New and Extended Results On Fourth-Order Differential Subordination for Univalent Analytic Functions

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**ABSTRACT**

In this paper, we introduce new concept that is fourth-order differential subordination associated with linear operator $I_{\lambda,\alpha,\mu}^\gamma$ for univalent analytic functions in the open unit disk. Here, we extended some lemmas. Also some interesting new results are obtained.

**1. Introduction**

Let $H(U)$ be the class of function which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} : |z| < 1 \}.$$

For $n \in N = \{1,2,3,\ldots\}$, and $a \in \mathbb{C}$, let

$$H[a,n] = \{ f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \},$$

and also, let $H_0 = [0,1]$.

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Let $\Sigma$ denote the subclass of $H(U)$ consisting of all univalent and analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).$$  \hfill (1.1)

Komatu [8] introduced and investigated a family of integral operator

$$J_\mu^\lambda: \Sigma \to \Sigma.$$  

That is obtained as follows

$$J_\mu^\lambda f(z) = z + \sum_{n=1}^{\infty} \left( \frac{\mu}{\mu+n-1} \right)^\lambda a_n z^n, \quad (z \in U^*, n > 1, \lambda \geq 0).$$  \hfill (1.2)

The Hurwitz - Lerch Zeta function

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(k+a)^s}, \quad a \in \mathbb{C}/z_0, s \in \mathbb{C} \text{ when } 0 < |z| < 1.$$  

Interms of Hadamard product (or convolution) where $G_{s,a}(z)$ is given by

$$G_{s,a}(z) = (1+a)^s[\phi(z, s, a) - a^{-s}], \quad (z \in U).$$  

Then a linear operator $I_{s,a,\mu}^\lambda f(z): \Sigma \to \Sigma$ (see[2]) is defined

$$I_{s,a,\mu}^\lambda f(z) = G_{s,a}(z) * J_\mu^\lambda f(z) = z + \sum_{n=2}^{\infty} \left( \frac{1+a}{k+a} \right)^s \left( \frac{\mu}{\mu+n-1} \right)^\lambda a_n z^n. \quad (1.3)$$

It is easily verified from (1.3) that

$$z \left( I_{s,a,\mu}^{\lambda+1} f(z) \right)' = \mu I_{s,a,\mu}^\lambda f(z) - (\mu - 1) I_{s,a,\mu}^{\lambda+1} f(z) \quad (1.4)$$

For several past years, there are many authors introduce and dealing with the theory of second-order differential subordination and superordination for example ([3,4,10]). Recently, many authors discussed the theory of third-order differential subordination and superordination for example ([5,6,7,11,12,13,14,15]). In the present paper, we investigated the extended theory of second-order differential subordination in the open unit disk introduced by Miller and Mocanu [9] to third-order case. Now, we extend this to fourth-order differential subordination and determined properties of functions $p$ that satisfy the following fourth-order differential subordination:
\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p''''(z);z): z \in U\}.

To prove our main results, we need the basic concepts in theory of the fourth-order.

**Definition (1.1):** [9]. Let \( f(z) \) and \( F(z) \) be members of the analytic function class \( H(U). \) The function \( f(z) \) is said to be subordinate to \( f(z) \) or \( F(z) \) is superordinate to \( f(z) \) if there exists a Schwarz function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \(|w(z)| < 1 \) \( (z \in U) \), and such that \( f(z) = F(w(z)) \). In such case, we write

\[
\psi < F, \text{ or } f(z) < F(z).
\]

If \( F(z) \) is univalent in \( U \), then \( f(z) < F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \).

**Definition (1.2):** [1]. Let \( Q \) be the set of all functions \( q \) that are analytic and univalent on \( \overline{U/E(q)} \), where

\[
E(q) = \left\{ \xi: \xi \in \partial U: \lim_{z \to \xi} q(z) = \infty \right\},
\]

and are such that \( \min|q'(\xi)| = p > 0 \) for \( \xi \in \partial U/E(q) \). Further, let the subclass of \( Q \) for which \( q(0) = a \) be denoted by \( Q(a) \) with

\[
Q(0) = Q_0 \text{ and } Q(1) = Q_1.
\]

**Definition (1.3):** Let \( \psi: \mathbb{C}^5 \times U \to \mathbb{C} \) and the function \( h(z) \) be univalent in \( U \). If the function \( p(z) \) is analytic in \( U \) and satisfies the following fourth-order differential subordination:

\[
\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p''''(z);z) < h(z), \tag{1.5}
\]

then \( p(z) \) is called a solution of the differential subordination. A univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination or more simply a dominant if \( p(z) < q(z) \) for all \( p(z) \) satisfying (1.5). A dominant \( \bar{q}(z) \) that satisfies \( \bar{q}(z) < q(z) \) for all dominants \( q(z) \) of (1.5) is said to be the best dominant.

