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ABSTRACT

In this paper, we scrutinize some fundamental features of a subclass of harmonic functions defined by a new operator. Like coefficient inequalities, convex combinations, distortion bounds.

MSC: 30C45, 30C50

1. Introduction

Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examinethe class of harmonic univalent functions, which is a subclass of harmonic functions. Let's show the open unit disk as $\mathbb{U}$. Let's show the family of continuous complex-valued harmonic functions that are harmonic in $\mathbb{U}$ as $\mathcal{H}$. $\mathcal{A}$ denotes the subclass of $\mathcal{H}$, including functions that are analytic in the open unit disk. f be a harmonic function in $\mathbb{U}$ and f may be written as $f = \mathfrak{h} + \overline{\mathfrak{g}}$; where $\mathfrak{h}$ and $\mathfrak{g}$ are in $\mathcal{A}$. We denominate $\mathfrak{h}$ is the analytic part of f, and $\mathfrak{g}$ is the co−analytic part of f. $\mathcal{S}$ denotes normalized analytic univalent functions in the open unit disk.

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Clunie and Sheil-Small [5] proved that f is sense−preserving and locally univalent in $\mathbb{U}$ if and only if $|\mathfrak{h}'(z)| > |\mathfrak{g}'(z)|$. So no loss of generality we can write $\mathfrak{h}(0) = 0$ and $\mathfrak{h}'(0) = 1$ since $\mathfrak{h}'(z) \neq 0$.

In this way, writing

$$\mathfrak{h}(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } \mathfrak{g}(z) = \sum_{k=1}^{\infty} b_k z^k \tag{1}$$

does not break generality.

Let $\mathcal{SH}$ indicate the family of functions $f = \mathfrak{h} + \bar{\mathfrak{g}}$ which are univalent, harmonic and sense−preserving in $\mathbb{U}$ for f functions that provide $f(0) = f_z(0) - 1 = 0$.

Consequently, $\mathcal{SH}$ includes the class of $\mathcal{S}$, which is normalized analytic univalent functions. Clearly, it will become apparent that although the analytic part $\mathfrak{h}$ of a function $f \in \mathcal{SH}$ is locally univalent, it need not be univalent. It can be easily demonstrated that the inequality of $|b_1| < 1$ must be achieved in order to have sense−preserving property. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ covers all functions in $\mathcal{SH}$ with $f_{\bar{z}}(0) = 0$ property. Clunie and Sheil-Small [5] examined $\mathcal{SH}$ as well as its subclasses and discovered some coefficient bounds. Thenceforward, there have been various associated articles on $\mathcal{SH}$ and its subclasses. For more details see ([1], [3], [4], [6], [7], [8], [10], [11], [13], [14] and [15] etc.). Furthermore, notice that $\mathcal{SH}$ reduces to the class $\mathcal{S}$, if the co−analytic part of f is identically zero.

For $f \in \mathcal{A}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Salagean differential operator D^n is defined by Salagean [9] $D^n: \mathcal{A} \to \mathcal{A}$,

$$D^0 f(z) = f(z)$$

$$\dots$$

$$D^{n+1} f(z) = z\big(D^n f(z)\big)'.$$

For

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

This operator was developed and modified by many researchers over time. As a simple example for $f = \mathfrak{h} + \bar{\mathfrak{g}}$ as stated in (1), Jahangiri et al. [8] described the modified Salagean operator of f like as

$$D^0 f(z) = f(z),$$

$$\dots$$

$$D^n f(z) = D^n \mathfrak{h}(z) + (-1)^n \overline{D^n \mathfrak{g}(z)},$$

where

$$D^n \mathfrak{h}(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

and

$$D^n \mathfrak{g}(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

Salagean [9] defined a linear operator denoted by I^n given by

$$I^0 f(z) = f(z),$$

$$I' f(z) = \frac{1}{z}\int_0^z \frac{f(t)}{t}\,dt$$

$$\ldots$$

$$I^n f(z) = I\big(I^{n-1} f(z)\big),$$

where $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $n \in \mathbb{N}_0$ and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k.$$

Then Pall-Szabo [2] defined a linear operator for $z \in \mathbb{U}, \lambda \geq 0, n \in \mathbb{N}_0$ denoted by $\mathcal{L}^n$ given by $\mathcal{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{L}^n f(z) = (1-\lambda) D^n f(z) + \lambda I^n f(z).$$

