New Type Of Double Sequence of Fuzzy Numbers Defined on Double Orlicz Function

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New type of Double Sequence of Fuzzy Numbers Defined on Double Orlicz Function

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ABSTRACT
In this paper we inserted the notion of $\Delta$-statistically pre-Cauchy double sequence of fuzzy numbers. Also, we establish a criterion for arbitrary double sequence of fuzzy numbers to becomes $\Delta$-statistically pre-Cauchy.

1. Introduction
Zadeh [35] introduced the concepts of fuzzy sets and fuzzy set operations. Maltoka [20] discussed bounded and convergent sequences of fuzzy numbers and studied their some properties. After on it, sequences of fuzzy numbers have been discussed by Dutta [7,8,9], Diamond and Kloeden [15], Nanda [21], Esi [13] and many others.

A fuzzy real number $X$ on $R$ is a function $X : R \rightarrow I = [0,1]$ associating each $t \in R$, with its grade of membership $X(t).$ The class of all fuzzy real numbers is denoted by $R(I).$ For $0 < \alpha \leq 1$, the $\alpha$ – level set $X^\alpha = \{ t \in R : X(t) \geq \alpha \}$, and the 0-level set $X^0 = \{ t \in R : X(t) > 0 \}$, is the clouser of strong 0-cut then it is compact.

Let $D$ denote to the set of all closed bounded intervals $B = [b_1,b_2].$ Define the relation $d$ on $D$ by $d(B,W) = \max \{ |b_1 - w_1|, |b_2 - w_2| \}$. Clearly $(D,d)$ is a complete metric space. (look Diamond and Kloeden [14], Nanda[21]).
The studying of Orlicz sequence spaces have been put newly by various authors (\([10], [11], [12], [25], [33], [34]\)).

Battor and Neamah [24] introduced a double sequence as follows: a double sequence \((X,Y) = (X_{rs},Y_{rs})\) is a double infinite matrix of elements \((X_{rs},Y_{rs})\), where \(X = X_{rs}\) is a double infinite matrix of elements \(X_{rs}\) and \(Y = Y_{rs}\) is infinite a double matrix of elements \(Y_{rs}\), which means \((X_{rs},Y_{rs})\) is complex double sequences and they defined a double Orlicz function on double sequence space in the following:

\[
M : [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty) \text{ such that } M(X,Y) = (M_1(X),M_2(Y)) \text{ and } (X,Y) = (X_{rs},Y_{rs})
\]

where \(M_1 : [0,\infty) \to [0,\infty), \quad M_2 : [0,\infty) \to [0,\infty)\), and \(M_1, M_2\) are two Orlicz functions which be continuous, non-decreasing, even, convex and satisfy the next conditions:

(i) \(M_1(0) = 0, M_2(0) = 0 \Rightarrow M(0,0) = (M_1(0), M_2(0)) = (0,0)\)

(ii) \(M_1(X) > 0, M_2(Y) > 0 \Rightarrow M(X,Y) = (M_1(X), M_2(Y)) > (0,0), \text{ for all } X, Y > 0\)

(iii) \(M_1(X) \to \infty, M_2(Y) \to \infty, \text{ as } X, Y \to \infty \Rightarrow M(X,Y) \to (\infty, \infty), \text{ as } (X,Y) \to (\infty, \infty)\).

**Remark 1.1** [24] If \(M\) is a double Orlicz function, then \(M_1(\lambda X) \leq \lambda M_1(X), M_2(\lambda Y) \leq \lambda M_2(Y), \text{ for all } X \geq 0, Y \geq 0 \text{ with } 0 < \lambda < 1, \text{ therefore } M(\lambda X,\lambda Y) = (M_1(\lambda X), M_2(\lambda Y)) \leq \lambda \left(M_1(X), M_2(Y)\right) = \lambda M(X,Y), \text{ for all } (X,Y) \geq (0,0)\).

If replaced the convexity of \(M\) by \(M(X + Y) \leq M(X) + M(Y)\), then it is said a modulus function (look [22]). A double Orlicz function may be bounded or unbounded, e.g. \(M_1(X) = X^p, M_2(Y) = Y^p\), \(0 < p \leq 1\) are unbounded, so consequently \(M(X,Y) = (X^p,Y^p), (0,0) < (p,p) \leq (1,1)\) is unbounded and \(M_1(X) = \frac{X}{x+1}, M_2(Y) = \frac{Y}{y+1}\) are bounded, so, \(M(X,Y) = \left(\frac{X}{x+1}, \frac{Y}{y+1}\right)\) is bounded.

