On Semi Feebly Open Set and its Properties

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1. Introduction

The topological idea from study this set is generalization the properties and using it's to prove the theorems. In [7] N. Leven (1963) gives the definition of semi open(s-open) set, semi closed (s-closed) set and studies the properties of it . He defined a set A named (s-open) set in topological space $\mathcal{X}$ if find an open set $O \subseteq A \subseteq \overline{O}$ where $\overline{O}$ denoted by the closure of $O$ in $\mathcal{X}$, the complement semi-open (s-open) set called semi-closed (s-closed) set. In (1971) S. G. Crossiey and S. K. Hildebrand defined the concept semi closure and they defined it, the semi closure of a set $A$ in topological space $\mathcal{X}$ is the smallest semi-closed (s-closed) set contained $A$ [2] and shortened by $Scl(A)$ or $\overline{A}^s$. The truth $\overline{A}^s$ is the intersection of all semi closed sets contained $A$, $\overline{A}^s \subseteq \overline{A}$ and $\overline{A}^s = \overline{A}^s$. Maheswari and Tapi (1978) in [3] defined feebly closed (f-closed), feebly open (f-open) set. A set $A$ in a topological space $\mathcal{X}$ named feebly open (f-open) set in $\mathcal{X}$ if find an open set $V$ such that $V \subseteq A \subseteq \overline{V}^s$. A set $A$ in a topological space $\mathcal{X}$ is feebly closed...
if it is complement is feebly open. Every open is (f-open)set, but the converse may be not true. Every closed is (f-closed) set, but the converse may be not true.

We will use the (tp-s) symbol to denote the topological space, (s-open) to semi open set, (s-closed) to semi closed set, (f-open) to feebly open set and (f-closed) to feebly closed set. wherever it is found in this paper.

2. Preliminaries

**Definition (2.1)**[7]: Assume that \( (X, t) \) is a tp-s \& \( A \subseteq X \). Then \( A \) is named s-open in \( X \) if there exists \( O \in t : O \subseteq A \subseteq \overline{O} \). Or equivalent [5], \( A \) called s-open in \( X \) \( \iff \) \( A \subseteq \overline{A} \), equivalent \( \overline{A} = \overline{\overline{A}} \), the complement of s-open is named s-closed.

**Definition (2.2)**[7]: Let \( (X, t) \) be tp-s \& \( A \subseteq X \) then \( A \) called s-closed in \( X \) if there exists a closed set \( F \) such that \( F^o \subseteq A \subseteq F \), or equivalent[5], \( A \) is s-closed in \( X \) \( \iff \) \( \overline{A} \subseteq \overline{A} \), equivalent \( A^o \overline{\overline{A}} \).

**Definition (2.3)**[5]: Let \( (X, t) \) be tp-s \& \( A \subseteq X \), then the intersection of all s-closed subset of \( X \) contained \( A \) named (s-closure) of \( A \) and the union of all s-open subset of \( X \) contained \( A \) named (s-interior) of \( A \) and are shortened by \( \overline{A}^s \), \( A^s \) respectively.

**Proposition (2.4)**[7]: Let \( \{ A_\lambda \}_{\lambda \in \Lambda} \) be a family of s-open in a tp-s \( X \) then \( \bigcup_{\lambda \in \Lambda} A_\lambda \) is s-open.

**Proposition (2.5)**[7]: Let \( X \) be a tp-s then the intersection of two s-open in \( X \) does not need to be s-open.

**Example (2.6)**: Let \( X = \{ k, v, h \}, t = \{ \{k\}, \{v\}, \{k, v\}, X, \emptyset \} \) then each of \{k, h\}, \{v, h\} are s-open, but \{k, h\} \( \cap \) \{v, h\} = \{h\} not s-open.

**Definition (2.7)**[4]: The intersection of every semi open subset of tp-s \( X \) contained a set \( A \) is named Semi kernel of \( A \) and shortened by \( (S \ker (A)) \).

Means that: \( S \ker (A) = \cap \{ U : U \text{ s-open and } A \subseteq U \} \).

