$$

Therefore denoting:

$\delta = \max\{\propto, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ we have $0 \leq \delta < 1$ and then, $\forall w_1, \dots, w_{n-1}, u, v \in H$

The following inequality

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq 2\delta\|w_1, \dots, w_{n-1}, u - Tu\| + \delta\|w_1, \dots, w_{n-1}, u - v\| \tag{2.23}$$

holds. In a similar manner we obtain

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq 2\delta\|w_1, \dots, w_{n-1}, u - Tv\| + \delta\|w_1, \dots, w_{n-1}, u - v\| \tag{2.24}$$

valid $\forall w_1, \dots, w_{n-1}, u, v \in H$.

From (2.23) it follows that card $F_T \leq 1$ we will show that T has a unique fixed point.

Let $u_\circ \in A$ be arbitrary and $\{u_n\}_{n=0}^{\infty}$, $u_n = T_{u_\circ}^n$, $n = 0,1,2, \dots$

be the Picard iteration associated to T if $u = u_n, v = u_{n-1}$ are two successive approximations,

then by (2.24) we have: $\|w_1, \dots, w_{n-1}, u_{n+1} - u_n\| \leq \delta\|w_1, \dots, w_{n-1}, u_n - u_{n-1}\|$

From this we deduce that $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and hence a convergent sequence. Let $u^* \in H$ be it is limit we have : $\lim_{n\to\infty}\|w_1, \dots, w_{n-1}, u_{n+1} - u_n\| = 0$.

By condition (N_4) and (2.23) we get :

$$\|w_1, \dots, w_{n-1}, u^* - Tu^*\| \leq \|w_1, \dots, w_{n-1}, u^* - u_{n+1}\| + \|w_1, \dots, w_{n-1}, Tu_n - Tu^*\|$$

$$\leq \|w_1, \dots, w_{n-1}, u^* - u_{n+1}\| + \delta\|w_1, \dots, w_{n-1}, u^* - u_n\| + 2\delta\|w_1, \dots, w_{n-1}, u_n - Tu_n\|,$$

which by letting $n \to \infty$ yields

$$\|w_1, \dots, w_{n-1}, u^* - Tu^*\| = 0 \Leftrightarrow u^* = Tu^*, \text{ since}$$

$\|w_1, \dots, w_{n-1}, u_n - Tu_n\| = \|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| \to 0$

And, therefore $F_T = \{u^*\}$ and $u_n \to u^*, (n \to \infty)$ for each $u_\circ \in$ H.

Definition(2.9): Let $(A, \|., \dots, . \|)$ be a n-normed space. A mapping T: H→H is called n-weak

contraction if there exists a constant $\alpha \in (0,1)$ and some $k \geq 0$ ∋

$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \alpha \|w_1, \dots, w_{n-1}, u - v\| + k\|w_1, \dots, w_{n-1}, v - Tu\|,$ (2.25)

$\forall u, v \in$ A.

Remark(2.10): Due to the symmetry of the distance, the n-weak contractive condition (2.25) implicitly includes the following dual one

$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \partial \|w_1, \dots, w_{n-1}, u - v\| + J\|w_1, \dots, w_{n-1}, u - Tv\|$ (2.26)

$\forall u, v \in$ A, $\partial \in (0,1)$ and $J \geq 0$.

Proposition(2.11): Any mapping satisfy condition (2.1), is a n-weak contraction.

Proof: By condition (2.1) and condition (N_4), we get

$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq h[\|w_1, \dots, w_{n-1}, u - Tu\| + \|w_1, \dots, w_{n-1}, v - Tv\|] \leq$

$h\{[\|w_1, \dots, w_{n-1}, u - v\| + \|w_1, \dots, w_{n-1}, v - Tu\|] +$

$[\|w_1, \dots, w_{n-1}, v - Tu\| + \|w_1, \dots, w_{n-1}, Tu - Tv\|]\}$

which yields:

$(1 - h)\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq h\|w_1, \dots, w_{n-1}, u - v\| + 2h\|w_1, \dots, w_{n-1}, v - Tu\|.$

And which implies :

$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \frac{h}{1-h}\|w_1, \dots, w_{n-1}, u - v\| + \frac{2h}{1-h}\|w_1, \dots, w_{n-1}, v - Tu\|,$

$\forall u, v \in$ A, and hence, if $0 < h < \frac{1}{2}$, (1) holds with $\alpha = \frac{h}{1-h}$ and $k = \frac{2h}{1-h}$.