A fuzzy double sequence \((X,Y) = (X_{rs},Y_{rs})\) is a double infinite matrix of elements \((X_{rs},Y_{rs})\) for all \(r,s \in N\), where \(X = X_{rs}\) is a double infinite matrix of elements \(X_{rs}\) and \(Y = Y_{rs}\) is infinite a double matrix of elements \(Y_{rs}\), where \(X_{rs}, Y_{rs} \in R(I), \text{ which means } (X_{rs},Y_{rs})\) is a double sequences of fuzzy real numbers, where \(X_{rs}, Y_{rs} \in R(I), \text{ for each } r,s \in N\).

The earliest works on double sequences of real or complex terms is found in Bromwich [1]. Later on it further studied by Basarir and Solancan [2], Moricz [23], Tripathy and Dutta [31], Tripathy and Sarma [32] and many others. The notion of regular convergence for double sequences of real or complex terms was introduced by Hardy [18]. Fast [16] was first introduced the concept of statistical convergence and it also independently by Buck [3] and Schoenberg [28]. Further it was studied by Salat [27], Fridy [17], Cannon [4,5] and many others.

The notion of \(\Delta\)-statistically pre-Cauchy double sequence of fuzzy numbers was introduced by Dutta and Reddy [29] and they established a standard for arbitrary double sequence of fuzzy numbers to become \(\Delta\)-statistically pre-Cauchy.

We concept this work to introduced the following notions:
A double sequences of fuzzy numbers \((X_{kl}), (Y_{kl})\) are said to be \(\Delta\)-statistically convergent to \(X_0, Y_0\) respectively if 
\[
\lim_{r,s \to \infty} \frac{1}{rs} \left\| \{(p, q): d(\Delta X_{pq}, X) \geq \varepsilon, \ p \leq r, q \leq s \} \right\| = 0
\]
and 
\[
\lim_{r,s \to \infty} \frac{1}{rs} \left\| \{(p, q): d(\Delta Y_{pq}, Y) \geq \varepsilon, \ p \leq r, q \leq s \} \right\| = 0,
\]
respectively,
therefore, \((X_{kl}, Y_{kl})\) is called \(\Delta\) -statistically convergent to \((X_0, Y_0)\) if 
\[
\lim_{r,s \to \infty} \frac{1}{rs} \left\| \{(p, q): d(\Delta X_{pq}, X), d(\Delta Y_{pq}, Y) \geq \varepsilon, \ p \leq r, q \leq s \} \right\| = 0,
\]
such that the vertical bars indicate to the number of elements in the set.

**Definition 1.2** A double sequence of fuzzy number \((X_{kl}, Y_{kl})\) is called \(\Delta\)-statistically pre-Cauchy if for all \(\varepsilon > 0\) there exist \(v(\varepsilon)\) and \(w(\varepsilon)\) where,
\[
\lim_{r,s \to \infty} \frac{1}{r^2 s^2} \left\| \{(p, q): d(\Delta X_{pq}, X_{vw}), d(\Delta Y_{pq}, Y_{vw}) \geq \varepsilon, \ p \leq r, q \leq s \} \right\| = 0.
\]

In actually, the first order difference operator \(\Delta\) may be represented as an infinite triangular matrix as following,
\[
\Delta = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & \ldots \\
0 & 1 & -1 & 0 & 0 & \ldots \\
0 & 0 & 1 & -1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