**Lemma (1.4):** Let \( z_0 \in U \) with \( r_0 = |z_0| \). For \( n \geq 1 \). Let

\[
f(z) = a_nz^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots
\]

be continuous on \( \overline{U_{r_0}} \) and analytic in \( U_{r_0} \cup \{z_0\} \), with \( f(z) \neq 0 \). If

\[
|f(z_0)| = \max\{|f(z)|: z \in \overline{U_{r_0}}\}, \tag{1.6}
\]

then there exists \( an, m \geq n \) such that

\[
\frac{z_0f'(z_0)}{f(z_0)} = m, \tag{1.7}
\]

\[
Re\left\{\frac{z_0f''(z_0)}{f'(z_0)} + 1\right\} \geq m, \tag{1.8}
\]

and
\[
Re \left\{ \frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)} \right\} \geq m^2.
\] 

(1.9)

Then
\[
Re \left\{ \frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right\} \geq m^3.
\] 

(1.10)

**Proof:** The relations (1.7), (1.8) were proved in Lemma [9, Chap 2, page 19] and (1.9) were proved in Lemma [9, Chap 6, page 322]. We only need to prove (1.10).

If we \( f(z) = R(r_0, \theta)e^{i\theta(r_0, \theta)}, \) for \( z = r_0 e^{i\theta}, \)
then
\[
\frac{zf'(z)}{f(z)} = \frac{\partial \emptyset}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta}.
\] 

(1.11)

Differentiating (1.11) with respect to \( \theta, \) we obtain
\[
i \frac{zf'(z)}{f(z)} \left[ \frac{z(zf'(z))'}{zf'(z)} - \frac{zf'(z)''}{zf'(z)} \right] = \frac{\partial^2 \emptyset}{\partial \theta^2} - i \left\{ \frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left( \frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right\}.
\] 

(1.12)

Another differentiation with respect \( \theta \) leads to
\[
\frac{zf'(z)}{f(z)} \left[ \frac{z(zf'(z))'}{zf'(z)} - 3 \frac{zf'(z)''}{zf'(z)} \cdot \frac{zf'(z)}{f(z)} + 2 \left( \frac{zf'(z)}{f(z)} \right)^2 \right] = - \frac{\partial^3 \emptyset}{\partial \theta^3} - i \frac{\partial}{\partial \theta} \left\{ \frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left( \frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right\}.
\] 

(1.13)

Also differentiation (1.13) with respect to \( \theta, \) we obtain
\[
i \frac{zf'(z)}{f(z)} \left[ \frac{z(z(zf'(z))')}{{zf'(z)}} - 4 \left( \frac{zf'(z)}{f(z)} \right)' \cdot \frac{zf'(z)}{f} - 3 \frac{zf'(z)}{f} \cdot \frac{zf'(z)}{f} \cdot \frac{zf'(z)}{f} + 12 \left( \frac{zf'(z)}{f} \right)^2 - 6 \left( \frac{zf'(z)}{f} \right)^3 \right] = - \frac{\partial^4 \emptyset}{\partial \theta^4} + i \frac{\partial^2}{\partial \theta^2} \left\{ \frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left( \frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right\}.
\]

Taking imaginary parts of this expression at \( z_0 \) and using (1.7) leads to
\[
Re \left[ \frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right] m - 4m^2 Re \left[ \frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)} \right] - 3m^2 Re \left[ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right]^2 + 12m^2 Re \left[ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right] - 6m^4 = \frac{\partial^2}{\partial \theta^2} \left[ \frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left( \frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right].
\]

