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On Differential Subordination Theorems of Analytic Multivalent Functions Defined by Generalized Integral Operator

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ABSTRACT

In this paper, we obtain some applications of second-order differential Subordination results involving a generalized integral operator for certain normalized analytic functions.

MSC: 30C45, 30C50

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0, p \in N = \{1,2,3, \dots .\}), \qquad (1.1)$$

which are analytic and $p - valent$ in the open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g in U,written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that

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$$f(z) = g\big(w(z)\big)\,, (z \in U).$$

In particular, if the function g is univalent in U, then $f \prec g$ if

$$f(0) = g(0), \text{ and } f(U) \subset g(U) \ \ ([9,18]).$$

For the function f given by (1.1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k\,.$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=P+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

The set of all functions f that are analytic and injective on $\overline{U}\ /\ E(f)$, Denote by Q where

$$E(f) = \left\{\zeta \in \partial U \ \ : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $\acute{f}(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f\ )$ (see [19]).

Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$, and h is univalent in U with $q \in Q$. Miller and Mocanu [12] consider the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), z\acute{p}(z), z^2\acute{\acute{p}}(z); z\right) \prec h(z) \tag{1.2}$$

implies $p(z) \prec q(z)$, for all functions $p(z) \in H[a,n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\}$, *where H be the linear space of all analytic functions in U , $a \in \mathbb{C}$ and $n \in Z^+$* that satisfy the differential subordination (1.2), moreover, they found conditions so that q is the smallest function with this property, called the best dominant of the subordination (1.2).

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$, and $h \in H$ with $q \in H[a,n]$. Recently Miller and Mocanu [13,14] studied the dual problem and determined conditions on ϕ such that

$$h(z) \prec \varphi\big(p(z), z\acute{p}(z), z^2\acute{\acute{p}}(z); z\big) \tag{1.3}$$

implies $q(z) \prec p(z)$, for all functions $p \in Q$ that satisfy the above superordination. They also found conditions so that the function q is the largest function with this property, called the best subordinant of the superordination (1.3). See [1,2,3,4,5,6], the authors studied differential subordination results for multivalent functions for other classes.

We define the integral operator $\mathcal{A}_p^q(\Psi,\Phi,\mathrm{T})f(z), f(z) \in A(p)$ as follows :

$$\mathcal{A}_p^0(\Psi,\Phi,\mathrm{T})f(z) = f(z)$$

$$\mathcal{A}_p^1(\Psi,\Phi,\mathrm{T})f(z) = \mathcal{A}_p(\Psi,\Phi,\mathrm{T})f(z) = \left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right) z^{p-\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)} \int_0^z t^{\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\ \right)-(p+1)} f(t)dt$$

$$\mathcal{A}_p^2(\Psi,\Phi,\mathrm{T})f(z)=\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)z^{p-\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)}\int_0^z t^{\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)-(p+1)}\mathcal{A}_p^1(\Psi,\Phi,\mathrm{T})f(t)dt$$

and, in general

$$\mathcal{A}_p^q(\Psi,\Phi,\mathrm{T})f(z)=\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)z^{p-\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)}\int_0^z t^{\left(\frac{p+\Phi}{\Psi+\mathrm{T}}\right)-(p+1)}\mathcal{A}_p^{q-1}(\Psi,\Phi,\mathrm{T})f(t)dt$$

$$(f(z)\in A(p);q\in N_0;z\in U). \qquad (1.4)$$

We see that for $f(z)\in A(p)$, we have that

$$\mathcal{A}_p^q(\Psi,\Phi,\mathrm{T})f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{p+\Phi}{p+\Phi+(\Psi+\mathrm{T})(k-p)}\right)^q a_k z^k,$$

$$0<\Psi<1,\Phi,\mathrm{T}\geq 0;p\in N,q\in N_0\,. \qquad (1.5)$$

From (1.5), it easy to verify that

$$(\Psi+\mathrm{T})z\left(\mathcal{A}_p^{q+2}f(z)\right)'=(\Phi+p)\left(\mathcal{A}_p^{q+1}f(z)\right)-\left(\Phi+p\left(1-(\Psi+\mathrm{T})\right)\right)\left(\mathcal{A}_p^{q+2}f(z)\right). \quad (1.6)$$

We note that :

1- $\mathcal{A}^q(\Psi,0,0)f(z)=I_\lambda^{-q}f(z)$ (see [17])

2- $\mathcal{A}_1^\alpha(1,1,0)f(z)=I^\alpha f(z)$ (see [11]).