Proposition(2.12): Any mapping T satisfying the contractive condition (2.10) is a n-weak contraction.

Proof: Using $\|w_1, \dots, w_{n-1}, u - Tv\| \leq \|w_1, \dots, w_{n-1}, u - v\| +$ $\|w_1, \dots, w_{n-1}, v - Tu\| + \|w_1, \dots, w_{n-1}, Tu - Tv\|$

we get $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \frac{h}{1-h}\|w_1, \dots, w_{n-1}, u - v\| + \frac{2h}{1-h}\|w_1, \dots, w_{n-1}, v - Tu\|,$

which is (2.25), with $\alpha = \frac{h}{1-h} < 1$ (since $k < \frac{1}{2}$) and $k = \frac{2h}{1-h} \geq 0$.

3. n −Zamfirescu mappings

An immediate consequence of Propositions (2.11) and (2.12) is the following.

Theorem(3.1): Let $(A, \|. , \ldots , . \|)$ be a n-Banach space and T: H→H be a n-weak contraction, i.e., a mapping satisfying (2.25) with $\alpha \in (0,1)$ and some $k \geq 0$. Then

1)$\text{Fix}(T) = \{u \in A: Tu = u\} \neq 0$.

2)For any $u_0 \in A$ the Picard iteration $\{u_n\}_{n=0}^{\infty}$ converges to some $u^* \in \text{Fix}(T)$.

3)The following estimates $\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \dfrac{\alpha^n}{1-\alpha} \|w_1, \ldots, w_{n-1}, u_0 - u_1\|, n = 0,1,2,.$ (3.1)

$$\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \frac{\alpha}{1-\alpha} \|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|, \quad n = 0,1,2, \ldots \tag{3.2}$$

hold, where α is the constant appearing in (2.25)

Proof: We shall prove that T has at least one fixed point in H. To this, let $u_0 \in H$ be arbitrary and let $\{u_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (2).

Take $u = u_{n-1}, v = u_n$ in (2.25) to obtain

$\|w_1, \ldots, w_{n-1}, Tu_{n-1} - Tu_n\| \leq \alpha \|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|$, which shows that

$$\|w_1, \ldots, w_{n-1}, u_n - u_{n+1}\| \leq \alpha \|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|, \tag{3.3}$$

Using (3.3), we obtain by induction

$\|w_1, \ldots, w_{n-1}, u_n - u_{n+1}\| \leq \alpha^n \|w_1, \ldots, w_{n-1}, u_0 - u_1\|, n = 0,1,2, \ldots$

And then

$\|w_1, \ldots, w_{n-1}, u_n - u_{n+p}\| \leq \alpha^n (1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}) \|w_1, \ldots, w_{n-1}, u_0 - u_1\| =$

$$\frac{\alpha^n}{1-\alpha} (1 - \alpha^p) \|w_1, \ldots, w_{n-1}, u_0 - u_1\|, n, p \in N, p \neq 0. \tag{3.4}$$

Since, $0 < \alpha < 1$, (3.4) shows that $\{u_n\}_{n=0}^{\infty}$ is n-Cauchy sequence and hence is

convergent. Denote $u^* = \lim\limits_{n \to \infty} u_n$ $\qquad\qquad\qquad\qquad\qquad\qquad$ (3.5)

then $\|w_1, \ldots, w_{n-1}, u^* - Tu^*\| \leq \|w_1, \ldots, w_{n-1}, u^* - u_{n+1}\| + \|w_1, \ldots, w_{n-1}, u_{n+1} - Tu^*\|$

$\quad = \|w_1, \ldots, w_{n-1}, u_{n+1} - u^*\| + \|w_1, \ldots, w_{n-1}, Tu_n - Tu^*\|.$

By (2.25) we have

$\|w_1, \ldots, w_{n-1}, Tu_n - Tu^*\| \leq \alpha \|w_1, \ldots, w_{n-1}, u_n - u^*\| + k\|w_1, \ldots, w_{n-1}, u^* - Tu_n\|$

And hence $\|w_1, \ldots, w_{n-1}, u^* - Tu^*\| \leq (1 + k)\|w_1, \ldots, w_{n-1}, u^* - u_{n+1}\| +$

$$\alpha \|w_1, \ldots, w_{n-1}, u_n - u^*\|, \ \forall n \geq 0 \tag{3.6}$$

Letting $n \rightarrow \infty$ in (3.6) we obtain $\|w_1, \ldots, w_{n-1}, u^* - Tu^*\| = 0$ i.e., u^* is a fixed point of T. The estimate (3.1) can be obtained from (3.4) by letting $p \rightarrow \infty$. To obtain (3.2), observe that by (3.3) we inductively obtain

$$\|w_1, \ldots, w_{n-1}, u_{n+h} - u_{n+h+1}\| \leq \alpha^{h+1}\|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|, \ h, n \in N$$

Similarly to deriving (3.4) we obtain

$$\|w_1, \ldots, w_{n-1}, u_n - u_{n+p}\| \leq \frac{\alpha(1-\alpha^p)}{1-\alpha}\|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|, \ n \geq 1, \ p \in N^* \tag{3.7}$$

Now letting $p \rightarrow \infty$ in (3.7),(3.2) follows.

Corollary(3.2): Any mapping satisfying the assumptions (z_1)-(z_3) in theorem (2.8), is a n-weak contraction.

Theorem(3.3): Let $(A, \|., \ldots, .\|)$ be a n-Banach space and T: H→H a weak contraction for which there exists $\mu \in (0,1)$ and some $k_1 \geq 0$ ∋

$$\|w_1, \ldots, w_{n-1}, Tu - Tv\| \leq \mu\|w_1, \ldots, w_{n-1}, u - v\| + k_1\|w_1, \ldots, w_{n-1}, u - Tu\|, \forall u, v \in H. \tag{3.8}$$

Then :

1)T has a unique fixed point, i.e., $F_T = \{u^*\}$

2)The Picard iteration $\{u_n\}_{n=0}^{\infty}$ given by (2) converges to u^*, for any $u_0 \in A$.

3)The a priori and a posteriori error estimates.

$$\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \frac{\alpha^n}{1-\alpha}\|w_1, \ldots, w_{n-1}, u_0 - u_1\|, \ n = 0,1,2, \ldots$$

$$\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \frac{\alpha}{1-\alpha}\|w_1, \ldots, w_{n-1}, u_{n-1} - u_n\|, \ n = 1,2, \ldots \ . \text{hold,}$$

4)The rate of convergence of the Picard iteration is given by

$$\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \mu\|w_1, \ldots, w_{n-1}, u_{n-1} - u^*\|, n = 1,2, \ldots \tag{3.9}$$

Proof: Assume T has two distinct fixed point $u^*, v^* \in A$. Then by (3.8) with $u = u^*, v = v^*$, we

get : $\|w_1, \ldots, w_{n-1}, u^* - v^*\| \leq \mu\|w_1, \ldots, w_{n-1}, u^* - v^*\| \Leftrightarrow (1 - \mu)\|w_1, \ldots, w_{n-1}, u^* - v^*\| \leq 0$

So contradicting $\|w_1, \ldots, w_{n-1}, u^* - v^*\| > 0$

Letting $u = u^*, v = u_n$ in (3.8), we obtain the estimate (3.9). The rest of the proof follows by Theorem (3.1).

Definition(3.4): [1]. Let $\sigma : R_+ \rightarrow R_+$ be a function, σ is called a comparison function if the following satisfies

$i_\sigma)\sigma$ is monotone increasing, i.e., $t_1 \leq t_2$ implies $\sigma(t_1) \leq \sigma(t_2)$

$ii_\sigma) \{\sigma^n(t)\}$ converges to 0, $\forall t \geq 0$.

Definition(3.5):[1]. Let $\sigma : R_+ \to R_+$ be a function, σ is called a (c) comparison function if the following satisfies

i_σ) σ is monotone increasing, i.e., $t_1 \le t_2$ implies $\sigma(t_1) \le \sigma(t_2)$

ii_σ) $\sum_{n=0}^{\infty} \sigma^n(t)$ converges $\forall\, t > 0$

Lemma(3.6): [1].

1. If σ is a comparison function then $\sigma(t) < t, \forall\, t > 0$.

2. Any (c)-comparison function is a comparison function.

3. If is a (c)-comparison function, then the function $\mu: R_+ \to R_+, \mu(t) = \sum_{h=0}^{\infty} \sigma^h(t)\, , t \in R_+$ is monotone increasing $\mu(0) = 0$.

Example(3.7): [13].

i. $\sigma(t) = \frac{1}{2}t$, if $0 \le t \le 1$ and $\sigma(t) = t - \frac{1}{2}$, if $t > 1$ is a comparison function

ii. $\sigma(t) = at, t \in R_+, a \in [0,1)$ then σ is (c)-comparison function**.**

Definition(3.8): Let $(A, \|.\, ,\dots ,\,.\,\|)$ be a n-Banach space. A self mapping T: A $\to$ A is said to be a n-weak σ-contraction or n- (α,k)-weak contraction, if there exist a comparison function σ and some $k \ge 0, \ni$

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \le \sigma\|w_1, \dots, w_{n-1}, u - v\| + k\|w_1, \dots, w_{n-1}, v - Tu\|, \forall\, u, v \in A. \qquad (3.10)$$

Remark(3.9):

i.   Any weak contraction is a n-weak σ-contraction, with $\sigma(t) = \alpha t,\ t \in R_+\, , 0 < \alpha < 1$ but the converse is not necessary true.

ii.   All $n-$contractions are weak σ-contractions if $k = 0$ in (2.25).

iii.   In the case of n-weak contractions, the fact that T satisfies (3.10) $\forall\, u, v \in$ A, does imply that the following dual inequality

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \le \sigma\|w_1, \dots, w_{n-1}, u - v\| + k\|w_1, \dots, w_{n-1}, u - Tv\|, \qquad (3.11)$$

obtained from (3.10) by formally replacing $\|w_1, \dots, w_{n-1}, Tu - Tv\|$ and

$\|w_1, \dots, w_{n-1}, u - v\|$ by $\|w_1, \dots, w_{n-1}, Tv - Tu\|$ and $\|w_1, \dots, w_{n-1}, v - u\|$, respectively, and then interchanging u and v, is also satisfied. So, to prove that a mapping T is a $n-$weak σ-contraction, we check the both (3.10) and (3.11).

Now, extend Theorem (3.1) and Theorem (3.3) to $n-$weak σ-contraction.