Finally, by applying (1.8), (1.9), we obtain the desired result (1.10).
Lemma (1.5): Let \( p \in H[a, b] \) and \( q \in Q \) with \( q(0) = a \) for \( z \in \overline{U_{r_0}} \). Let
\[
S = q^{-1}[p(z)] = f(z). \tag{1.14}
\]
If there exists points \( z_0 \in U \) and \( s_0 \in \partial U / E(q) \) such that \( p(z_0) = q(s_0) \) and \( P(U_{r_0}) \subseteq q(U) \),
\[
\text{Re} \left\{ \frac{s_0 q''(s_0)}{q'(s_0)} \right\} \geq 0, \quad \left| \frac{z p'(z)}{q'(s)} \right| \leq k \tag{1.15}
\]
and
\[
\text{Re} \left\{ \frac{s_0^2 q^{(3)}(s_0)}{q'(s_0)} \right\} \geq 0, \quad \left| \frac{z^2 p''(z)}{q'(s)} \right| \leq k^2, \tag{1.16}
\]
where \( r_0 = |z_0| \). Then there exists an \( m \geq n \geq 1 \) such that
\[
z_0 p'(z_0) = m s_0 q'(s_0) \tag{1.17}
\]
\[
\text{Re} \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} \right\} + 1 \geq m \text{Re} \left\{ \frac{s_0 q''(s_0)}{q'(s_0)} + 1 \right\} \tag{1.18}
\]
and
\[
\text{Re} \left\{ \frac{z_0 p'(z_0) + 3z_0^2 p''(z_0) + z_0^3 p^{(3)}(z_0)}{z_0 p'(z_0)} \right\} \geq m^2 \text{Re} \left\{ \frac{s_0 q'(s_0) + 3s_0^2 q''(s_0) + s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} \right\}. \tag{1.19}
\]
Then
\[
\text{Re} \left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} \geq m^3 \text{Re} \left\{ \frac{s_0 q'(s_0) + 7s_0^2 q''(s_0) + 6s_0^3 q^{(3)}(s_0) + s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)} \right\}, \tag{1.20}
\]
or
\[
\text{Re} \left\{ \frac{z_0^3 p^{(4)}(z_0)}{p'(z_0)} \right\} \geq k^3 \text{Re} \left\{ \frac{s_0^3 q^{(4)}(s_0)}{q'(s_0)} \right\}. \tag{1.21}
\]
Proof: The relations (1.17), (1.18) proved in [9, Chap. 2, page 22] and (1.19) proved in [9, Chap. 6, pages 325 and 327]. We only need to prove (1.20). Note that \( q \) is univalent at \( s_0 \) and hence \( q'(s_0) \neq 0 \). Since \( p \) is analytic in \( U \), the set \( p(U_{r_0}) \) is a bounded set and \( p(U_{r_0}) \subseteq q(\overline{U}) / E(q) \).
From (1.14) we see that \( f \) is analytic in \( \overline{U_{r_0}} \) and satisfies \( |f(z_0)| = |s_0| = 1, f(0) = 0 \) and \( |f(z)| \leq 1 \) for \( |z| \leq r_0 \). A further calculation show that \( f^{(k)}(0) = p^{(k)}(0) \) for \( k = 1, 2, \ldots, n - 1 \). Thus \( f \) satisfies the conditions of Lemma (1.4) and deduce that there exists an \( m, n \) such that
\[
\frac{z_0 f'(z_0)}{f(z_0)} = m, \tag{1.22}
\]
and
\[ \text{Re}\left\{ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right\} \geq m. \]  
(1.23)

Since from (1.14) we have \( p(z) = q(s) \), with \( s = f(z) \), we obtain
\[ z p'(z) = q'(s)z f'(z). \]  
(1.24)

Differentiating (1.20) leads to
\[ z\left(\frac{zp'(z)}{zp(z)}\right)' = \frac{zq''(s)}{q'(s)} \cdot \frac{zf'(z)}{f(z)} + \frac{z(zf'(z))'}{zf'(z)}. \]
(1.25)