3- $\mathcal{A}_p^q(1,1,0)f(z)=I_p^q f(z)$ (see [19]).

4- $\mathcal{A}_1^q(1,1,0)f(z)=D^q f(z)$ (see [16]).

5- $\mathcal{A}_1^q(1,1,0)f(z)=I^q f(z)$ (see [10]).

6- $\mathcal{A}_1^q(1,0,0)f(z)=I^q f(z)$ (see [18]).

Also we note that :

1- $\mathcal{A}_p^q(1,0,0)f(z)=J_p^q f(z)$

$$=\left\{f(z):J_p^q f(z)=z^p+\sum_{k=n+p}^{\infty}\left(\frac{p}{k}\right)^q a_k z^k, q\in N_0, z\in U\right\}.$$

2- $\mathcal{A}_p^q(1,l,0)f(z)=J_p^q(l)f(z)=$

$$\left\{f(z):J_p^q(l)f(z)=z^p+\sum_{k=n+p}^{\infty}\left(\frac{p+l}{k+l}\right)^q a_k z^k, q\in N_0, l>0, z\in U\right\}$$

3- $\mathcal{A}_p^q(\lambda,0,0)f(z)=J_{p,\lambda}^q f(z)$

$$=\left\{f(z):J_{p,\lambda}^q f(z)=z^p+\sum_{k=n+p}^{\infty}\left(\frac{p}{k+\lambda(k-p)}\right)^q a_k z^k, q\in N_0, \lambda\geq 0, z\in U\right\}.$$

4- $\mathcal{A}_p^q(\lambda,\alpha\delta,0)f(z)=I_p^q(\lambda,\alpha,\delta)f(z)$ [1]

$$\left\{f(z):J_{p,\lambda}^q f(z)=z^p+\sum_{k=n+p}^{\infty}\left(\frac{p+\alpha\delta}{p+\alpha\delta+\lambda(k-p)}\right)^q a_k z^k, q\in N_0, \lambda\geq 0, \alpha,\delta>0, z\in U\right\}.$$

In this paper, we shall determine some properties on the admissible functions defined with operator $\mathcal{A}_p^q(\Psi,\Phi,\mathrm{T})$.

2- Preliminaries:

In order to prove our results, we shall make use of the following known results.

Lemma (2.1)[9] : Let q be univalent in $U, \zeta\in\mathbb{C}^*$ and suppose that

$$Re\left\{1+\frac{z\acute{q}(z)}{\acute{q}(z)}\right\}>\max\left\{0,-Re\left(\frac{1}{\zeta}\right)\right\}. \tag{2.1}$$

If $p(z)$ is analytic in U, with $p(0)=q(0)$ and

$$p(z)+\zeta z\acute{p}(z)\prec q(z)+\zeta z\acute{q}(z), \tag{2.2}$$

then $p(z)\prec q(z)$, and $q(z)$ is the best dominant.

Lemma (2.2)[17] : Let the function $q(z)$ be univalent in the unit disk, and let θ,φ be analytic in domain D containing $q(U)$ with $\varphi(w)\neq 0$ when $w\in q(U)$. Set

$Q(z)=z\acute{q}(z)\varphi(q(z))$ and $h(z)=\theta\big(q(z)\big)+Q(z)$. Suppose that

1- Q is starlike univalent in U.

2- $Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\}>0$ for $z\in U$.

If p is analytic with $p(0)=q(0), p(U)\subseteq D$ and

$$\theta\big(p(z)\big)+z\acute{p}(z)\varphi\big(p(z)\big)\prec\theta\big(q(z)\big)+z\acute{q}(z)\varphi\big(q(z)\big), \tag{2.3}$$

then $p\prec q$, and $q(z)$ is the best dominant.

