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ABSTRACT

In this paper, a characterization of weakly inward mapping is introduced in CAT(0) spaces using triangle comparison property, and used this characterization to prove the existence of fixed point for single mapping in CAT(0) spaces, Moreover use X., Ding definition to prove the existence of fixed point for weakly inward contraction multi-valued mapping in CAT(0) spaces.

MSC: 30C45, 30C50

1. Introduction

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [1][2][3][4].

The first theorem to prove the existence and uniqueness of fixed point for contraction self-mapping class $T: M \to M$ in general metric space was Banach theorem [5] . After that there are many directions to generalized this principle with respect to various suitable conditions For some generalizations, The most important one was J.Caristi-Kirk [6], it was significantly more than Banach theorem because the last theorem deals only with contraction mapping class, Banach and Caristi did not need the linearity

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construction in their work, But the weakness with these theorems is that they just deal with self-mapping $T: M \to M$,and don't approach to find a fixed point for $T: N \to M$ where $N \subseteq M$.

After that, Halpern[7] introduced the inwardness concept on mapping $T: N \to M$ is one which affirms that, in some sense, A to maps points of N to N. This condition weaker than Kirk's condition on T maps ∂N to N such that ∂N denotes the boundary points of N.

J.Caristi[8] proved the theorem that considered a significant tool and gave us an important generalization of the Banach contraction concept to fined fixed point in metric spaces because Banach theorem deals only with contraction mapping.

Theorem(I): **[8](Caristi theorem)**" Let (M, d) be a complete metric space, N a closed subset of M, Assume $f: N \to N$ an arbitrary function and $T: N \to M$ is continuous function. If

$$d\left(f(x), T\left(f(x)\right)\right) \le d\left(x, T(x)\right) + \lambda d\left(x, f(x)\right) \qquad , \forall x \in N \tag{0.1}$$

Then f has a fixed point, Where $\lambda < 0$ is a real number ."

J.Caristi[8] introduced a significant characterization for inward and weakly inward mapping in linear space M , that is, $T: M \to M$ is weakly inward if it satisfying

$$\lim_{t \to 0^+} t^{-1} d\left((1-t)x + tT(x), M\right) = 0, \qquad \forall x \in M \tag{0.2}$$

This approach fails to take into account the weakly inward mapping in metric space and the case of multivalued mapping.

In 1999, X.p. DING and Y.r. HE[9] introduced the new definition of weakly inward for multivalued mapping and gave an important characterization for weakly inward mapping. This characterization states that, when M is a metric space and 2^M denotes the power set of M then $T: N \to 2^M$ is weakly inward multi-valued mapping if $x \in N$ with $x \notin T(x)$ and $\forall \varepsilon \in (0,1)$ there exist $y \in N$ and $y \neq x$ such that

$$\varepsilon d(x, y) + d\left(y, T(x)\right) \le d\left(x, T(x)\right) \tag{0.3}$$

Moreover, they removed the convexity condition to find a fixed point for multivalued contraction mapping.

In 2005, S. Dhompongsa[10] proved the existence of a fixed point of nonexpansive multivalued mapping in CAT(0) space

In this paper, the characterization of weakly inward mapping is extended to CAT(0) space for single value mapping sense because it carry the most important properties of Banach spaces like convexity and uniformly convex property and used it to reduce the multivalued mapping to single valued mapping by adding a condition includes there exist a point in domain the weakly inward mapping $T: N \to 2^M$ such that $y \in T(x)$ and $d(x, y) = d(x, T(x))$ to proves the existence of a fixed point for multivalued mapping by using Carsiti theorem, these conditions is equivalent to conditions of theorem (2.1)[9] .

2. Preliminaries

Before stating precise definition concerns the geodesic metrics spaces and the most important properties of the CAT(0) spaces we give a brief review of inward and weakly inward concepts in normed spaces.

Let M be a normed space and N is a subset of M. The inward set $I_N(x)$ of x relative to N it defined as $I_N(x)=\{x+k(z-x):k\geq 1, z\in N\}$ and if N is convex then $k\geq 0$. Geometrically, it is the union of all rays starting at x and passing through other points $z\in N$, $\overline{I_N(x)}$ denotes the closure of $I_N(x)$.