Another differentiation leads to
\[ z\left(\frac{z(zf'(z))'}{zp(z)}\right)' = \frac{z}{zf'(z)} \left( \frac{z(zf'(z))'}{zf'(z)} \right)' + 4 \frac{s^2 q''(s)}{f'(z)} \cdot \frac{zf'(z)}{f(z)} + 6 \frac{s^3 q^{(3)}(s)}{f'(z)} \cdot \frac{zf'(z)}{f(z)}. \]
(1.26)

While another differentiation leads to
\[ z\left(\frac{z(zf'(z))'}{zp(z)}\right)' = \frac{z}{zf'(z)} \left( \frac{z(zf'(z))'}{zf'(z)} \right)' + 3 \frac{s^2 q''(s)}{f'(z)} \cdot \frac{zf'(z)}{f(z)} + 4 \frac{s^3 q^{(3)}(s)}{f'(z)} \cdot \frac{zf'(z)}{f(z)}. \]

If we evaluate the real part of this expression at \( z_0 \) and use (1.23) and (1.25), we obtain
\[ \text{Re}\left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} = \text{Re}\left\{ \frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right\} + 4m \frac{s_0^3 q''(s_0)}{s_0 q'(s_0)} \cdot \text{Re}\left\{ \frac{z_0^2 p'(z_0)}{z_0 f'(z_0)} \right\} + 3m \frac{s_0^3 q''(s_0)}{s_0 q'(s_0)} \cdot \text{Re}\left\{ \frac{z_0^2 f'(z_0)}{f'(z_0)} \right\}^2 + 6m^2 \frac{s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} \cdot \text{Re}\left\{ \frac{z_0 f'(z_0)}{f'(z_0)} + 1 \right\} + \frac{m^3 s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)}.
\]

By use condition (1.9) and (1.10) in Lemma(1.4) and use (1.7), (1.8), we obtain
\[ \text{Re}\left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} \geq m^3 + 4m^3 \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} + 3m^3 \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} + 6m^3 \frac{s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} + m^3 \frac{s_0^3 q^{(4)}(s_0)}{s_0 q'(s_0)}, \]

hence
\[
\text{Re}\left\{ z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0) \right\} \\
\geq m^3 \left[ s_0 q'(s_0) + 7s_0^2 q''(s_0) + 6s_0^3 q^{(3)}(s_0) + s_0^4 q^{(4)}(s_0) \right] / s_0 q'(s_0)
\]

or \[
\text{Re}\left\{ \frac{z_0^3 p^{(4)}(z_0)}{p'(z_0)} \right\} \geq k^3 \text{Re}\left\{ \frac{s_0^3 q^{(4)}(s_0)}{q'(s_0)} \right\}.
\]

**Definition (1.6):** Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in Q \) and \( n \in N/\{2\} \). The class of admissible function \( A_n[\Omega, q] \) consists of those function \( \psi : \mathbb{C}^5 \times U \rightarrow \mathbb{C} \) that satisfies the following admissibility condition \( \psi(r, s, t, w, b; z) \in \Omega \), whenever

\[
r = q(\xi), s = k\xi q''(\xi), \mathfrak{R}\left( \frac{s}{t} + 1 \right) \geq k\mathfrak{R}\left( \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right), \mathfrak{R}\left( \frac{w}{s} \right) \geq k^2 \mathfrak{R}\left( \frac{\xi q^{(3)}(\xi)}{q'(\xi)} \right)
\]

and \( \mathfrak{R}\left( \frac{b}{s} \right) \geq k^3 \mathfrak{R}\left( \frac{\xi^3 q^{(4)}(\xi)}{q'(\xi)} \right) \),

where \( z \in U \), \( \xi \in \partial U / E(q) \) and \( k \geq n \).

The next theorem is a foundation result in the theory of fourth-order differential subordinations. Its proof is very short because of we use Lemma (1.5) and the special conditions given in the definition of the class of admissible functions \( A_n[\Omega, q] \).