Lemma (2.3)[7] :Let $q(z)$ be convex in $U, q(0)=a$ and $\zeta\in\mathbb{C}, Re(\zeta)>0$.

If $p\in H[a,1]$ and $p(z)+\gamma z\acute{q}(z)$ is univalent in U, then

$$q(z)+\zeta z\acute{q}(z)\prec p(z)+\zeta z\acute{p}(z), \tag{2.4}$$

implies $q(z) \prec p(z)$, and $q(z)$ is the best subordinant.

Lemma (2.4)[8] : Let $q(z)$ be convex univalent in the unit disk U and let θ, φ be analytic in a domain D containing $q(U)$. Suppose that

1- $Re\left\{\frac{\acute{\theta}(q(z))}{\varphi(q(z))}\right\} > 0$, for $z \in U$.

2- $z\acute{q}(z)\varphi(q(z))$ is starlike univalent in U.

If $p(z) \in H[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + z\acute{p}(z)\varphi(p(z))$ is univalent in U, and

$$\theta\big(q(z)\big) + z\acute{q}(z)\varphi\big(q(z)\big) \prec \theta\big(p(z)\big) + z\acute{p}(z)\varphi\big(p(z)\big), \qquad (2.5)$$

then $q(z) \prec p(z)$, and $q(z)$ is the best subordinant..

3- Main results:

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\Psi > 0, \Phi, \mathrm{T} \geq 0; p \in N, q \in N_0 = N \cup \{0\}; z \in U$ and the powers are understood as principle values.

Theorem (3.1): Let $q(z)$ be univalent in U with $q(0) = 0, \gamma > 0$ and suppose that

$$Re\left\{1 + \frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > max\left\{0, -Re\left(\frac{\sigma(\Phi + p)}{\mathrm{T} + \Psi}\right)\right\}. \qquad (3.1)$$

If $f \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} + \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1\right) \prec q(z) + \frac{\mathrm{T} + \Psi}{\sigma(\Phi + p)} z\acute{q}(z), \qquad (3.2)$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof : If we consider the analytic function

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma}, \sigma > 0 \ , \ z \in U \ . \qquad (3.3)$$

Differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{z\acute{p}(z)}{p(z)}=\frac{\sigma(\Phi+p)}{\Psi+T}\left(\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)}-1\right), \tag{3.4}$$

that is

$$\frac{\Psi+T}{\sigma(\Phi+p)}z\acute{p}(z)=\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\right)^{\sigma}\left(\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)}-1\right).$$

Thus, the subordination (3.2) is equivalent to

$$p(z)+\frac{\Psi+T}{\sigma(\Phi+p)}z\acute{p}(z)\prec q(z)+\frac{\Psi+T}{\sigma(\Phi+p)}z\acute{q}(z). \tag{3.5}$$

Applying Lemma (2.1), with $\zeta=\frac{\Psi+T}{\sigma(\Phi+p)}$, the proof of Theorem (3.1) is complete.

Taking the convex function $q(z)=\frac{1+Az}{1+Bz}$, in the Theorem (3.1), we have the following corollary.

Corollary (3.1): Let $A, B\in\mathbb{C}, A\neq B, |B|<1$, $\mathrm{Re}(\zeta)>0$ and $\sigma>0$. If $f(z)\in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\right)^{\sigma}+\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\right)^{\sigma}\left(\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)}-1\right)\prec\frac{1+Az}{1+Bz}+\frac{\Psi+T}{\sigma(\Phi+p)}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\right)^{\sigma}\prec\frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q=0$ in Theorem (3.1), we obtain the following result:

Corollary (3.2): Let $q(z)$ be univalent in U, with $q(0)=1, \sigma>0$, and suppose that (3.1) holds. If $f(z)\in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{2}f(z)}{z^p}\right)^{\sigma}+\left(\frac{\mathcal{A}_p^{2}f(z)}{z^p}\right)^{\sigma}\left(\frac{\mathcal{A}_p^{1}f(z)}{\mathcal{A}_p^{2}f(z)}-1\right)\prec q(z)+\frac{\Psi+\mathrm{T}}{\sigma(\Psi+p)}z\acute{q}(z),$$

then

$$\left(\frac{\mathcal{A}_p^{2}f(z)}{z^p}\right)^{\sigma}\prec q(z).$$

and q(z) is the best dominant.