Let $T:N\to M$ a mapping N into M. An inwardness concept on the mapping T is one which to force that, in some sense, T maps points of N toward N such that $T(x)\in I_N(x)$ for all $x\in N$ and T is called inward mapping, T is called weakly inward mapping if $T(x)\in \overline{I_N(x)}$ for all $x\in N$.

Now we give some precise definitions which are required in this study, the below definition describes a type of metric space which is a path connected

Definition(2.1):[11] Let (M,d) be metric space and x, y belongs to M an isometry mapping $\alpha:[0,l]\to M$ joining x to y is called geodesic curve if $\alpha(0)=x$ and nd $\alpha(l)=y$ such that $d(x,y)=l$ and for each $t_1,t_2\in[0,l]$ then $d\left(\alpha(t_1),\alpha(t_2)\right)=|t_1-t_2|$. The image of α is called the geodesic segment and denoted by $[x,y]$ .and define as $z\in[x,y], z=\alpha(t)$,$\exists\, t\in[0,l]$ such that $d(x,z)+d(z,y)=d(x,y)$.

Definition (2.2): Let (M,d) be metric space, M is called geodesic space if for each x,y belongs to M there exist a geodesic curve joining them and it's called unique geodesic space if there is exist one geodesic curve joining each distinct two points.

The Definition below talk about important property in geodesic spaces, it is called betweenness, some authors called it Menger convexity.

Definition(2.3): [12] Let (M,d) be metric space, $z\in M$ is called between x and y if $z\in[x,y]$, we observe that $d(x,y)=d(x,z)+d(z,y)$, this property is called betweenness and considered a generalization of convexity in normed space and it's called convexity in Menger[13] sense. In unique geodesic space for each, $t\in[0,l]$ we denote the image of $\alpha(t)=x_t$ such that $x_t\in[x,y]$ and we write $x_t=(1\text{-}t)x\oplus ty$ and the distance $d(x,x_t)=td(x,y)$.

Remark (2.4): In geodesic space (M,d) every three-point (x,y,z) construct geodesic triangle denoted by $\triangle(x,y,z)$ such that consists (x,y,z) as vertices and three geodesic segments $[x,y]$,$[x,z]$ and $[y,z]$ as edges of $\triangle(x,y,z)$. A comparison triangle for the triangle $\triangle(x,y,z)$ is a triangle $\bar{\triangle}(x,y,z)$ $=\triangle(\bar{x},\bar{y},\bar{z})$ in R^2 such that

$$\{d(x,y)=d(\bar{x},\bar{y}), d(x,z)=d(\bar{x},\bar{z}) \text{ and } d(y,z)=d(\bar{y},\bar{z})\}$$

and for each point $p\in[x,y]$ then there exist $\bar{p}\in[\bar{x},\bar{y}]$ such that $d(x,y)=d(\bar{x},\bar{y})$, comparison points for points in $[x,z]$ and $[y,z]$ defined in the same way.

Definition(2.5):[11] A geodesic space (M,d) is called CAT(0) space if each geodesic triangle $\triangle(x,y,z)$ in (M,d) satisfying the CAT(0) comparison axiom, that is for every $p,q\in\triangle(x,y,z)$ and their comparison points $\bar{p},\bar{q}\in\triangle(\bar{x},\bar{y},\bar{z})$,respectively, then $d(x,y)\le d(\bar{x},\bar{y})$.

Remark (2.6): in [10][13] Let (M,d) be CAT(0) space satisfying the following facts.