**Theorem (1.7):** Let \( p \in H[a, n] \) with \( n \in N/\{2\} \). Also, let \( q \in Q(a) \) and satisfy the following conditions:

\[
\mathfrak{R}\left( \frac{\xi^2 q'''(\xi)}{q'(\xi)} \right) \geq 0 \quad \quad \quad \frac{z^2 p''(z)}{q'(\xi)} \leq k^2,
\]

where \( z \in U \), \( \xi \in \partial U / E(q) \) and \( k \geq n \). If \( \Omega \) a set in \( \mathbb{C} \), \( \psi \in A_n[\Omega, q] \) and

\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p^{(3)}(z), z^4 p^{(4)}(z); z) \in \Omega
\]

then

\[
p(z) < q(z) \quad (z \in U).
\]

**Proof:** If we assume that \( p < q \), then there exist point \( z_0 = r_0 e^{i\theta_0} \in U \) and \( s_0 \in \partial U / E(q) \) such that \( p(z_0) = q(s_0) \) and \( p(U_{r_0}) \subset q(U) \). From (1.27), we see that the condition (1.16) of Lemma (1.5) is satisfied when \( z \in U_{r_0} \) and \( s \in \partial U / E(q) \). Since all the conditions of that Lemma are satisfied, conclusions (1.17), (1.18), (1.19) and (1.21) follow. Using these last four results of Definition (1.6) leads to

\[
\psi(p(z_0), zp'(z_0), z^2 p''(z_0), z^3 p^{(3)}(z_0), z^4 p^{(4)}(z_0); z) \notin \Omega,
\]

since this contradicts (2.8), we must have \( p < q \).
2. Fourth-order differential subordination with $I_{s,a,\mu}^2$

We first define the following class of admissible function, which are required in proving the differential subordination theorem involving the operator $I_{s,a,\mu}^2 f(z)$ defined by (1.3).

**Definition (2.1):** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap H_0$. The class function $B_I[\Omega, q]$ consists of those function $\phi: \mathbb{C} \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(x), \quad v = \frac{k\xi q'(x) + (\mu - 1)q(x)}{\mu},$$

$$\mathcal{R}\left(\frac{x\mu^2 - (\mu - 1)^2u}{v\mu - u(\mu - 1)} + (2 - 2\mu)\right) \geq \mathcal{R}\left(\frac{\xi q''(x)}{q'(x)} + 1\right)$$

and

$$\mathcal{R}\left(\frac{y\mu^3 - 3x\mu^3 + (2\mu + 1)(\mu - 1)^2}{v\mu - u(\mu - 1)} + (3\mu^2 - 1)\right) \geq k^2\mathcal{R}\left(\frac{\xi^2 q'''(x)}{q'(x)}\right).$$

**Theorem (2.2):** Let $\phi \in B_I[\Omega, q]$. If the function $f(z) \in \Sigma$ and $q \in \mathbb{Q}_0$ and satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi^2 q'''(x)}{q'(x)}\right) \geq 0 \quad \frac{|I_{s,a,\mu}^{\lambda-1} f(z)|}{|q'(x)|} \leq k^2,$$

and

$$\{\phi(I_{s,a,\mu}^{\lambda+1} f(z), I_{s,a,\mu}^{\lambda} f(z), I_{s,a,\mu}^{\lambda-1} f(z), I_{s,a,\mu}^{\lambda-2} f(z), I_{s,a,\mu}^{\lambda-3} f(z)); z \in U\} \subset \Omega,$$

then

$$I_{s,a,\mu}^{\lambda+1} f(z) < q(z) \quad (z \in U).$$

**Proof:** Define the analytic function $p(z)$ in $U$ by

$$p(z) = I_{s,a,\mu}^{\lambda+1} f(z), \quad (z \in U).$$

Then, differentiating (2.3) with respect to $z$ and using (1.4), we have
Further computations show that

\[ I^{\lambda-1}_{s,a,\mu}f(z) = \frac{z^2p''(z) + (2\mu - 1)zp'(z) + (\mu - 1)^2p(z)}{\mu^2} \] (2.5)

and

\[ I^{\lambda-2}_{s,a,\mu}f(z) = \frac{z^3p'''(z) + 3\mu z^2p''(z) + (3\mu^2 - 3\mu + 1)zp'(z) + (\mu - 1)^3p(z)}{\mu^3} \] (2.6)

\[ I^{\lambda-3}_{s,a,\mu}f(z) = \frac{z^4p''''(z) + 2(2\mu + 1)z^3p'''(z) + (6\mu^2 + 1)z^2p''(z) + (4\mu^3 - 3\mu^2 + \mu - 1)zp'(z) + (\mu - 1)^4p(z)}{\mu^4} \] (2.7)