Therefore, if $f(z) \in \mathcal{A}$ and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

then

$$\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k. \tag{2}$$

Now we define the Salagean integro differential operator for functions $f = h + \bar{g}$ as stated in (1),

$$\mathcal{L}^0 f(z) = (1-\lambda) D^0 f(z) + \lambda I^0 f(z)$$

$$\ldots$$

$$\begin{aligned}\mathcal{L}^n f(z) &= (1-\lambda) D^n f(z) + \lambda I^n f(z)\\ &= (1-\lambda)\big(D^n \mathfrak{h}(z) + (-1)^n \overline{D^n \mathfrak{g}(z)}\big) + \lambda\big(I^n \mathfrak{h}(z) + (-1)^n \overline{I^n \mathfrak{g}(z)}\big),\end{aligned}$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$. Therefore

$$\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty}\left[k^n(1-\lambda) + \lambda\frac{1}{k^n}\right]a_k z^k + (-1)^n \sum_{k=1}^{\infty}\left[k^n(1+\lambda) + \lambda\frac{1}{k^n}\right]\overline{b_k z^k}. \quad (3)$$

Let show the subclass of $\mathcal{SH}$ containing functions f as stated in (1) which provide the following condition as $\mathcal{SH}(\lambda, n, \mu)$

$$\text{Re}\left(\frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)}\right) \geq \mu, \quad 0 \leq \mu < 1 \quad (4)$$

where $\mathcal{L}^n f(z)$, as stated in (2).

We denote the subclass $\overline{\mathcal{SH}}(\lambda, n, \mu)$ containing harmonic functions $f_n = \mathfrak{h} + \overline{\mathfrak{g}_n}$ in $\mathcal{SH}$ so that h and g_n be defined as follows

$$\mathfrak{h}(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad \mathfrak{g}_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad (5)$$

$a_k, b_k \geq 0$.

If the parameters are chosen specially, $\mathcal{SH}(\lambda, n, \mu)$ classes are reduced to different subclasses of harmonic univalent functions (For example Jahangiri [7], Jahangiri et al. [8], Silverman [10], Silverman and Silvia [11], Uralegaddi and Somanatha [12], Cho and Srivastava [4], Bayram and Yalcin [13], Atshan and Wanas [16]).

2. Main Results

In Theorem 2.1, we present a sufficient coefficient condition for harmonic univalent functions that are the members of $\mathcal{SH}^0(\lambda, n, \mu)$.

2.1 Theorem

Let $f = \mathfrak{h} + \overline{\mathfrak{g}}$. In here let $\mathfrak{h}$ and $\mathfrak{g}$ are as stated in (1) with $b_1 = 0$. Let

$$\sum_{k=2}^{\infty}\left[k^n(1-\lambda) + \lambda\frac{1}{k^n}\right]\left(k^n(1-\lambda) + \lambda\frac{1}{k^n} - \mu\right)|a_k| + \sum_{k=2}^{\infty}\left[k^n(1+\lambda) + \lambda\frac{1}{k^n}\right]\left(k^n(1+\lambda) + \lambda\frac{1}{k^n} + \mu\right)|b_k| \leq 1 - \mu, \quad (6)$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$. Then f is sense−preserving and harmonic univalent in $\mathbb{U}$ so we can say that $f \in \mathcal{SH}^0(\lambda, n, \mu)$.

As a special notation for convenience, sometimes, we write

$$A_n = \left[k^n(1-\lambda) + \lambda\frac{1}{k^n}\right]\left(k^n(1-\lambda) + \lambda\frac{1}{k^n} - \mu\right)$$

and

$$B_n = \left[k^n(1+\lambda) + \lambda\frac{1}{k^n}\right]\left(k^n(1+\lambda) + \lambda\frac{1}{k^n} + \mu\right)$$

and

$$C_n = k^n(1-\lambda) + \lambda\frac{1}{k^n}.$$

in this article.

Proof. If we accept that $z_1 \neq z_2$, we obtain

$$\left|\frac{f(z_1)-f(z_2)}{\mathfrak{h}(z_1)-\mathfrak{h}(z_2)}\right| \geq 1-\left|\frac{\mathfrak{g}(z_1)-\mathfrak{g}(z_2)}{\mathfrak{h}(z_1)-\mathfrak{h}(z_2)}\right| = 1-\left|\frac{\sum_{k=2}^{\infty} b_k(z_1^k - z_1^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k - z_1^k)}\right|$$

$$> 1-\frac{\sum_{k=2}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|}$$

$$\geq 1-\frac{\sum_{k=2}^{\infty} \frac{B_n}{1-\mu}|b_k|}{1-\sum_{k=2}^{\infty} \frac{A_n}{1-\mu}|a_k|} \geq 0$$

that proving univalence. Pay attention f is sense—preserving in $\mathbb{U}$. To show this feature;