Taking $\Phi = \Psi = 1$ in the Theorem (3.1), we have the following result.

Corollary (3.3): Let $q(z)$ be univalent in U, with $q(0) = 1, \beta \in \mathbb{C}^*, \sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} + \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1\right) \prec q(z) + \frac{1+T}{\sigma(1+p)} z\acute{q}(z),$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} \prec q(z).$$

and $q(z)$ is the best dominant.

Theorem (3.2): Let $q(z)$ be univalent in U, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, let $\lambda, \sigma \in \mathbb{C}^*$, $f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \tag{3.6}$$

and

$$Re\left\{1 + \frac{z\acute{\acute{q}}(z)}{\acute{q}(z)} - \frac{z\acute{q}(z)}{q(z)}\right\} > 0, \quad (z \in U). \tag{3.7}$$

If

$$\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} \prec 1 + \frac{(\Psi + \mathrm{T}) z q\acute{(}z)}{\sigma(\Phi + p) q(z)}, \tag{3.8}$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} \prec q(z)$$

and $q(z)$ is the best dominant of (3.6).

proof : Let

$$p(z) = \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma}, \quad z \in U\,. \tag{3.9}$$

According to (3.4) the function $p(z)$ is analytic in U, and differentiating (3.9) logarithmically with respect to z, we obtain

$$\frac{z\acute{p}(z)}{p(z)}=\frac{\sigma(\Phi+p)}{\Psi+\mathrm{T}}\left(\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)}-1\right). \tag{3.10}$$

In order to prove our result we will use Lemma (2.2). In this lemma consider

$$\theta(w)=1 \text{ and } \varphi(w)=\frac{\Psi+\mathrm{T}}{\sigma(\Phi+p)w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z)=z\acute{q}(z)\varphi\big(q(z)\big)=\frac{(\Psi+\mathrm{T})z\acute{q}(z)}{\sigma(\Phi+p)q(z)},$$

and

$$h(z)=\theta\big(q(z)\big)+Q(z)=1+\frac{(\Psi+\mathrm{T})z\acute{q}(z)}{\sigma(\Phi+p)q(z)},$$

from (3.7), we see that $Q(z)$ is a starlike function in U. We also have

$$Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\}=Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}-\frac{z\acute{q}(z)}{q(z)}\right\}>0,\quad (z\in U)$$

and then, by using Lemma (2.2), we deduce that the subordination (3.6) implies

$$p(z)\prec q(z)$$

and the function $q(z)$ is the best dominant of (3.8).

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.4): Let $\sigma\in\mathbb{C}^*$. Let $f(z)\in A(p)$ and suppose that

$$\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\neq 0,\quad (z\in U).$$

If

$$\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)}\prec 1+\frac{(\Psi+\mathrm{T})z(A-B)}{\sigma(\Phi+p)(1+Az)(1+Bz)},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p}\right)^{\sigma}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q(z)=\frac{1+z}{1-z}$ in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.5): Let $\sigma \in \mathbb{C}^*$, $f(z) \in A(p)$ and suppose that

$$\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \quad (z \in U).$$

If

$$\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} \prec 1+\frac{2(\Psi+\mathrm{T})z}{\sigma(\Phi+p)(1-z)(1+z)},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p}\right)^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z)=\frac{1+z}{1-z}$ is the best dominant.

Theorem (3.3): Let $q(z)$ be univalent in U, with $q(0)=1$, let $\sigma \in \mathbb{C}^*$, and let $\psi, \nu, \eta \in \mathbb{C}$ with $\nu+\eta \neq 0$. Let $f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{\nu \mathcal{A}_p^{q+1} f(z)+\eta \mathcal{A}_p^{q+2} f(z)}{(\nu+\eta) z^p} \neq 0, \quad (z \in U) \tag{3.11}$$

and

$$Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > max\{0,-Re(\psi)\}, \quad (z \in U). \tag{3.12}$$