We now define the transformation from \( \mathbb{C}^5 \) to \( \mathbb{C} \) by

\[
\begin{align*}
  u(r, s, t, w, b) &= r, \\
  v(r, s, t, w, b) &= \frac{s + (\mu - 1)r}{\mu}, \\
  x(r, s, t, w, b) &= \frac{t + (2\mu - 1)s + (\mu - 1)^2r}{\mu^2}, \\
  y(r, s, t, w, b) &= \frac{w + 3\mu t + (3\mu^2 - 3\mu + 1)s + (\mu - 1)^3r}{\mu^3}, \\
  g(r, s, t, w, b) &= \frac{b + 2(2\mu + 1)w + (6\mu^2 + 1)t + (4\mu^3 - 3\mu^2 + \mu - 1)s + (\mu - 1)^2r}{\mu^4}.
\end{align*}
\] (2.8)

Let

\[
\psi(r, s, t, w, b; z) = \phi(u, v, x, y, g; z) = \phi\left(\frac{s + (\mu - 1)r}{\mu}, \frac{t + (2\mu - 1)s + (\mu - 1)^3r}{\mu^2}, \frac{w + 3\mu t + (3\mu^2 - 3\mu + 1)s + (\mu - 1)^3r}{\mu^3}, \frac{b + 2(2\mu + 1)w + (6\mu^2 + 1)t + (4\mu^3 - 3\mu^2 + \mu - 1)s + (\mu - 1)^2r}{\mu^4}; z\right). \tag{2.9}
\]

The proof will make use of Theorem(1.7). Using the equations (2.3) to (2.7), we have from (2.9) that

\[
\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) = \phi\left(I^{\lambda+1}_{s,a,\mu}f(z), I^{\lambda}_{s,a,\mu}f(z), I^{\lambda-1}_{s,a,\mu}f(z), I^{\lambda-2}_{s,a,\mu}f(z), I^{\lambda-3}_{s,a,\mu}f(z)\right). \tag{2.10}
\]

Hence, clearly, (2.2) becomes

\[
\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega,
\]
we note that

\[
\frac{t}{s} + 1 = \frac{x\mu^2 - (\mu - 1)^2u}{\nu\mu - u(\mu - 1)} + (2 - 2\mu)
\]

and

\[
\frac{w}{s} = \frac{y\mu^3 - 3x\mu^3 + (2\mu + 1)(\mu - 1)^2u}{\nu\mu - u(\mu - 1)} + (3\mu^2 - 1),
\]

and

\[
b = \frac{g\mu - (\mu - 1)^4 + (4\mu^4 - 2\mu^3)y + (\mu - 1)^3(4\mu + 2)u + (12\mu^2 + 6\mu)(x\mu^2 - (\mu - 1)^2u}{\nu\mu - u(\mu - 1)}
\]

\[-(12\mu^3 + 6\mu^2 + 8\mu - 2).
\]

Therefore, the admissibility condition for \( \phi \in B_I[\Omega, q] \) in Definition (2.1) is equivalent to admissibility condition for \( \psi \in A_3[\Omega, q] \) as given in Definition (1.6) with \( n = 3 \).

Therefore, by using (2.1) and Theorem (1.7), we obtain

\[
p(z) = I_{s, a, \mu}^{(1\ldots 1)} f(z) < q(z).
\]

This completes the proof of Theorem (2.2).

Our next corollary is an extension of Theorem (2.2) to the case when the behavior of \( q(z) \) on \( \partial U \) is not known.

**Corollary (2.3):** Let \( \Omega \subset \mathbb{C} \) and let function \( q(z) \) be univalent in \( U \) with \( q(0) = 0 \). Let \( \phi \in B_I[\Omega, q] \) for some \( p \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( f \in \Sigma \) and \( q_\rho \) satisfies the following conditions:

\[
\Re \left( \frac{\xi^2 q''_\rho(\xi)}{q'_\rho(\xi)} \right) \geq 0 \quad \left| \frac{I_{s, a, \mu}^{(1\ldots 1)} f(z)}{q'_\rho(\xi)} \right| \leq k^2, \quad (z \in U; k \geq 2; \xi \in \partial U/E(q_\rho)),
\]

and

\[
\phi(I_{s, a, \mu}^{(1\ldots 1)} f(z), I_{s, a, \mu}^{(1\ldots 1)} f(z), I_{s, a, \mu}^{(1\ldots 1)} f(z), I_{s, a, \mu}^{(1\ldots 2)} f(z), I_{s, a, \mu}^{(1\ldots 2)} f(z); z) < h(z),
\]

then

\[
I_{s, a, \mu}^{(1\ldots 1)} f(z) < q(z) \quad (z \in U).
\]