The fuzzy double sequence \((X_{rs}, Y_{rs})\) may be expressed as an infinite matrix of fuzzy numbers as following,
\[
(X_{rs}, Y_{rs}) = \begin{bmatrix}
X_{11}, Y_{11} & X_{12}, Y_{12} & X_{13}, Y_{13} & \ldots & X_{1n}, Y_{1n} & \ldots \\
X_{21}, Y_{21} & X_{22}, Y_{22} & X_{23}, Y_{23} & \ldots & X_{2n}, Y_{2n} & \ldots \\
X_{31}, Y_{31} & X_{32}, Y_{32} & X_{33}, Y_{33} & \ldots & X_{3n}, Y_{3n} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Then, for any fuzzy double sequence by \((X, Y) = (X_{rs}, Y_{rs})\), we have
\[
\Delta(X_{rs}, Y_{rs}) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & \ldots \\
0 & 1 & -1 & 0 & 0 & \ldots \\
0 & 0 & 1 & -1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \begin{bmatrix}
X_{11}, Y_{11} & X_{12}, Y_{12} & X_{13}, Y_{13} & \ldots & X_{1n}, Y_{1n} & \ldots \\
X_{21}, Y_{21} & X_{22}, Y_{22} & X_{23}, Y_{23} & \ldots & X_{2n}, Y_{2n} & \ldots \\
X_{31}, Y_{31} & X_{32}, Y_{32} & X_{33}, Y_{33} & \ldots & X_{3n}, Y_{3n} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_{11} - X_{21}, Y_{11} - Y_{21} & X_{12} - X_{22}, Y_{12} - Y_{22} & \ldots & X_{1n} - X_{2n}, Y_{1n} - Y_{2n} & \ldots \\
X_{21} - X_{31}, Y_{21} - Y_{31} & X_{22} - X_{32}, Y_{22} - Y_{32} & \ldots & X_{2n} - X_{3n}, Y_{2n} - Y_{3n} & \ldots \\
X_{31} - X_{41}, Y_{31} - Y_{41} & X_{32} - X_{42}, Y_{32} - Y_{42} & \ldots & X_{3n} - X_{4n}, Y_{3n} - Y_{4n} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]
This way about construction on difference double sequences is useful according to discuss some properties on the spaces of such sequences.

Lindenstrauss and Tzafriri [19] took the idea of Orlicz function and defined the sequence space $L_M$ of single sequences, later on Battor and Neamah [24] used that idea to construct a double sequence space:

$$2L_M = \left\{ (X_{rs}, Y_{rs}) \in 2\omega : \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left( M_1 \left( \frac{|X_{rs}|}{\rho} \right) + M_2 \left( \frac{|Y_{rs}|}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space under the norm:

$$\| (X_{rs}, Y_{rs}) \|_M = \inf \left\{ \rho > 0 : \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left( M_1 \left( \frac{|X_{rs}|}{\rho} \right) + M_2 \left( \frac{|Y_{rs}|}{\rho} \right) \right) \leq 1 \},$$

which is said a double Orlicz of a double sequence space such that $2\omega$ is a family of all $\mathbb{R}^2$ or $\mathbb{C}^2$ double sequence, that is, $(X_{rs})$ and $(Y_{rs})$ are complex or real double sequence and conclusion that the double Orlicz of a double sequence space $2L_M$ be closely related to the space $2L_p$, which is, a double Orlicz double sequence space with $M(X, Y) = (M_1(X), M_2(Y)) = (X^p, Y^p)$, for $(1,1) \leq (p,p) < (\infty, \infty)$ such that $M_1(X) = X^p$, for $1 \leq p < \infty$, and $M_2(Y) = Y^p$, for $1 \leq p < \infty$.

Connor, Fridy and Kline in [6] and Dutta and Reddy in [37] showed the condition of bounded sequence to becomes statistically pre-Cauchy, so we can establish the follows criterion for arbitrary double sequence of fuzzy terms to be statistically pre-Cauchy for a double Orlicz $M = (M_1, M_2)$.

2. Main results

**Theorem 2.1** Let $(X, Y) = (X_{pq}, Y_{pq})$ be a double sequence of fuzzy number and $M = (M_1, M_2)$ be a bounded double Orlicz function, then $(X, Y)$ is $\Delta$- statistically pre-Cauchy if and only if

$$\lim_{r,s \to \infty} \frac{1}{r^2 s^2} \sum_{p \leq r} \sum_{q \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right) = 0,$$

for some $\rho > 0$.

**Proof:** Suppose

$$\lim_{r,s \to \infty} \frac{1}{r^2 s^2} \sum_{p \leq r} \sum_{q \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right) = 0,$$

for some $\rho > 0$. For each $\varepsilon > 0$, $\rho > 0$ and $m, n \in N$, we get

$$\frac{1}{r^2 s^2} \sum_{p \leq r} \sum_{q \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right) =$$

$$\frac{1}{r^2 s^2} \sum_{p \leq r} \sum_{q \leq s} d(\Delta_{pq} X_{vw}) < \varepsilon \sum_{q, w \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right)$$

and

$$\frac{1}{r^2 s^2} \sum_{p \leq r} \sum_{q \leq s} d(\Delta_{pq} X_{vw}) \geq \varepsilon \sum_{q, w \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right)$$

