A Subclass of Harmonic Univalent Functions Defined by Salagean Integro Differential Operator

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1. Introduction

Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examine the class of harmonic univalent functions, which is a subclass of harmonic functions. Let's show the open unit disk as $\mathbb{U}$. Let's show the family of continuous complex-valued harmonic functions that are harmonic in $\mathbb{U}$ as $\mathcal{H}$. $\mathcal{A}$ denotes the subclass of $\mathcal{H}$, including functions that are analytic in the open unit disk. $f$ be a harmonic function in $\mathbb{U}$ and $f$ may be written as $f = \mathfrak{h} + \overline{\mathfrak{g}}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are in $\mathcal{A}$. We denominate $\mathfrak{h}$ is the analytic part of $f$, and $\mathfrak{g}$ is the co-analytic part of $f$. $\mathcal{S}$ denotes normalized analytic univalent functions in the open unit disk.

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ABSTRACT

In this paper, we scrutinize some fundamental features of a subclass of harmonic functions defined by a new operator. Like coefficient inequalities, convex combinations, distortion bounds.

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Clunie and Sheil-Small [5] proved that \( f \) is sense-preserving and locally univalent in \( \mathbb{U} \) if and only if \( |h'(z)| > |g'(z)| \). So no loss of generality we can write \( h(0) = 0 \) and \( h'(0) = 1 \) since \( h'(z) \neq 0 \).

In this way, writing
\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k
\]
does not break generality.

Let \( \mathcal{SH} \) indicate the family of functions \( f = h + \overline{g} \) which are univalent, harmonic and sense-preserving in \( \mathbb{U} \) for \( f \) functions that provide \( f(0) = f_\mathbb{C}(0) - 1 = 0 \).

Consequently, \( \mathcal{SH} \) includes the class of \( \mathcal{S} \), which is normalized analytic univalent functions. Clearly, it will become apparent that although the analytic part \( h \) of a function \( f \in \mathcal{SH} \) is locally univalent, it need not be univalent. It can be easily demonstrated that the inequality of \( |b_1| < 1 \) must be achieved in order to have sense-preserving property.

The subclass \( \mathcal{SH}^0 \) of \( \mathcal{SH} \) covers all functions in \( \mathcal{SH} \) with \( f_\mathbb{C}(0) = 0 \) property. Clunie and Sheil-Small [5] examined \( \mathcal{SH} \) as well as its subclasses and discovered some coefficient bounds. Thenceforward, there have been various associated articles on \( \mathcal{SH} \) and its subclasses. For more details see ([1], [3], [4], [5], [6], [7], [8], [10], [11], [13], [14] and [15] etc.). Furthermore, notice that \( \mathcal{SH} \) reduces to the class \( \mathcal{S} \), if the co-analytic part of \( f \) is identically zero.

For \( f \in \mathcal{A} \), \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the Salagean differential operator \( D^n \) is defined by Salagean [9] \( D^n : \mathcal{A} \to \mathcal{A} \),
\[
D^0 f(z) = f(z) \\
\vdots \\
D^{n+1} f(z) = z D^n f(z)'.
\]

For
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
we get
\[
D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.
\]

This operator was developed and modified by many researchers over time. As a simple example for \( f = h + \overline{g} \) as stated in (1), Jahangiri et al. [8] described the modified Salagean operator of \( f \) like as
\[
D^0 f(z) = f(z), \\
\vdots \\
D^n f(z) = D^n h(z) + (\overline{g}) \overline{D^n (\overline{g})}(\overline{z}),
\]
where
\[
D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k
\]
and

\[ D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k. \]

Salagean [9] defined a linear operator denoted by \( I^n \) given by

\[
\begin{align*}
I^0 f(z) &= f(z), \\
I' f(z) &= \frac{1}{z} \int_{0}^{z} \frac{f(t)}{t} dt, \\
&\quad \text{\ldots} \\
I^n f(z) &= I(I^{n-1} f(z)),
\end{align*}
\]

where \( f(z) \in \mathcal{A} \), \( z \in \mathbb{U} \), \( n \in \mathbb{N}_0 \) and

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

then

\[ I^n f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \]

Then Pall-Szabo [2] defined a linear operator for \( z \in \mathbb{U}, \lambda \geq 0, n \in \mathbb{N}_0 \) denoted by \( \mathcal{L}^n \) given by \( \mathcal{L}^n : \mathcal{A} \rightarrow \mathcal{A} \),

\[ \mathcal{L}^n f(z) = (1 - \lambda) D^n f(z) + \lambda I^n f(z). \]

Therefore, if \( f(z) \in \mathcal{A} \) and

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]

then

\[ \mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[ k^n (1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^k. \tag{2} \]

Now we define the Salagean integro differential operator for functions \( f = h + \bar{g} \) as stated in (1),

\[
\begin{align*}
\mathcal{L}^0 f(z) &= (1 - \lambda) D^0 f(z) + \lambda I^0 f(z) \\
&\quad \text{\ldots} \\
\mathcal{L}^n f(z) &= (1 - \lambda) D^n f(z) + \lambda I^n f(z) \\
&= (1 - \lambda) \left( D^n h(z) + (-1)^n \bar{D}^n g(z) \right) + \lambda \left( I^n h(z) + (-1)^n \bar{I}^n g(z) \right),
\end{align*}
\]

where \( n \in \mathbb{N}_0, 0 \leq \lambda \leq 1 \). Therefore
\( \mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[ k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[ k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right] b_k z^k. \) \hspace{1cm} (3)

Let show the subclass of \( \mathcal{SH} \) containing functions \( f \) as stated in (1) which provide the following condition as \( \mathcal{SH}(\lambda, n, \mu) \)

\[ \text{Re} \left( \frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \right) \geq \mu, \quad 0 \leq \mu < 1 \] \hspace{1cm} (4)

where \( \mathcal{L}^n f(z) \), as stated in (2).

We denote the subclass \( \overline{\mathcal{SH}}(\lambda, n, \mu) \) containing harmonic functions \( f_n = h + \overline{g_n} \) in \( \mathcal{SH} \) so that \( h \) and \( g_n \) be defined as follows

\[ h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \] \hspace{1cm} (5)

\( a_k, b_k \geq 0. \)

If the parameters are chosen specially, \( \mathcal{SH}(\lambda, n, \mu) \) classes are reduced to different subclasses of harmonic univalent functions (For example Jahangiri [7], Jahangiri et al. [8], Silverman [10], Silverman and Silvia [11], Uralegaddi and Somanatha [12], Cho and Srivastava [4], Bayram and Yalcin [13], Atshan and Wanas [16]).

2. Main Results

In Theorem 2.1, we present a sufficient coefficient condition for harmonic univalent functions that are the members of \( \mathcal{SH}^0(\lambda, n, \mu) \).

2.1 Theorem

Let \( f = h + \overline{g} \). In here let \( h \) and \( g \) are as stated in (1) with \( b_1 = 0. \) Let

\[ \sum_{k=2}^{\infty} \left[ k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right] \left[ k^n(1 - \lambda) + \lambda \frac{1}{k^n} - \mu \right] |a_k| + \sum_{k=2}^{\infty} \left[ k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right] \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} + \mu \right) |b_k| \leq 1 - \mu, \] \hspace{1cm} (6)

where \( n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1. \) Then \( f \) is sense–preserving and harmonic univalent in \( \mathbb{U} \) so we can say that \( f \in \mathcal{SH}^0(\lambda, n, \mu) \).

As a special notation for convenience, sometimes, we write

\[ A_n = \left[ k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right] \left( k^n(1 - \lambda) + \lambda \frac{1}{k^n} - \mu \right) \]

and

\[ B_n = \left[ k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right] \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} + \mu \right) \]

and

\[ C_n = k^n(1 - \lambda) + \lambda \frac{1}{k^n}. \]
in this article.

**Proof.** If we accept that \( z_1 \neq z_2 \), we obtain

\[
\frac{|f(z_1) - f(z_2)|}{|h(z_1) - h(z_2)|} \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \frac{\sum_{k=2}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}
\]

\[
> 1 - \frac{\sum_{k=2}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|}
\]

\[
\geq 1 - \frac{\sum_{k=2}^{\infty} \frac{b_n}{1-\mu}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{a_n}{1-\mu}|a_k|} \geq 0
\]

that proving univalence. Pay attention \( f \) is sense–preserving in \( \mathbb{U} \). To show this feature;