**Definition (2.8)**[8]: A set \( A \) in a tp-s \( X \) called f-open in \( X \) if there exists an open set \( V \) such that \( V \subseteq A \subseteq \overline{V}^s \), or equivalent, A set \( A \) called f-open in \( X \) if and only if \( A \subseteq \overline{A}^s \), the complement of f-open is called f-closed that \( \overline{A}^s \subseteq A \).

**Remark (2.9)**[6]: Let \( (X, t) \) be tp-s \& \( A \subseteq X \) then \( A \) is f-open and \( A^c \) is f-closed.

But the converse is not true in general as in the next example.

**Example (2.10)**: Assume that \( X = \{1,2,3,4,5\} \) and \( t = \{ \emptyset, X, \{1\}, \{2,4\}, \{1,2,4\} \} \) then, f-open sets are \{\emptyset, X, \{1\}, \{2,4\}, \{1,2,4\}, \{1,2,3,4\} \}, f-closed sets are \{\emptyset, X, \{2,3\}, \{1,3,5\}, \{3,5\}, \{5\} \}.

Take \( A = \{1,2,3,4\} \) is f-open, but it not open set & \( A^c = \{5\} \) f-closed, but it is not closed.

**Proposition (2.11)**[9]: Assume that \( (X, t) \) is a tp-s & \( A, B \) subsets of \( X \) then:
1. $A^f$ f-closed.
2. $A \subseteq A^f$.
3. $A$ is (f-closed) $\iff A = \overline{A^f}$.
4. $A \subseteq B \Rightarrow \overline{A^f} \subseteq \overline{B^f}$.
5. If $\{ A_\lambda \}_{\lambda \in \Lambda}$ be a collection of subset of $X$ then $\bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f = \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f$.
6. If $\{ A_\lambda \}_{\lambda \in \Lambda}$ be a collection of subset of $X$ then $\bigcap_{\lambda \in \Lambda} \overline{A_\lambda}^f \subseteq \bigcap_{\lambda \in \Lambda} \overline{A_\lambda}^f$.
7. $A^f = \overline{A}^f$.
8. $\overline{A}^f \subseteq A$.
9. $\overline{A}^f = \overline{\overline{A}}^f = \overline{A}$.
10. $\overline{A^f} = A \cup A'$.
11. $\overline{A}^f = A \cup \overline{A}$.
12. $x \in \overline{A}^f \iff$ any f-open $G$ contained $x$, $A \cap G \neq \emptyset$.

**Proposition (2.12)[9]:** Let $X$ be a $tp$-s $A, B$ subset of $X$ where $B$ f-open, If $x \in B$ and $A \cap B = \emptyset$ then $x \notin \overline{A}^f$.

**Definition (2.13)[10]:** Let $X$ be a $tp$-s a subset $A$ of $X$ is said to be

i. Dense or (every dense) in $X$ $\iff \overline{A} = X$.

ii. Nowhere dense or (non-dense) in $X$ iff $(\overline{A})^* = \emptyset$.

**Definition (2.14)[5]:** Let $(X, t)$ be a $tp$-s and $A \subseteq X$, $A$ is named preopen (p-open) if $A \subseteq \overline{A}$ and $A^c$ is named pre closed (p-closed) that $\overline{A}^c \subseteq A$.

**Lemma (2.15)[4]:** Every singleton $\{ x \}$ in a $tp$-s $X$ is either nowhere dense or preopen.

3. The Main Results

**Definition (3.1):** Assume that $(X, t)$ is a $tp$-s then a subset $A$ in a space $X$ is named semi feebly open ($sf$-open) set in a space $X$ if $A \subseteq U$ where $U$ semi open set in $X$ then $\overline{A}^f \subseteq U$. The complement of semi feebly open is called semi feebly closed ($sf$-closed) it is as follows $U \subseteq A^{sf}$ where $U$ semi closed set in $X$. 
Example (3.2): Let $X = \{k, v, h\}$, $\tau = \{X, \emptyset, \{k\}\}$ then

open set: $\{X, \emptyset, \{k\}\}$, closed set: $\{\emptyset, X, \{v, h\}\}$

$s$-open: $\{\emptyset, X, \{k\}, \{k, v\}, \{k, h\}\}$, $s$-closed: $\{\emptyset, X, \{v, h\}, \{h\}, \{v\}\}$