Theorem(3.10): Let $(A, \|.\, ,\dots ,\,.\,\|)$ be a n-Banach space and T: A $\to$A be a weak σ-contraction with σ a (m)-comparison function. Then:

i)$F(T) = \{u \in A : Tu = u\} \neq \emptyset$

ii)For any $u_0 \in A$ the Picard iteration $\{u_n\}_{n=0}^{\infty}$ defined by $u_0 \in A$ and $u_{n+1} = Tu_n, \ n = 0,1,2,\ldots$ converges to a fixed point u^* of T.

iii)The following estimate $\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \mu(\|w_1, \ldots, w_{n-1}, u_n - u_{n+1}\|), n = 0,1,2,\ldots$ holds, where $\mu(t)$ in Definition (3.4)

Theorem(3.11): Let A and T be as in Theorem (3.8). Suppose T also satisfies the following condition, there exists a comparison function σ and some

$k_1 \geq 0 \ni \|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \sigma(\|w_1, \ldots, w_{n-1}, u - v\|) + k_1 \|w_1, \ldots, w_{n-1}, u - Tu\|,$

holds, $\forall u, v \in A.$ Then

i)T has a unique fixed point, i.e., $F(T) = \{u^*\}$.

ii)The estimate (Theorem (3.3)-3) holds

iii)The rate of convergence of the Picard iteration is given by :

$\|w_1, \ldots, w_{n-1}, u_n - u^*\| \leq \sigma(\|w_1, \ldots, w_{n-1}, u_{n-1} - u^*\|), \ n = 1,2,\ldots.$

The proofs of Theorem (3.8) and (3.9) are essentially similar to those of Theorem (3.1) and(3.3).

References

[1] S. S. Abed, Fixed point principles in general b −metric spaces and b −menger probabilistic spaces, Al-Qadisiyah J. for comp. science, and Math., 10(2)(2018), 10.2.366**.**

[2] S. S. Abed, Approximating fixed points of the general asymptotic set valued mappings, J .of Adva. in Math., 18 (2020), 52-59.

[3] S. S. Abed, H. A. Jabbar, Coupled points for total weakly contraction mappings via ρ distance, International Journal of Basic and Applied Sciences, 5(3)(2016), 164-171.

[4] S. S. Abed, and H. A. Jabbar, Coupled points for total weakly contraction mappings in -m spaces, IJASTR, 6(3)(2016), 64-79.

 [5] A. Amini-Harandi, M. Fakhar, and H. R. Hajisharifi, Approximate fixed points of α-nonexpansive mappings, J. Math. Anal. Appl., 467(2)(2018), 1168–1173.

[6] H. Dutta, On sequence spaces with elements in a sequence of real linear n-normed spaces, Appl. Math. Lett., 23(2010), 1109–1113.

[7] H. Gunawan, On n-inner products, n-norms, and the Cauchy-Schwarz inequality, Soochow J. of Math., 28(4)(2002), 389-398.

[8] H. Gunawan, M. Mashadi, On n-normed spaces, IJMMS, 27(10)(2001), 631–639.

[9] M. A. Khamsi, W. A. Kirk, An introduction to metric fixed point theorems, (2001)Wiley, New York.

[10] J. Meng, M. Song, Fixed point theorems in normal n- normed spaces, J. Math. Comput. Sci., 7(1)(2017), 84-91.

[11] B. E. Rhoades, Fixed point theorems for set-valued mappings[J]. Math Semin Notes, (1982),10.

[12] B. E. Rhoades, Common fixed points of compatible set-valued mappings [J], Publ Math Debrecen, 48(3 - 4)(1996), 237 - 240.

[13] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis, 47(2001), 2683–2693.

[14] N. Shahzad, A.F.R.L. Hierro, F. Khojasteh, Some fixed point theorems under (A, S)-contractivity conditions, Racsam, 111(2)(2017), 307-324.

[15] E. Sukaesih, H. Gunawan, O. Neswan, Fixed point theorems on bounded sets in an n-normed space, J. Math. Anal., 3(2015), 51-58.

[16] N. S. Taresh, S.S. Abed, On Stability of iterative sequences with error, Mathematics, 7(765)(2019), 1-11.

[17] X. Wu, Fixed point problems for nonexpansive mappings in bounded sets of Banach spaces, Adva. Math.l Phys. 2020, Article ID 918, (2016), 1-6.

[18] E. Zidler, Non-linear functional analysis and application, (I-fixed point theorems), 1985. Springer **V**erlag, New York.