\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k| |z|^{k-1}
\]

\[
> 1 - \sum_{k=2}^{\infty} \frac{A_n}{1-\mu}|a_k|
\]

\[
\geq \sum_{k=2}^{\infty} \frac{B_n}{1-\mu}|b_k|
\]

\[
> \sum_{k=2}^{\infty} k|b_k| |z|^{k-1}
\]

\[
\geq |g'(z)|.
\]

Using the fact that \( \text{Re } w \geq \alpha \iff |1 - \alpha + w| \geq |1 + \alpha - w| \), it suffices to show the following inequality

\[
|(1-\mu)\mathcal{L}^n f(z) + \mathcal{L}^{n+1} f(z)| - |(1+\mu)\mathcal{L}^n f(z) - \mathcal{L}^{n+1} f(z)| \geq 0. \tag{7}
\]

Replacing for \( \mathcal{L}^n f(z) \) and \( \mathcal{L}^{n+1} f(z) \) in (7), we obtain

\[
|(1-\mu)\mathcal{L}^n f(z) + \mathcal{L}^{n+1} f(z)| - |(1+\mu)\mathcal{L}^n f(z) - \mathcal{L}^{n+1} f(z)|
\]

\[
\geq 2(1-\mu)|z| - \sum_{k=2}^{\infty} C_n(C_n + 1 - \mu)|a_k| |z|^k - \sum_{k=2}^{\infty} C_n(C_n - 1 + \mu)|b_k| |z|^k
\]

\[
- \sum_{k=2}^{\infty} C_n(C_n - 1 - \mu)|a_k| |z|^k - \sum_{k=2}^{\infty} C_n(C_n + 1 + \mu)|b_k| |z|^k
\]

\[
> 2(1-\mu)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} - \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} \right\}.
\]

This last phrase is not negative with (6), and therefore the proof is complete.
2.2 Theorem

Let $f_n = h + \frac{\partial_n}{\partial_n}$ be as stated in (4) with $b_1 = 0$. Then $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$, if and only if

$$\sum_{k=2}^{\infty} A_n |a_k| + \sum_{k=2}^{\infty} B_n |b_k| \leq 1 - \mu, \quad (8)$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$.

**Proof.** Notice that $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is a subclass of $\mathcal{SH}^0(\lambda, n, \mu)$. Then the “if” part can be easily proved from 2.1 Theorem. For the “only if” part, we must demonstrate that $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ if the inequality (8) is not valid. Here is an issue to be considered; a necessary and sufficient condition for $f_n = h + \frac{\partial_n}{\partial_n}$ be as stated in (5), to be in $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is that the condition (4) must be provided.

The above stipulation must be valid for all $z$ complex numbers where, $|z| = r < 1$. After selecting the $z$ values from the positive real axis where $0 \leq |z| = r < 1$, we obtain

$$\text{Re} \left\{ \frac{(1 - \mu)z - \sum_{k=2}^{\infty} A_n a_k z^k - \sum_{k=2}^{\infty} B_n b_k \bar{z}^k}{z - \sum_{k=2}^{\infty} C_n a_k z^k + \sum_{k=2}^{\infty} C_n b_k \bar{z}^k} \right\} \geq 0.$$

This is equivalent to

$$\left\{ \frac{(1 - \mu) - \sum_{k=2}^{\infty} A_n a_k r^{k-1} - \sum_{k=2}^{\infty} B_n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} C_n a_k r^{k-1} + \sum_{k=2}^{\infty} C_n b_k r^{k-1}} \right\} \geq 0. \quad (9)$$

If the inequality (8) is not valid, then the expression in (9) is negative for $r$ values approaching 1. Hence there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (9) is negative. This shows the required condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ and so the proof is complete.

2.3 Theorem

Let $f_n = h + \frac{\partial_n}{\partial_n}$ be as stated in (5). At that case a necessary and sufficient condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is

$$f_n(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_{nk}(z) \right),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1 - \mu}{A_n} z^k$$

and

$$g_{n1}(z) = z, \quad g_{nk}(z) = z + (-1)^n \frac{1 - \mu}{B_n} z^k$$

for $X_k \geq 0$, $Y_k \geq 0$, $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$, $k = 2, 3, \ldots, n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$.

Notably, the extreme points of $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ are $\{h_k\}$ and $\{g_{nk}\}$.

**Proof.** For $f_n$ functions which are as stated in (5), we obtain
\[ f_n(z) = \sum_{k=1}^{\infty} \left( X_k b_k(z) + Y_k g_{nk}(z) \right) \]

\[ = \sum_{k=1}^{\infty} (X_k + Y_k)z - \sum_{k=2}^{\infty} \frac{1-\mu}{A_n} X_k z^k + (-1)^n \sum_{k=2}^{\infty} \frac{1-\mu}{B_n} Y_k z^k. \]

Then

\[ \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} \left( \frac{1-\mu}{A_n} X_k \right) + \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} \left( \frac{1-\mu}{B_n} Y_k \right) \]

\[ = \sum_{k=2}^{\infty} X_k + \sum_{k=2}^{\infty} Y_k = 1 - X_1 - Y_1 \leq 1 \]

and so \( f_n \in \mathcal{SH}(\lambda, n, \mu) \). Moreover, if \( f_n \in \mathcal{SH}(\lambda, n, \mu) \), then

\[ a_k \leq \frac{1-\mu}{A_n} \]

and

\[ b_k \leq \frac{1-\mu}{B_n} \]

Set

\[ X_k = \frac{A_n}{1-\mu} a_k, \]

\[ Y_k = \frac{B_n}{1-\mu} b_k, \]

for \( k = 2, 3, \ldots \) and

\[ X_1 + Y_1 = 1 - \left( \sum_{k=2}^{\infty} X_k + Y_k \right) \]

where \( X_k \geq 0, Y_k \geq 0 \). Therefore, in accordance with, we have

\[ f_n(z) = (X_1 + Y_1)z + \sum_{k=2}^{\infty} \left( X_k b_k(z) + Y_k g_{nk}(z) \right) \]

\[ = \sum_{k=1}^{\infty} \left( X_k b_k(z) + Y_k g_{nk}(z) \right). \]

### 2.4 Theorem

Let \( f_n \in \mathcal{SH}(\lambda, n, \mu) \). Then for \( |z| = r < 1 \) and \( n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1 \), we have
\[ r - \frac{1 - \mu}{A_n} r^2 \leq |f_n(z)| \leq r + \frac{1 - \mu}{A_n} r^2. \]

**Proof.** Here, we will only prove the right side since the proving of the left side, and the right side of the inequality is very similar. Let \( f_n \in \mathcal{SH}^\sigma(\lambda, n, \mu) \). Taking the absolute value of \( f_n \) we can easily see

\[ |f_n(z)| \leq r + \sum_{k=2}^{\infty} (a_k + b_k)r^k \leq r + \frac{(1 - \mu)r^2}{A_n} \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} |a_k| + \frac{(1 - \mu)r^2}{A_n} \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} |b_k| \leq r + \frac{1 - \mu}{A_n} r^2. \]

The left side can be shown in a similar way.

**2.5 Theorem**

The class \( \mathcal{SH}^\sigma(\lambda, n, \mu) \) is closed under convex combination.

**Proof.** Let \( f_{nj} \in \mathcal{SH}^\sigma(\lambda, n, \mu) \) for \( j = 1, 2, \ldots \), where \( f_{nj} \) is given by

\[ f_{nj}(z) = z - \sum_{k=2}^{\infty} a_{kj}z^k + (-1)^n \sum_{k=2}^{\infty} b_{kj}z^k. \]

Then by (7),

\[ \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} a_{kj} + \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} b_{kj} \leq 1. \]  

(10)

For \( \sum_{j=1}^{\infty} p_j = 1, 0 \leq p_j \leq 1 \), the convex combination of \( f_{nj} \) can be written as

\[ \sum_{j=1}^{\infty} p_j f_{nj}(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{\infty} p_j a_{kj} \right) z^k + (-1)^n \sum_{k=2}^{\infty} \left( \sum_{j=1}^{\infty} p_j b_{kj} \right) z^k. \]

Then by (10),

\[ \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} \left( \sum_{j=1}^{\infty} p_j a_{kj} \right) + \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} \left( \sum_{j=1}^{\infty} p_j b_{kj} \right) \leq \sum_{j=1}^{\infty} p_j = 1. \]

This is the situation required by inequality (8). In this way \( \sum_{j=1}^{\infty} p_j f_{nj}(z) \in \mathcal{SH}^\sigma(\lambda, n, \mu) \).

**References**