$f$-open: $\{\emptyset, X, \{k\}, \{k, v\}, \{k, h\}\}$, $f$-closed: $\{\emptyset, X, \{v, h\}, \{h\}, \{v\}\}$

$s f$-open $\emptyset$: $\{\emptyset, X, \{v\}, \{h\}, \{v, h\}\}$

we notes that $\{(k), \{k, v\}, \{k, h\}\}$ not $sf$-open because $\{k\} \subseteq \{k\}$ where $\{k\}$ $s$-open, but $\{k\}^f = X \varsubsetneq \{k\}$, $\{k\}$ is not $sf$-open.

$\{k, v\} \subseteq \{k, v\}$ where $\{k, v\}$ $s$-open, but $\{k, v\}^f = X \varsubsetneq \{k, v\}$, $\{k, v\}$ is not $sf$-open.

$\{k, h\} \subseteq \{k, h\}$ where$\{k, h\}$ $s$-open, but $\{k, h\}^f = X \varsubsetneq \{k, h\}$, $\{k, h\}$ is not $sf$-open.

Remark (3.3): Each $f$-closed is $sf$-open.

Proof: Let $A$ be $f$-closed set in a $tp$-$s$ $X$. $A \subseteq U$, $U$ $s$-open, $A$ is ($f$-closed) set then $A = \overline{A}^f$ and $A = \overline{A}^f \subseteq U$ $\Rightarrow A$ is $(sf$-open$)$ set.

The converse of (Remark(3-3)) is not true in general, as in the next example shows:

Example (3.4): Let $X = \{1,2,3,4,5\}$, $\tau = \{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}\}$ and $A = \{1,2,4,5\}$ then $A$ is $sf$-open not $f$-closed.

Proof: The open sets are $\{X, \emptyset, \{1\}, \{3,4\}, \{13,4\}, \{2,3,4,5\}\}$,

the closed sets are $\{\emptyset, X, \{2,3,4,5\}, \{1,2,5\}, \{2,5\}, \{1\}\}$ and

$s$-open sets are $\{X, \emptyset, \{1\}, \{3,4\}, \{13,4\}, \{2,3,4,5\}, \{2,3,4\}, \{3,4,5\}, \{1,2,3,4\}, \{13,4,5\}\}$

Let $A = \{1,2,4,5\}$ and $U = X$ which is $s$-open, then $A \subseteq U$

$\overline{A}^f = \{1,2,4,5\} \cup \overline{\{1,2,4,5\}}^g$ [Proposition (2.11)(11)]

$\overline{A}^f = \{1,2,4,5\} \cup X = X \subseteq U = X$ $\Rightarrow A$ $sf$-open set, but $A$ not $f$-closed because $\overline{\{1,2,4,5\}}^g = X \varsubsetneq \{1,2,4,5\}$.

Remark (3.5): Every closed set is $sf$-open set.

Proof: Let $A$ closed set $\Rightarrow A$ is $f$-closed by [Remark (1.15)] $A$ is $sf$-open set.

But the converse of (Remark(3-5)) in general is not true as in our [Example (3.4)] shows:

$A = \{1,2,4,5\}$ $sf$-open, $\overline{A} = X \neq A \Rightarrow \overline{A} \neq A$ then $A$ is not closed.
The next diagram explains the relationship these types of sets.

\[
\begin{align*}
\text{Open} & \quad \leftrightarrow \quad f\text{-open} & \quad \leftrightarrow \quad s\text{-open} \\
\uparrow & & \downarrow \\
S_f\text{-open} & & \\
\downarrow & & \uparrow \\
\text{Closed} & \quad \leftrightarrow \quad f\text{-closed} & \quad \leftrightarrow \quad s\text{-closed}
\end{align*}
\]

**Notes (3.6)**

For each \( tp\)- set \( \emptyset, X \) are \( sf\)-open.