If

$$\mathrm{K}(z)=\Upsilon\left[\frac{\nu \mathcal{A}_p^{q+1} f(z)+\eta \mathcal{A}_p^{q+2} f(z)}{(\nu+\eta) z^p}\right]^{\sigma}+\sigma\left[\left(\frac{\nu z(\mathcal{A}_p^{q+1} f(z))\acute{}+\eta z(\mathcal{A}_p^{q+2} f(z))\acute{}}{\nu \mathcal{A}_p^{q+1} f(z)+\eta \mathcal{A}_p^{q+2} f(z)}-p\right)\right] \tag{3.13}$$

and

$$\mathrm{K}(z) \prec \Upsilon q(z)+\frac{z\acute{q}(z)}{q(z)}, \tag{3.14}$$

then

$$\left[\frac{\nu A_p^{q+1} f(z)+\eta A_p^{q+2} f(z)}{(\nu+\eta) z^p}\right]^{\sigma} \prec q(z)$$

and $q(z)$ is the best dominant of (3.11).

Proof : Let

$$p(z) = \left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu+\eta) z^p}\right]^\sigma , \quad z \in U. \tag{3.15}$$

According to (3.8) the function $p(z)$ is analytic in U, and differentiating (3.15) logarithmically with respect to z, we obtain

$$\frac{z\acute{p}(z)}{p(z)} = \sigma\left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{A+1} f(z) + \eta A_p^{A+2} f(z)} - p\right], \tag{3.16}$$

and hence

$$z\acute{p}(z) = \sigma\left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu+\eta) z^p}\right]^\sigma \left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p\right].$$

In order to prove our result, we will use Lemma (2.2). In this lemma consider

$$\theta(w) = \Upsilon w \ \ and \ \ \varphi(w) = \frac{1}{w},$$

then θ is analytic in C and $\varphi(w) \neq 0$ is analytic in C^*. Also if we let

$$Q(z) = z\acute{q}(z)\varphi\big(q(z)\big) = \sigma\left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p\right]$$

and

$$\begin{aligned} h(z) &= \theta\big(q(z)\big) + Q(z) \\ &= \Upsilon\left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu+\eta) z^p}\right]^\sigma + \sigma\left[\left(\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p\right)\right] \end{aligned}$$

from (3.11), we see that $Q(z)$ is a starlike function in U. We also have

$$Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\} = Re\left\{\Upsilon + 1 + \frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > 0, \qquad (z \in U)$$

and then, by using Lemma (2.2), we deduce that the subordination (3.14) implies

$$p(z) \prec q(z).$$

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem (3.3) and according to(3.4), the condition (3.12) becomes

$$max\{0, -Re(\Upsilon)\} \leq \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu = 1$ and $\eta = 0$, we obtain the following result.

Corollary (3.6) : Let $\Upsilon \in C$ with

$$max\{0, -Re(\Upsilon)\} \leq \frac{1-|B|}{1+|B|}.$$

Let $f(z) \in A(p)$ and suppose that

$$\frac{A_p^{q+1} f(z)}{z^p} \neq 0, \quad (z \in U).$$

If

$$\Upsilon \left[\frac{\nu A_p^{q+1} f(z)}{z^p}\right]^{\sigma} + \sigma \left[\left(\frac{z(A_p^{q+1} f(z))'}{A_p^{q+1} f(z)} - p\right)\right] \prec \Upsilon \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{A_p^{q+1} f(z)}{z^p}\right)^{\gamma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $p = \nu = q = 1, \eta = 0 \; and \; q(z) = \frac{1+z}{1-z}$ in Theorem (3.3), we obtain the next result.

Corollary (3.7) : Let $f(z) \in A(p)$ and suppose that

$$\frac{A_p^2 f(z)}{z^p} \neq 0, \quad (z \in U).$$

and $\sigma \in C^*$. If

$$\Upsilon \left[\frac{A^2 f(z)}{z}\right]^{\sigma} + \sigma \left[\left(\frac{z(A^2 f(z))'}{A^2 f(z)} - 1\right)\right] \prec \Upsilon \frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)},$$

then

$$\left(\frac{A^2 f(z)}{z}\right)^{\gamma} \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

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