**Proof:** By using Theorem (2.2), yield

\[
I_{s, a, \mu}^{(1\ldots 1)} f(z) < q_\rho(z) \quad (z \in U),
\]

then, we obtain the result from

\[
q_\rho(z) < q(z) \quad (z \in U).
This completes the proof of Corollary (1).

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$. In this case, the class $B_{t}[h(U),q]$ is written as $B_{t}[h,q]$. The following two results are immediate consequence of Theorem (2.2) and corollary (2.3).

**Theorem (2.4):** Let $\phi \in B_{t}[h,q]$ . If the function $f \in \Sigma$ and $q \in \mathbb{Q}_{0}$ satisfy the following conditions (2.1) and

$$
\phi(I_{s,a,\mu}^{l+1}f(z), I_{s,a,\mu}^{l}f(z), I_{s,a,\mu}^{l-1}f(z), I_{s,a,\mu}^{l-2}f(z), I_{s,a,\mu}^{l-3}f(z); z) < h(z),
$$

then

$$I_{s,a,\mu}^{l+1}f(z) < q(z) \quad (z \in U).$$

**Corollary (2.5):** Let $\Omega \subset \mathbb{C}$ and let function $q$ be univalent in $U$ with $q(0) = 0$. Also Let $\phi \in B_{t}[h,q_{p}]$ for some $p \in (0,1)$, where $q_{p}(z) = q(\rho z)$ . If the function $f \in \Sigma$ and $q_{p}$ satisfies the conditions (2.11), and

$$
\phi(I_{s,a,\mu}^{l+1}f(z), I_{s,a,\mu}^{l}f(z), I_{s,a,\mu}^{l-1}f(z), I_{s,a,\mu}^{l-2}f(z), I_{s,a,\mu}^{l-3}f(z); z) < h(z),
$$

then

$$I_{s,a,\mu}^{l+1}f(z) < q(z) \quad (z \in U).$$

The following result yield the best dominant of differential subordination (2.12).

**Theorem (2.6):** Let the function $h$ be univalent in $U$. Also let $\phi: \mathbb{C}^{5} \times U \to \mathbb{C}$ and Suppose that following differential equation:

$$
\phi\left(\begin{array}{c}
q(z), \frac{2q'(z) + (\mu - 1)q(z)}{\mu} z^2q''(z) + (2\mu - 1)2q'(z) + (\mu - 1)^2q(z), \\
z^3q'''(z) + \frac{3\mu z^2q''(z) + (3\mu^2 - 3\mu + 1)2q'(z) + (\mu - 1)^3q(z)}{\mu^3}, \\
z^4q''''(z) + 2(2\mu + 1)z^3q'''(z) + (6\mu^2 + 1)z^2q''(z) + (4\mu^4 - 3\mu^2 + \mu - 1)2q'(z) + (\mu - 1)^4q(z); z
\end{array}\right) = h(z),
$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies the condition (2.1). If $f \in \Sigma$ satisfies the condition (2.12) and if

$$
\phi(I_{s,a,\mu}^{l+1}f(z), I_{s,a,\mu}^{l}f(z), I_{s,a,\mu}^{l-1}f(z), I_{s,a,\mu}^{l-2}f(z), I_{s,a,\mu}^{l-3}f(z); z),
$$

is analytic in $U$, then

$$I_{s,a,\mu}^{l+1}f(z) < q(z) \quad (z \in U)$$

and $q(z)$ is the best dominant.

**Proof:** Using Theorem(2.2), that $q(z)$ is a dominant of (2.12). Since $q(z)$ satisfies (2.14), it is also a solution of (2.12). Therefore, $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.
References