Every subset of discreet or indiscreet \( tp\)- set is \( sf\)-open.

Every closed interval in \( (R, U) \) where \( U \) is usual topology is \( sf\)-open.

**Proposition (3.7):** Let \( X \) be \( tp\)- set then the union of all \( sf\)-open sets in \( X \) is also \( sf\)-open set.

**Proof:** Let \( \bigcup_{\lambda \in \Lambda} A_\lambda \subseteq U, U \) semi-open in a topological space \( X \) then

\[
A_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda \subseteq U \Rightarrow A_\lambda \subseteq U \quad \Rightarrow \quad \overline{A_\lambda}^f \subseteq U \Rightarrow \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f \subseteq U
\]

since \( \{A_\lambda\}_{\lambda \in \Lambda} \) be a collection of all subset of \( X \) then \( \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f = \bigcup_{\lambda \in \Lambda} A_\lambda^f \leftarrow U \)

\[
\Rightarrow \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f \subseteq U \quad \Rightarrow \quad \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f \subseteq U \quad \text{is} \quad \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f \quad sf\text{-Open}.
\]

**Proposition (3.8):** Let \( A \) be nowhere dense in a \( tp\)- set \( X \) then \( A \) is \( f\)-closed.
Proof: Assume that $A$ is a subset of $X$ so that $A$ is nowhere dense then $(\overline{A})^* = \emptyset$ and $(\overline{A})^0 = \emptyset$, but $\emptyset \subseteq A \Rightarrow A$ is $f$-closed set.

**Lemma (3.9):** A subset $A$ of a $tp$-$s$ $X$ is $sf$-open iff $\overline{A}^f \subseteq S\ker(A)$.

Proof: ($\Rightarrow$) Let $A$ be $sf$-open in $X$, then $\overline{A}^f \subseteq U$ when $A \subseteq U$ and U is $sf$-open in $X$, this emplace $\overline{A}^f \subseteq \cap \{U : A \subseteq U$ and $U \in s$-open$\}) = S\ker(A)$. ($\Leftarrow$) Conversely, assume that $\overline{A}^f \subseteq S\ker(A)$ $\Rightarrow \overline{A}^f \subseteq \cap \{U : A \subseteq U$ and $U \in s$-open$\}) \Rightarrow \overline{A}^f \subseteq U$ for all $s$-open set $U$ in $X$.

**Proposition (3.10):** Let $X$ be $tp$-$s$, then the arbitrary intersection of $sf$-open sets in $X$ is $sf$-open set.

Proof: Let $\{A_{\lambda} : \lambda \in \Lambda \}$ be arbitrary collection of $sf$-open sets in a space $X$, and let $A = \cap A_{\lambda}$, let $x \in \overline{A}^f$ then by [Lemma (2.15)] we consider the following two cases.

**Case 1:** \{x\} is nowhere dense

If $x \notin A$ then for some $\alpha \in \Lambda$ we have $x \notin A$, as nowhere dense subset are feebly closed [Proposition (3.8)] there for $x \notin S\ker(A)$.

On the other hand by [Lemma (3.9)] $A_{\alpha}$ is $sf$-open, then $x \in \overline{A}^f \subseteq \overline{A_{\alpha}}^f \subseteq S\ker(A)$ paradoxically $x \in A$ and hence $x \in S\ker(A)$ $\Rightarrow \overline{A}^f \subseteq S\ker(A)$, $A$ $sf$-open set.

**Case 2:** \{x\} is preopen, Let $F = \overline{\{x\}}^0$ and $x \notin S\ker(A)$, $\exists$ semi closed set $C$ containing $X$, so that $C \cap A = \emptyset$, $x \in F = \overline{\{x\}}^0 \subseteq \overline{C} \subseteq C$. As $F$ is an open set containing $x$ and $x \in \overline{A}^f$ therefore, $F \cap A \neq \emptyset$ as $F \subseteq C \Rightarrow C \cap A = \emptyset$ paradoxically $x \in S\ker(A) \Rightarrow A$ $sf$-open set.

**Proposition (3.11):** If $A$ is $sf$-open and $B$ $f$-closed in a $tp$-$s$ $X$ then $A \cap B$ is $sf$-open in $X$.

Proof: Assume that $A \cap B \subseteq U$ where U is $s$-open set then $A \cap B \cap U^c = \emptyset$

$\Rightarrow A \cap (B \cap U^c) = \emptyset \Rightarrow A \subseteq B^c \cup U$ ,but $B^c \cup U$ $s$-open $\overline{A}^f \subseteq B^c \cup U$

$\Rightarrow \overline{A}^f \cap (B^c \cup U)^c = \emptyset \Rightarrow \overline{A}^f \cap B \subseteq U \Rightarrow \overline{A \cap B}^f \subseteq U \Rightarrow A \cap B$ $sf$-open.

**Proposition (3.12):** Assume that $X$ is a $tp$-$s$ & $A \subseteq X$ then $\overline{A}^f$ is $sf$-open set.

Proof: Let $\overline{A}^f \subseteq G$ where $G$ is $s$-open set, since $\overline{\overline{A}^f} = \overline{A}^f \subseteq U$

$\Rightarrow \overline{A}^f \subseteq G \Rightarrow \overline{A}^f$ $sf$-open set.

**Proposition (3.13):** Assume that $X$ is a $tp$-$s$ & $A \subseteq X$ then $\overline{A}$ is $sf$-open set.
Proof: Let $\overline{A} \subseteq U$ where $U$ is s-open set, since $\overline{\overline{A}} = \overline{A} = \overline{A}$ [Proposition(2.11)(9)]. Then $\overline{\overline{A}} = \overline{A} \Rightarrow \overline{\overline{A}} \subseteq U$ where $U$ is s-open $\Rightarrow \overline{A}$ is sf-open set.

**Proposition(3.14):** Assume that $X$ is an $tp-s$ & $A \subseteq X$, if $A$ is s-closed and pre closed then $A$ is sf-open set.

Proof: Let $A$ is s-closed then $A^o = \overline{A}$, since $A$ pre closed then $\overline{A^o} \subseteq A$, but $A^o = \overline{A}$ then $\overline{\overline{A}} \subseteq A \Rightarrow A$ f-closed by using [Remark(3.3)] $A$ is sf-open set.

**Definition(3.15):** Assume that $X$ is a $tp-s$ & $A \subseteq X$ .Then the intersection of all sf-closed of $X$ which containing $A$ is named sf-closure of $A$ and shortened by $A^{sf}$, that means $A^{sf} = \cap \{F: F \text{ is sf-closed in } X\}$.

**Lemma(3.16):** Assume that $X$ is a $tp-s$ & $A \subseteq X$. Then $x \in A^{sf}$ iff for all sf-open set $G$ and $x \in G$, $G \cap A \neq \emptyset$.

Proof:($\Rightarrow$) Assume that $x \notin A^{sf}$ then $x \notin \{F: F \text{ is sf-closed in } X\}$ and $A \subseteq F$, then $x \in [\cap F]^C$, $[\cap F]^C$ sf-open containing $x$. Hence $[\cap F]^C \cap A \subseteq [\cap F]^C \cap [\cap F] = \emptyset$.($\Leftarrow$) Conversely, Suppose that $\exists$ sf-open set $G$ so that $x \in G$, $G \cap A = \emptyset$ then $A \subseteq G^C$, $G^C$ is sf-closed hence $x \notin A^{sf}$.

**Definition(3.17):** Let $X$ be a $tp-s$ , $x \in X$ & $A \subseteq X$ .The point $x$ is called sf-limit point of $A$ if each sf-open set containing $U$, contains a point of $A$ distinct from $x$. We shall call the set of all sf-limit point of $A$ the sf-derivative set of $A$ and denoted by $A^{isf}$. Therefore $x \in A^{isf}$ if for every sf-open set $U$ in $X$ such $x \in V$ implies that $\cap(A - \{x\}) \neq \emptyset$.

**Proposition(3.18):** Let $X$ be a $tp-s$ and $A \subseteq B \subseteq X$. Then:

1. $A^{sf} = A \cup A^{isf}$.
2. $A$ is an sf-closed set iff $A^{isf} \subseteq A$.
3. $A^{isf} \subseteq B^{isf}$.

Proof: 1- By definition $A \subseteq \overline{A}^{sf}$ ....(1). Let $x \in A^{isf} \Rightarrow x \notin A$. Then $\forall$ sf-open set $U$ contained $x$, then $(U \cap A) - \{x\} \neq \emptyset$. Then $\forall$ sf-open set in $U$ contained $x$, then $U \cap A \neq \emptyset$ by [Lemma(3.16)]. Then $x \in \overline{A^{isf}} \Rightarrow A^{isf} \subseteq A^{sf}$ ....(2). From (1) and (2) $A \cup A^{isf} \subseteq \overline{A^{sf}}$.

Let $x \in \overline{A^{sf}}$. Since $A \subseteq \overline{A^{sf}}$ by definition and $\forall x \in \overline{A^{sf}}$ Then either $x \in A$ or $x \notin A$. If $x \in A \Rightarrow x \in A \cup A^{isf}$ and if $x \notin A$. Since $x \in \overline{A^{sf}} \Rightarrow \forall$ sf-open set $U$ contained $x$, then $U \cap A \neq \emptyset$, Since $x \notin A$ then $(U \cap A) - \{x\} \neq \emptyset$. Then $x \in A^{isf} \Rightarrow x \in A \cup A^{isf}$ then $\overline{A^{sf}} \subseteq A \cup A^{isf}$ then $\overline{A^{sf}} \subseteq A \cup A^{isf}$.
2- (⇒) Let $A^{sf} \subseteq A$. $\overline{A}^sf = A \cup A^{sf} \subseteq A$, since $A \subseteq \overline{A}$ then $A = \overline{A}^sf$, then $A$ is an $sf$-closed set.

(⇐) Let $A$ be $sf$-closed set. Thus $A = \overline{A}^sf$ from [proposition (3.18)(1)]. $A = A \cup A^{sf}$ then $A^{sf} \subseteq A$.

3- Let $A \subseteq B$ and let $x \in A^{sf}$, $\forall U$ is $sf$-open set contained $x$ then $(U \cap A) - \{x\} \neq \emptyset$. Since $A \subseteq B \Rightarrow (U \cap B) - \{x\} \neq \emptyset$. Then $x \in B^{sf}$ then $A^{sf} \subseteq B^{sf}$.

**Remark(3.19):** Assume that $\chi$ is a $tp$-$s$ & $A \subseteq \chi$, then $\overline{A}^sf$ is smallest $sf$-closed set containing $A$.

**proof:** Suppose that $B$ is $sf$-closed set contend such that $A \subseteq B$ since $\overline{A}^{sf} = A \cup A^{sf}$. And $\overline{A}^{sf} \subseteq B^{sf}$, $A \subseteq B$, then $\overline{A}^{sf} = A \cup A^{sf} \subseteq A \cup A^{sf} \subseteq B$, then $\overline{A}^{sf} \subseteq B$ therefore $\overline{A}^{sf}$ is smallest $sf$-closed set contained $A$.

**Proposition(3.20):** Let $\chi$ be a $tp$-$s$ & $A, B$ are subset of $\chi$ with $B$ $sf$-open set. If $x \in B$ and $B \cap A = \emptyset$ then $x \notin \overline{A}^{sf}$.

**proof:** Suppose $x \in \overline{A}^{sf}$, then either $x \in A$ or $x \in A^{sf}$. If $x \in A$, then $B \cap A \neq \emptyset$ which contradicts the assumption and if $x \in A^{sf}$ and $x \notin A$, then $(B \cap A) - \{x\} \neq \emptyset$ for every $sf$-open $G$ in $\chi$ containing $x$ and hence $G \cap A \neq \emptyset$ which is a contradiction since $B$ is $sf$-open set containing $x$ and $B \cap A = \emptyset$ and hence $x \notin \overline{A}^{sf}$.