1. The CN inequality of Bruhat and Titts, That is for every three points $x,y_1,y_2\in\mathrm{M}$ $d(x,\frac{1}{2}(y_1\oplus y_2))^2\le d(x,y_1)^2+d(x,y_2)^2-\frac{1}{4}d(y_1,y_2)^2$.
2. Let $p\in[x,y]$ in $\triangle(x,y,z)$ then $d(x,p)\le d(\bar{x},\bar{p})$ for comparison point $\bar{p}$ in $\triangle(\bar{x},\bar{y},\bar{z})$.
3. Let N be closed convex subset in M and the mapping $P_{\mathrm{N}}:\mathrm{M}\to\mathrm{N}$ defined as $\mathrm{P}_{\mathrm{N}}(x)=\{y\in\mathrm{N}:d(x,y)=d(x,\mathrm{N})\}$ is called projection map from M to N then for each $x\in\mathrm{M}$ then $P_{\mathrm{N}}(x)$ is unique such that $d(x,P_{\mathrm{N}}(x))=d(x,\mathrm{N})=\inf_{y\in\mathrm{N}}\{d(x,y)\}$.
4. For each $y\in[x,\mathrm{P}_{\mathrm{N}}(x)]$ then $\mathrm{P}_{\mathrm{N}}(y)=\mathrm{P}_{\mathrm{N}}(x)$
5. In $\triangle(x,y,z)$, let m_1 and m_2 belong to $[x,y]$ and $[x,z]$ respectively and $t\in(0,1)$ such that $d(x,m_1)=t(x,y)$ and $d(x,m_2)=t(x,z)$ then $d(m_1,m_2)\le td(y,z)$.

Let (M,d) be geodesic metric space we denote $G_{c(\mathrm{M})},G_{cc(\mathrm{M})},G_{k(\mathrm{M})}$ and $G_{kc(\mathrm{M})}$ by a family of closed, closed convex, compact and compact convex subsets respectively.

3. Main results

In this section, the definition of inward subset is extended into geodesic sense and give a characterization of weakly inward in CAT(0) space and proves the multivalued version. The definitions below we show the inward concept in geodesic spaces.

Definition (3.1) :Let (M,d) geodesic space, N subset of M and $x\in\mathrm{N}$, The inward subset of x with respect to N as $\mathrm{I}_{\mathrm{N}(x)}=\{z\in\mathrm{M}:(x,z]\cap\mathrm{N}\neq\phi\}\cup\{x\}$.

That means any geodesic segment initiated from x to z in M contains at least one point from N except x. On another hand that means there exist $t\in(0,1)$ and $x_t\in(x,z]\cap\mathrm{N}$ such that $d(x,x_t)+d(x_t,z)=d(x,z)$.

Definition (3.2): let (M,d) be geodesic metric space and N a subset of M, $\mathrm{T}:\mathrm{N}\to\mathrm{M}$ is called inward mapping if $\mathrm{T}(x)\in\mathrm{I}_{\mathrm{N}(x)}$ for each $x\in\mathrm{N}$, T is called weakly inward if $\mathrm{T}(x)\in\overline{I_{\mathrm{N}(x)}}$ $\forall x\in\mathrm{N}$.

When T is a multivalued function, that is, $T:N \to 2^M$,where 2^M denotes to the power set of M , T is called inward mapping if $T(x) \subseteq I_{N(x)}$ for each $x \in N$, and T is called weakly inward if

$T(x) \in \overline{I_{N(x)}} \quad \forall x \in N$.

The most advantage of the inward concept that we can generalize the contraction concept in weaker conditions without need to generate Picard iteration.

The next theorem is a cornerstone in this paper, we extend the weakly inward mapping to CAT(0) spaces by using convexity property .

Theorem (3.3): Let (M,d) be CAT(0) space, N is convex subset of M and let $T:N \to M$ be a weakly inward mapping then for each $x \in N$

$$\inf_{t\in(0,1)} d\{(1-t)x \oplus tT(x), N\} = 0 \tag{3.1}$$

Proof: let T is weakly inward, $T(x) \in \overline{I_{N(x)}} \quad \forall x \in N$, then

$\forall \epsilon > 0, \exists z \in I_{N(x)}$ such that $d(z,T(x)) < \epsilon/t$ for some $t \in (0,1)$ and $z \in I_{N(x)}$.

, then there exist $u_t \in (x,z] \cap N$ such that $u_t = (1-t)x \oplus tz$ and $d(x,u_t) = td(x,z)$.

let $v_t \in (x,T(x))$ such that $v_t = (1-t)x \oplus tT(x)$ and $d(x,v_t) = td(x,T(x))$.

In $\triangle(x,T(x),z)$, $d(v_t,u_t) \le td(z,T(x))$, but $u_t \in N$ then

$$d(v_t,N) \le d(v_t,u_t) \le td(z,T(x)) \le \frac{\epsilon}{t} t = \epsilon$$

And it leads

$$\inf_{t\in(0,1)} \{d(1-t)x \oplus tT(x), N\} = 0 \qquad \blacksquare$$

Remark (3.4): In CAT(0) space if $z \in \overline{I_{N(x)}}$ then every point $x_t = (1-t)x \oplus tz$, $x_t \in \overline{I_{N(x)}}$.
In the case of multivalued mapping X.p. DING[9][14] gave a significant characterization for weakly inward, that is, $T:N \to 2^M$ is weakly inward if for each $x \in N$ with $x \notin T(x)$ and for each $r \in (0,1)$ $\exists u \in N$ *such that* $u \ne x \quad rd(x,u) + d(u,A(x)) \le d(x,A(x))$.

Theorem (3.5): Let (M,d) be complete CAT(0) space, $N \subseteq M$ is closed and convex. Let $T:N \to M$ be a weakly inward contraction mapping with Lipschitz constant $k < 1$, then T has a fixed point.

Proof: Assume T has no fixed point then $d(x,\mathrm{T}(x))>0 \quad \forall x\in \mathrm{N}$.

and choose $\varepsilon>0$ such that $k<\frac{1-\varepsilon}{1+\varepsilon}$.

T is weakly inward then by theorem (3.3) $\inf_{t\in(0,1)}\{d(1-t)x\oplus t\mathrm{T}(x),\mathrm{N}\}=0$ for each $x\in\mathrm{N}$, then there exist $t\in(0,1)$ such that $x_t=(1-t)x\oplus t\mathrm{T}(x)$

such $d(x_t,\mathrm{N})\le\varepsilon t d(x,\mathrm{T}(x))$. where $d(x_t,\mathrm{N})=\inf_{y\in\mathrm{N}}\{d(x_t,y\}$

Clear that x_t satisfying $d(x,x_t)=td(x,\mathrm{T}(x)) \quad and \; d(x,\mathrm{T}(x)=d(x,x_t)+d(x_t,\mathrm{T}(x))$

then

$$\exists y\in\mathrm{N} \; such\ that\ d(x_t,y)<\varepsilon t d(x,\mathrm{T}(x))=\varepsilon d(x,x_t) \qquad (3.2)$$

Define

$$g:\mathrm{N}\to\mathrm{N} \qquad by \qquad g(x)=y \qquad (3.3)$$

Such that y satisfying the equation 3.2 clear that $x\neq y$

$$\frac{d(y,x)}{d(x,x_t)}\le\frac{d(y,x_t)+d(x_t,x)}{d(x,x_t)}=1+\frac{d(y,x_t)}{d(x,x_t)}$$

From equation 3.2 we get

$$\frac{d(y,x_t)}{d(x,x_t)}<1+\varepsilon \text{ then we get}$$

$$d(x,x_t)>\frac{d(y,x_t)}{1+\varepsilon}$$

$$d(y,\mathrm{T}(y))\le d(y,x_t)+d(x_t,\mathrm{T}(x))+d(\mathrm{T}(x),\mathrm{T}(y))$$

$$d(y,\mathrm{T}(y))\le d(y,x_t)+d(x_t,\mathrm{T}(x))+d(\mathrm{T}(x),\mathrm{T}(y))$$

$$\le d(y,x_t)+d(x,\mathrm{T}(x))-d(x,x_t)-k(x,y)$$

$$<\varepsilon(x,x_t)-(x,x_t)+d(x,\mathrm{T}(x))+k(x,y)$$

$$=(\varepsilon-1)d(x,x_t)+d(x,\mathrm{T}(x))+kd(x,y)$$

$$d(y,\mathrm{T}(y))<\frac{d(x,y)}{(1+\varepsilon)}(\varepsilon-1)+d(x,\mathrm{T}(x))+kd(x,y)$$

$$d(y,\mathrm{T}(y))<d(x,\mathrm{T}(x))+(k-\frac{1-\varepsilon}{1+\varepsilon})d(x,y) \qquad (3.4)$$