and

$$\varepsilon d(\Delta_{pq} X_{vw}) < \varepsilon \sum_{q, w \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right)$$

and

$$\varepsilon d(\Delta_{pq} Y_{vw}) \geq \varepsilon \sum_{q, w \leq s} \left( M_1 \left( \frac{d(\Delta_{pq} X_{vw})}{\rho} \right), M_2 \left( \frac{d(\Delta_{pq} Y_{vw})}{\rho} \right) \right)$$
\[ \geq \frac{1}{r^2s^2} \sum_{p,v,s,r} d(\Delta x_{pq}, \Delta x_{qw}) \geq \varepsilon \sum_{q,w} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, \Delta x_{qw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, \Delta y_{qw})}{\rho} \right) \right) \right] \]

\[ \geq M(\varepsilon) \left[ \frac{1}{r^2s^2} \left( (p,q) : \left( d(\Delta x_{pq}, \Delta x_{qw}), d(\Delta y_{pq}, \Delta y_{qw}) \right) \geq \varepsilon, p \leq r, q \leq s \right) \right] \]

\[ \geq 0, \text{ where } M = (M_1, M_2) \]

Next, suppose that \((X,Y)\) is \(\Delta\)-statistically pre-Cauchy and that \(\varepsilon\) has been given.

Let \(\varepsilon > 0\) such that \(M_1(\delta) < \frac{\varepsilon}{2}\) and \(M_2(\delta) < \frac{\varepsilon}{2}\) and consequently, \(M(\delta) < \frac{\varepsilon}{2}\).

Since \(M\) is bounded, there exist an integer \(A\) such that \(M_1(X) < \frac{A}{2}\) and \(M_2(Y) < \frac{A}{2}\) for all \(X, Y \geq 0\) and consequently, \(M(X, Y) < \frac{A}{2}\) for all \(X, Y \geq 0\). Note that, for all \(n \in N\).

\[ \frac{1}{r^2s^2} \sum_{p,v,s,r} \sum_{q,w} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, \Delta x_{qw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, \Delta y_{qw})}{\rho} \right) \right) \right] = \]

\[ \frac{1}{r^2s^2} \sum_{p,v,s,r} d(\Delta x_{pq}, \Delta x_{qw}) < \delta \sum_{q,w} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, \Delta x_{qw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, \Delta y_{qw})}{\rho} \right) \right) \right] \]

\[ \leq M(\delta) + \frac{1}{r^2s^2} \sum_{p,v,s,r} d(\Delta x_{pq}, \Delta x_{qw}) \geq \delta \sum_{q,w} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, \Delta x_{qw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, \Delta y_{qw})}{\rho} \right) \right) \right] \]

\[ \leq \frac{\varepsilon}{2} + \frac{A}{2} \left[ \left( (p,q) : \left( d(\Delta x_{pq}, \Delta x_{qw}), d(\Delta y_{pq}, \Delta y_{qw}) \right) \geq \delta, p \leq r, q \leq s \right) \right] \]

\[ \leq \varepsilon + A \left[ \left( (p,q) : \left( d(\Delta x_{pq}, \Delta x_{qw}), d(\Delta y_{pq}, \Delta y_{qw}) \right) \geq \delta, p \leq r, q \leq s \right) \right] \quad \text{(1)} \]

Since, \((X,Y)\) is \(\Delta\)-statistically pre-Cauchy, there is, \(N\) such that the right hand side of equation (1) is less than \(\varepsilon\) for each \(n \in N\). Hence,

\[ \lim_{r,s \to \infty} \frac{1}{r^2s^2} \sum_{p,v,s,r} \sum_{q,w} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, \Delta x_{qw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, \Delta y_{qw})}{\rho} \right) \right) \right] = 0. \]

**Theorem 2.2** Let \((X,Y) = (X_{pq}, Y_{pq})\) be a double sequence of fuzzy number and let \(M\) be a bounded double Orlicz function, then \((X,Y)\) is \(\Delta\)-statistically convergent to \((X_0, Y_0)\) if and only if

\[ \lim_{r,s} \frac{1}{r^2s^2} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, X_0)}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, Y_0)}{\rho} \right) \right) \right] = 0. \]

**Proof:** consider that

\[ \lim_{r,s} \frac{1}{r^2s^2} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( M_1 \left( \frac{d(\Delta x_{pq}, X_0)}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta y_{pq}, Y_0)}{\rho} \right) \right) \right] = 0, \]
with a double Orlicz function \( M = (M_1, M_2) \), then \((X, Y)\) is \(\Delta\) - statistically convergent to \((X_0, Y_0)\) see[22].