**Definition(3.21):** Assume that $\chi$ is a $tp$-$s$ & $B \subseteq X$. An $sf$-neighborhood of $B$ is any subset of $\chi$ which contains an $sf$-open set containing $B$. The $sf$-neighborhood of a subset $\{x\}$ is also called $sf$-neighborhood of the point $x$.

**Definition(3.22):** Assume that $A$ is a subset of a $tp$-$s$ $\chi$. For each $x \in \chi$, then $x$ is said to be $sf$-boundary point of $A$ if each $sf$-neighborhood $U_x$ of $x$, we have $U_x \cap A \neq \emptyset$ and $U_x \cap A^c \neq \emptyset$. The set of all $sf$-boundary point of $A$ is denoted by $b_{sf}(A)$.

**Proposition(3.23):** Assume that $\chi$ is a $tp$-$s$ and $A, B \subseteq \chi$, then

1. $A$ is an $sf$-closed set $\iff A = \overline{A}^{sf}$.
2. $\overline{A}^{sf} \subseteq \overline{A}$.
3. $\overline{A}^{sf} = \overline{A}^{sf}$.
4. If $A \subseteq B$ then $\overline{A}^{sf} \subseteq B^{sf}$.

**proof:** 1- (⇒) Let $A$ is an $sf$-closed set. Since $A \subseteq \overline{A}^{sf}$. Then $\overline{A}^{sf} \subseteq A$ (since $\overline{A}^{sf}$ is the smallest $sf$-closed set containing $A$), then $A = \overline{A}^{sf}$. 


\( (\Leftarrow) \) Let \( \overline{A}^{sf} = A \). Then \( \overline{A}^{sf} \) is an \( sf \)-closed set. As \( A = \overline{A}^{sf} \Rightarrow A \) is a \( sf \)-closed set.

2- Let \( x \in \overline{A}^{sf} \) and \( A \) is a \( sf \)-closed set, then \( A = \overline{A}^{sf} \Rightarrow x \in A \subseteq \overline{A} \). Then \( x \in \overline{A} \). Therefore \( \overline{A}^{sf} \subseteq \overline{A} \).

3- Since \( \overline{A}^{sf} \) is \( sf \)-closed set, then \( \overline{A}^{sf} = \overline{A}^{sf} \overline{A}^{sf} \) by (2).

4- Let \( A \subseteq B \) and \( B \subseteq \overline{B}^{sf} \), then \( A \subseteq \overline{B}^{sf} \Rightarrow \overline{B}^{sf} \) is a \( sf \)-closed set containing \( A \). Since \( \overline{A}^{sf} \) is smallest \( sf \)-closed set containing \( A \). Then \( \overline{A}^{sf} \subseteq \overline{B}^{sf} \).

**Definition (3.24):** Assume that \( \chi \) is \( tp-s \) and \( A \subseteq \chi \). The union of all \( sf \)-open sets of \( \chi \) contained in \( A \) is named \( sf \)-Interior of \( A \), shortened by \( A^{sf} \) or \( sf-\text{In}_t(A) \), that means \( sf-\text{In}_t(A) = \cup \{ B : B \subseteq A \} \text{ is } sf \text{-open in } X \text{ and } B \subseteq A \).

**Proposition (3.25):** Assume that \( \chi \) is \( tp-s \) and \( A \subseteq X \). Then \( \overline{A}^{sf} = \left( A^{c^{sf}} \right)^c \).

Proof: Since \( A \subseteq \overline{A}^{sf} \Rightarrow \overline{A}^{sf} \subseteq A^c \Rightarrow \overline{A}^{sf} \subseteq A^c \Rightarrow A^{c^{sf}} \subseteq A^{c^{sf}} \Rightarrow A^{c^{sf}} \subseteq A^{c^{sf}} \subseteq \overline{A}^{sf} \) ....(1).