And clear that $(k-\frac{1-\varepsilon}{1+\varepsilon})<0$

Now to apply Caristi theorem above on the map g in the equation 0.1 by substitute $y=g(x)$ in the equation 3.4 we get

$$d(g(x),\mathrm{T}(g(x)) < d(x,\mathrm{T}(x)) + (k - \frac{1-\varepsilon}{1+\varepsilon})d(x,g(x))$$

and this implies g has fixed point but this contradicts with assumption $g(x)=y$ and $x \neq y \ \forall\, x \in \mathrm{N}$ and this implies T has fixed point. ■

Theorem (3.6): Let (M,d) be complete CAT(0) space and N is a closed subset of N . $\mathrm{T}:\mathrm{N} \to \mathrm{M}$ be inward nonexpansive mapping then T has a fixed point.
Proof: let $x \in \mathrm{N}$ then there exist

$t_0 \in (0,1)$ such that $x_0 \in [x,\mathrm{T}(x)] \cap \mathrm{N}$

And $$d(x,\mathrm{T}(x)) = d(x,x_0) + d(x_0,\mathrm{T}(x))$$

$$d(x_0,\mathrm{T}(x_0)) \leq d(x_0,\mathrm{T}(x)) + d(\mathrm{T}(x),\mathrm{T}(x_0))$$

$$< d(x_0,\mathrm{T}(x)) + d(x_0,x) = d(x,\mathrm{T}(x))$$

Let

$t_1 \in (0,1)$ such that $x_1 \in [x_0,\mathrm{T}(x_0)] \cap \mathrm{N}$

$$d(x_1,\mathrm{T}(x_1)) \leq d(x_1,\mathrm{T}(x_0)) + d(\mathrm{T}(x_0),\mathrm{T}(x_1))$$

$$< d(x_1,\mathrm{T}(x_0)) + d(x_0,x_1) = d(x_0,\mathrm{T}(x_0))$$

By this method, we construct the sequence (x_n) satisfying $d(x_{n+1},\mathrm{T}(x_{n+1})) < d(x_n,\mathrm{T}(x_n))$

Then $\inf_{n\to\infty} \{d(x_n,\mathrm{T}(x_n)\} = 0$, by theorem 15[11] then T has an approximation fixed point sequence .

Next theorem plays an important role in fixed point theorem for weakly inward multivalued mapping class, we use it to prove the existence of fixed point by using Caristi theorem[8]

Theorem(3.7): [9] Let (M,d) be complete CAT(0) metric space and $\phi \neq \mathrm{N} \subseteq \mathrm{M}$. $\mathrm{T}:\mathrm{N} \to 2^{\mathrm{M}}$ is weakly inward if for each $x \notin \mathrm{T}(x),\ \exists z \in \mathrm{T}(x) \cap \overline{\mathrm{I}_{\mathrm{N}(x)}}$ *such that* $d(x,z) = d(x,\mathrm{T}(x))$.

Proof: $z \in \mathrm{T}(x) \cap \overline{\mathrm{I}_{\mathrm{N}(x)}}$ *and* $x \notin \mathrm{T}(x)$ then $d(x,z) > 0$ and by theorem (3.3) above we get

$$\inf_{t\in(0,1)} \{d(1-t)x \oplus tz,\mathrm{N}\} = 0, \quad \forall\, x \in \mathrm{N}$$

Then $\exists\, t(0,1)$ such that $d((1-t)x \oplus tz,\mathrm{N}) \leq t\varepsilon d(x,z)$, $\varepsilon > 0$

Let $x_t = (1-t)x \oplus tz$ then x_t satisfying the equation below (3.5)

$$d(x,z)=d(x,x_t)+d(x_t,z) \text{ and } d(x,x_t)=td(x,z) \quad (3.5)$$

$$\exists y\in E \text{ such that } d(x_t,y)<\varepsilon td(x,z) \quad (3.6)$$

by equation 3.5 we get $d(x_t,y)<\varepsilon d(x,x_t)$

$$\frac{d(y,x)}{d(x,x_t)}\le\frac{d(y,x_t)+d(x_t,x)}{d(x,x_t)}=1+\frac{d(y,x_t)}{d(x,x_t)}$$

$$< \frac{d(y,x)}{d(x,x_t)}<1+\varepsilon$$

And this leads

$$d(x,x_t)>\frac{d(y,x)}{1+\varepsilon} \quad (3.7)$$

$$d(y,z)\le d(y,x_t)+d(x_t,z)$$

$$d(y,z)<\varepsilon d(x,x_t)+d(x_t,z)$$

by equation 3.5 we get

$$d(y,z)<\varepsilon d(x,x_t)+d(x,z)-d(x,x_t)=(\varepsilon-1)d(x,x_t)+d(x,z)$$

$$d(y,z)<d(x,z)-(1-\varepsilon)d(x,x_t)$$

by equation 3.7

$$d(y,z)=d(x,z)-\frac{(1-\varepsilon)}{(1+\varepsilon)}d(y,x)$$

let

$$r=\frac{1-\varepsilon}{1+\varepsilon}<1$$

Then

$$d(y,z)<d(x,z)-rd(y,x)$$

Then

$$rd(x,y)+d(y,z)<d(x,z)$$

also $z\in \mathrm{T}(x)$ *implies* $d(y,\mathrm{T}(x))<d(y,z)$ and $d(x,z)=d(x,\mathrm{T}(x))$

then we get

$$rd(x,y)+d(y,\mathrm{T}(x))<kd(x,y)+d(y,z)<d(x,\mathrm{T}(x))$$

and this coincides with the definition[9] above then T is wein. ■

Remark (3.8): S. Dhompongsa[10] proved the existence of a fixed point for multivalued mapping in CAT(0) space.

Remark (3.9): The next corollaries give condition asserts that for each $x\in \mathrm{N}$ there exists a point z in $\mathrm{T}(x)\cap \overline{\mathrm{I}_{\mathrm{N}(x)}}$ satisfying. $d(x,z)=d(x,\mathrm{T}(x))$

Corollary (3.10): Let (M,d) be complete CAT(0) metric space and $\mathrm{N}\subseteq \mathrm{M}$. $\mathrm{T}:\mathrm{N}\to G_{k(M)}$ contraction mapping such that $\forall x\in \mathrm{N},\exists z\in \mathrm{T}(x)\cap \overline{\mathrm{I}_{\mathrm{N}(x)}}$ then T has a fixed point.

Proof. For each $x\in \mathrm{N}$ define the function $g:\mathrm{N}\to \mathbb{R}$ such that $g(x)=d(x,\mathrm{T}(x))$.

Clear that g is continuous and its domain is compact then it has minimum value, that is, there there exist $z\in \mathrm{T}(x)$ such that $d(x,\mathrm{T}(x)=d(x,z)$ and T satisfying the condition of the theorem above is hold then T is weakly inward.

Corollary (3.11): Let (M,d) be complete CAT(0) metric space and $\mathrm{N}\subseteq \mathrm{M}$. $\mathrm{A}:\mathrm{N}\to G_{bc(\mathrm{M})}$ contraction mapping such that $\forall x\in \mathrm{N},\exists z\in \mathrm{T}(x)\cap \overline{\mathrm{I}_{\mathrm{N}(x)}}$ and $\mathrm{T}(x)$ is proximal then A has a fixed point.

Proof: $\mathrm{T}(x)$ is proximal for each x in N ,then $\exists z\in \mathrm{T}(x)$ such that $d(x,z)=d(x,\mathrm{T}(x))$ and that leads T satisfying the condition of the theorem above is hold then T is weakly inward mapping and it has a fixed point.

4. Conclusion

Inward and weakly inward mapping is an important tool to deal with mapping $\mathrm{T}:\mathrm{N}\to \mathrm{M}$, $\mathrm{N}\subseteq \mathrm{M}$ it asserts T to maps N to N and finding fixed point without generate iteration procedure and it a strong tool to find the fixed point for nonexpansive mapping. Moreover, weakly inward mapping provides conditions weaker than Kirk condition on boundary points of N to find fixed point for non-self mapping

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