$$|\mathfrak{h}'(z)| \geq 1-\sum_{k=2}^{\infty} k|a_k|\,|z|^{k-1}$$

$$> 1-\sum_{k=2}^{\infty} \frac{A_n}{1-\mu}|a_k|$$

$$\geq \sum_{k=2}^{\infty} \frac{B_n}{1-\mu}|b_k|$$

$$> \sum_{k=2}^{\infty} k|b_k|\,|z|^{k-1}$$

$$\geq |\mathfrak{g}'(z)|.$$

Using the fact that $\mathrm{Re}w \geq \alpha \Leftrightarrow |1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show the following inequality

$$|(1-\mu)\mathcal{L}^n f(z)+\mathcal{L}^{n+1} f(z)| - |(1+\mu)\mathcal{L}^n f(z)-\mathcal{L}^{n+1} f(z)| \geq 0. \tag{7}$$

Replacing for $\mathcal{L}^n f(z)$ and $\mathcal{L}^{n+1} f(z)$ in (7), we obtain

$$|(1-\mu)\mathcal{L}^n f(z)+\mathcal{L}^{n+1} f(z)| - |(1+\mu)\mathcal{L}^n f(z)-\mathcal{L}^{n+1} f(z)|$$

$$\geq 2(1-\mu)|z| - \sum_{k=2}^{\infty} C_n(C_n+1-\mu)|a_k|\,|z|^k - \sum_{k=2}^{\infty} C_n(C_n-1+\mu)|b_k|\,|z|^k$$

$$-\sum_{k=2}^{\infty} C_n(C_n-1-\mu)|a_k|\,|z|^k - \sum_{k=2}^{\infty} C_n(C_n+1+\mu)|b_k|\,|z|^k$$

$$> 2(1-\mu)|z|\left\{1-\sum_{k=2}^{\infty}\frac{A_n|a_k|}{1-\mu}-\sum_{k=2}^{\infty}\frac{B_n|b_k|}{1-\mu}\right\}.$$

This last phrase is not negative with (6), and therefore the proof is complete.

2.2 Theorem

Let $f_n = \mathfrak{h} + \overline{\mathfrak{g}_n}$ be as stated in (4) with $b_1 = 0$. Then $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$, if and only if

$$\sum_{k=2}^{\infty} A_n |a_k| + \sum_{k=2}^{\infty} B_n |b_k| \leq 1 - \mu, \tag{8}$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$.

Proof. Notice that $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is a subclass of $\mathcal{SH}^0(\lambda, n, \mu)$. Then the "if" part can be easily proved from 2.1 Theorem. For the "only if" part, we must demonstrate that $f_n \notin \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ if the inequality (8) is not valid. Here is an issue to be considered; a necessary and sufficient condition for $f_n = \mathfrak{h} + \overline{\mathfrak{g}_n}$ be as stated in (5), to be in $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is that the condition (4) must be provided.

The above stipulation must be valid for all z complex numbers where, $|z| = r < 1$. After selecting the z values from the positive real axis where $0 \leq |z| = r < 1$, we obtain

$$\operatorname{Re}\left\{\frac{(1-\mu)z - \sum_{k=2}^{\infty} A_n a_k z^k - \sum_{k=2}^{\infty} B_n b_k \bar{z}^k}{z - \sum_{k=2}^{\infty} C_n a_k z^k + \sum_{k=2}^{\infty} C_n b_k \bar{z}^k}\right\} \geq 0.$$

This is equivalent to

$$\left\{\frac{(1-\mu) - \sum_{k=2}^{\infty} A_n a_k r^{k-1} - \sum_{k=2}^{\infty} B_n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} C_n a_k r^{k-1} + \sum_{k=2}^{\infty} C_n b_k r^{k-1}}\right\} \geq 0. \tag{9}$$

If the inequality (8) is not valid, then the expression in (9) is negative for r values approaching 1. Hence there exists $z_0 = r_0$ in (0,1) for which the quotient in (9) is negative. This shows the required condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ and so the proof is complete.

2.3 Theorem

Let $f_n = \mathfrak{h} + \overline{\mathfrak{g}_n}$ be as stated in (5). At that case a necessary and sufficient condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is

$$f_n(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k \mathfrak{g}_{n_k}(z)\right),$$

where

$$\mathfrak{h}_1(z) = z, \;\; \mathfrak{h}_k(z) = z - \frac{1-\mu}{A_n} z^k$$

and

$$\mathfrak{g}_{n_1}(z) = z, \mathfrak{g}_{n_k}(z) = z + (-1)^n \frac{1-\mu}{B_n} \bar{z}^k$$

for $X_k \geq 0$, $Y_k \geq 0, \sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $k = 2,3,\cdots, n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1$.

Notably, the extreme points of $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ are $\{\mathfrak{h}_k\}$ and $\{\mathfrak{g}_{n_k}\}$.

Proof. For f_n functions which are as stated in (5), we obtain

$$f_n(z) = \sum_{k=1}^{\infty} \left(X_k \mathfrak{h}_k(z) + Y_k \mathfrak{g}_{n_k}(z) \right)$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\mu}{A_n} X_k z^k + (-1)^n \sum_{k=2}^{\infty} \frac{1-\mu}{B_n} Y_k \bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty} \frac{A_n}{1-\mu} \left(\frac{1-\mu}{A_n} X_k \right) + \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} \left(\frac{1-\mu}{B_n} Y_k \right)$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=2}^{\infty} Y_k = 1 - X_1 - Y_1 \leq 1$$

and so $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$. Moreover, if $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$, then

$$a_k \leq \frac{1-\mu}{A_n}$$

and

$$b_k \leq \frac{1-\mu}{B_n}.$$

Set

$$X_k = \frac{A_n}{1-\mu} a_k,$$

$$Y_k = \frac{B_n}{1-\mu} b_k,$$

for $k = 2,3,\cdots$ and

$$X_1 + Y_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + Y_k \right)$$

where $X_k \geq 0,\ Y_k \geq 0$. Therefore, in accordance with, we have

$$f_n(z) = (X_1 + Y_1) z + \sum_{k=2}^{\infty} \left(X_k \mathfrak{h}_k(z) + Y_k \mathfrak{g}_{n_k}(z) \right)$$

$$= \sum_{k=1}^{\infty} \left(X_k \mathfrak{h}_k(z) + Y_k \mathfrak{g}_{n_k}(z) \right).$$

2.4 Theorem

Let $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$. Then for $|z| = r < 1$ and $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$, we have

$$r-\frac{1-\mu}{A_n}r^2 \leq |f_n(z)| \leq r+\frac{1-\mu}{A_n}r^2.$$

Proof. Here, we will only prove the right side since the proving of the left side, and the right side of the inequality is very similar. Let $f_n \in \overline{\mathcal{SH}}^0(\lambda,n,\mu)$. Taking the absolute value of f_n we can easily see

$$|f_n(z)| \leq r+\sum_{k=2}^{\infty}(a_k+b_k)r^k \leq r+\frac{(1-\mu)r^2}{A_n}\sum_{k=2}^{\infty}\frac{A_n}{1-\mu}|a_k|+\frac{(1-\mu)r^2}{A_n}\sum_{k=2}^{\infty}\frac{B_n}{1-\mu}|b_k| \leq r+\frac{1-\mu}{A_n}r^2.$$

The left side can be shown in a similar way.

2.5 Theorem

The class $\overline{\mathcal{SH}}^0(\lambda,n,\mu)$ is closed under convex combination.

Proof. Let $f_{n_j} \in \overline{\mathcal{SH}}^0(\lambda,n,\mu)$ for Let j= 1,2, ..., where f_{n_j} is given by

$$f_{n_j}(z)=z-\sum_{k=2}^{\infty}a_{k_j}z^k+(-1)^n\sum_{k=2}^{\infty}b_{k_j}\bar{z}^k.$$

Then by (7),

$$\sum_{k=2}^{\infty}\frac{A_n}{1-\mu}a_{k_j}+\sum_{k=2}^{\infty}\frac{B_n}{1-\mu}b_{k_j} \leq 1. \tag{10}$$

For $\sum_{j=1}^{\infty}p_j=1$, $0 \leq p_j \leq 1$, the convex combination of f_{n_j} can be written as

$$\sum_{j=1}^{\infty}p_j f_{n_j}(z)=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{\infty}p_j a_{k_j}\right)z^k+(-1)^n\sum_{k=2}^{\infty}\left(\sum_{j=1}^{\infty}p_j b_{k_j}\right)\bar{z}^k.$$

Then by (10),

$$\sum_{k=2}^{\infty}\frac{A_n}{1-\mu}\left(\sum_{j=1}^{\infty}p_j a_{k_j}\right)+\sum_{k=2}^{\infty}\frac{B_n}{1-\mu}\left(\sum_{i=1}^{\infty}p_j b_{k_j}\right)$$

$$=\sum_{j=1}^{\infty}p_j\sum_{k=2}^{\infty}\frac{A_n}{1-\mu}a_{k_j}+\sum_{j=1}^{\infty}p_j\sum_{k=2}^{\infty}\frac{B_n}{1-\mu}b_{k_j}$$

$$\leq \sum_{j=1}^{\infty}p_j=1.$$

This is the situation required by inequality (8). In this way $\sum_{j=1}^{\infty}p_j f_{n_j}(z) \in \overline{\mathcal{SH}}^0(\lambda,n,\mu)$.

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