Since \( A^{c^{sf}} \subseteq A^c \Rightarrow A \subseteq A^{c^{sf}} \Rightarrow \overline{A}^{sf} \subseteq A^{c^{sf}} \subseteq A^{c^{sf}} \subseteq \overline{A}^{sf} \) ....(2). From (1) and (2) we get \( \overline{A}^{sf} = \left( A^{c^{sf}} \right)^c \).

**Proposition (3.26):** Assume that \( \chi \) is \( tp-s \) and \( A \subseteq \chi \). Then \( x \in A^{sf} \) iff there is an \( sf \)-open set \( U \) containing \( x \) so that \( x \in U \subseteq A \).

Proof: Assume that \( x \in A^{sf} \Leftrightarrow x \in \bigcup \{ U : U \subseteq A \text{ such that } U \text{ is } sf \text{-open in } X \} \Leftrightarrow \exists U \text{ is } sf \text{-open in } X \text{ so that } x \in U \subseteq A \).

**Proposition (3.27):** Assume that \( \chi \) \( tp-s \) and \( A \subseteq B \subseteq \chi \), then:

1. \( A^{sf} \) is an \( sf \)-open set.
2. \( A \) is an \( sf \)-open set iff \( A = A^{sf} \).
3. \( A^{sf} = A^{sf^{sf}} \).
4. If \( A \subseteq B \) then \( A^{sf} \subseteq B^{sf} \).

Proof: 1- \( A^{sf} = \bigcup \{ B : B \text{ is } sf \text{-open and } B \subseteq A \} \), by [proposition (3.7)]. Then \( A^{sf} \) is an \( sf \)-open set.

2- \( \Rightarrow \) Let \( A \) be an \( sf \)-open set from definition. \( A^{sf} \subseteq A, A^{sf} = \bigcup \{ U : U \subseteq A, U \text{ is an } sf \text{-open set in } X \} \). Since \( A \) is \( sf \)-open set in \( X \). Then \( A \subseteq A^{sf} \Rightarrow A = A^{sf} \).

\( \Leftarrow \) Let \( A = A^{sf} \), since \( A^{sf} \) is the union \( sf \)-open sets and since \( A^{sf} = A \Rightarrow A \) is a \( sf \)-open set.
3- Let $A^{\text{sf}} = \bigcup \{ B : B \text{ is an sf-open set in } X \text{ and } B \subseteq A \}$. Then $A^{\text{sf}} = A^{\text{sf}}$. By (2) $A^{\text{sf}} = A^{\text{sf}}$.

4- Let $A \subseteq B$ & $x \in A^{\text{sf}}$. Then $\exists$ sf-open $U$ in $X$ such that $x \in U \subseteq A$. Since $A \subseteq B$. Then $\exists$ sf-open $U$ in $X$ such that $x \in U \subseteq A \subseteq B$. $x \in U \Rightarrow x \in B^{\text{sf}}$. Then $A^{\text{sf}} \subseteq B^{\text{sf}}$.

**Proposition (3.28):** Assume that $\chi$ is a tp-$s$ & $A \subseteq \chi$. Then:

1. $b_{\text{sf}}(A) = A^{\text{sf}} \cap \overline{A}^{\text{sf}}$.
2. $A^{\text{sf}} = A - b_{\text{sf}}(A)$.
3. $\overline{A}^{\text{sf}} = A \cup b_{\text{sf}}(A)$.

Proof: Clear

**Proposition (3.29):** Assume that $\chi$ is a tp-$s$ & $A \subseteq \chi$. Then:

1. $A^{\text{sf}} = A^{\text{sf}} \cup b_{\text{sf}}(A)$.
2. $A$ is an sf-open set $\Leftrightarrow b_{\text{sf}}(A) \subseteq A^c$.
3. $(A^{\text{sf}})^c = \left( \overline{A}^{\text{sf}} \right)^c$.

Proof: Clear

**References**