Conversely, suppose that \((X, Y)\) is \(\Delta\) - statistically convergent to \((X_0, Y_0)\). In the same manner to theorem (2.1) and using that \( M \) be a double Orlicz function we can prove that

\[
\lim_{r,s} \frac{1}{r^s} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( M_1 \left( \frac{d(X_{pq}, X_0)}{\rho} \right) \right), \left( M_2 \left( \frac{d(Y_{pq}, Y_0)}{\rho} \right) \right) \right] = 0,
\]

Corollary 2.3 Let \((X, Y) = (X_{pq}, Y_{pq})\) be a double sequence of fuzzy number, then \((X, Y)\) is \(\Delta\)-statistically pre-Cauchy, if and only if

\[
\lim_{r,s} \frac{1}{r^s} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( d(\Delta X_{pq}, \Delta X_{vw}), d(\Delta Y_{pq}, \Delta Y_{vw}) \right) \right] = 0.
\]

Proof: Let \( B_1 = \sup_{p,q} d(\Delta X_{pq}, \bar{0}), B_2 = \sup_{p,q} d(\Delta Y_{pq}, \bar{0}) \) and define

\[
M(X, Y) = \left( \frac{1+B_1}{1+X}, \frac{1+B_2}{1+Y} \right).\text{Then,}
\]

\[
\left[ \left( M_1 \left( \frac{d(\Delta X_{pq}, \Delta X_{vw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta Y_{pq}, \Delta Y_{vw})}{\rho} \right) \right) \right] \leq \left( 1 + 2B_1 \right) d(\Delta X_{pq}, \Delta X_{vw}), \left( 1 + 2B_2 \right) d(\Delta Y_{pq}, \Delta Y_{vw})
\]

and

\[
\left[ \left( M_1 \left( \frac{d(\Delta X_{pq}, \Delta X_{vw})}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta Y_{pq}, \Delta Y_{vw})}{\rho} \right) \right) \right] = \left( 1 + 2B_1 \right) \frac{d(\Delta X_{pq}, \Delta X_{vw})}{1 + d(\Delta X_{pq}, \Delta X_{vw})}, \left( 1 + 2B_2 \right) \frac{d(\Delta Y_{pq}, \Delta Y_{vw})}{1 + d(\Delta Y_{pq}, \Delta Y_{vw})}
\]

\[
\geq \frac{(1 + 2B_1) d(\Delta X_{pq}, \Delta X_{vw})}{1 + d(\Delta X_{pq}, \Delta X_{vw})}, \frac{(1 + 2B_2) d(\Delta Y_{pq}, \Delta Y_{vw})}{1 + d(\Delta Y_{pq}, \Delta Y_{vw})}
\]

Thus, \( \lim_{r,s} \frac{1}{r^s} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( d(X_{pq}, X_{vw}), d(Y_{pq}, Y_{vw}) \right) \right] = 0 \), if and only if

\[
\lim_{r,s} \frac{1}{r^s} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( M_1 \left( \frac{d(\Delta X_{pq}, X_0)}{\rho} \right) \right), \left( M_2 \left( \frac{d(\Delta Y_{pq}, Y_0)}{\rho} \right) \right) \right] = 0, \text{and direct application of theorem (2.1) the proof completes.}
\]

Corollary 2.4 Let \((X, Y) = (X_{pq}, Y_{pq})\) be a double sequence of fuzzy number, then \((X, Y)\) is \(\Delta\)-statistically convergent to \((X_0, Y_0)\), if and only if

\[
\lim_{r,s} \frac{1}{r^s} \sum_{p=1}^{r} \sum_{q=1}^{s} \left[ \left( d(\Delta X_{pq}, X_0), d(\Delta Y_{pq}, Y_0) \right) \right] = 0.
\]
Proof: Let $B_1 = \sup_{p,q} d\left(\Delta X_{pq}, \tilde{0}\right)$, $B_2 = \sup_{p,q} d\left(\Delta Y_{pq}, \tilde{0}\right)$ and define $M(X,Y) = \left(\frac{1+B_1+X_0}{1+X}, \frac{1+B_2+Y_0}{1+Y}\right)$. Then in the same manner of the proof of corollary (2.3) we can get the prove. ■

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References:
