On Differential Subordination Theorems of Analytic Multivalent Functions Defined by Generalized Integral Operator

Waggas Galib Atshan  
*Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq,*  
waggas.galib@qu.edu.iq

Ali Hussein Battor  
*Department of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf, Iraq,*  
alih.battoor@uokufa.edu.iq

Abeer Farhan Abaas  
*Department of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf, Iraq,*  
abeerfarhan688@gmail.com

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**Recommended Citation**  
DOI: 10.29350/2411-3514.1193  
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1. Introduction

Let \( A(p) \) denote the class of functions of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0, \ p \in N = \{1,2,3, \ldots \}),
\]

which are analytic and \( p-\)valent in the open unit disk \( U = \{z: z \in \mathbb{C}, |z| < 1\} \). If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \) in \( U \), written \( f < g \) or \( f(z) < g(z) \), if there exists a Schwarz function \( w(z) \) analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that
\[ f(z) = g(w(z)), (z \in U). \]

In particular, if the function \( g \) is univalent in \( U \), then \( f < g \) if

\[ f(0) = g(0), \text{ and } f(U) \subset g(U) \text{ ([9,18])}. \]

For the function \( f \) given by (1.1) and \( g \in A(p) \) given by

\[ g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k. \]

The Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[ (f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z). \]

The set of all functions \( f \) that are analytic and injective on \( \overline{U} / E(f) \), Denote by \( Q \) where

\[ E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}, \]

and are such that \( \dot{f}(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \) (see [19]).

Let \( \psi : \mathbb{C} \times U \to \mathbb{C} \), and \( h \) is univalent in \( U \) with \( q \in Q \). Miller and Mocanu [12] consider the problem of determining conditions on admissible functions \( \psi \) such that

\[ \psi(p(z), z\dot{p}(z), z^2\ddot{p}(z); z) < h(z) \] (1.2)

implies \( p(z) < q(z) \), for all functions \( p(z) \in H[a, n] = \{ f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \} \), where \( H \) be the linear space of all analytic functions in \( U \), \( a \in \mathbb{C} \) and \( n \in \mathbb{Z}^+ \) that satisfy the differential subordination (1.2), moreover, they found conditions so that \( q \) is the smallest function with this property, called the best dominant of the subordination (1.2).

Let \( \phi : \mathbb{C} \times U \to \mathbb{C} \), and \( h \in H \) with \( q \in H[a,n] \). Recently Miller and Mocanu [13,14] studied the dual problem and determined conditions on \( \phi \) such that

\[ h(z) < \phi(p(z), z\dot{p}(z), z^2\ddot{p}(z); z) \] (1.3)

implies \( q(z) < p(z) \), for all functions \( p \in Q \) that satisfy the above superordination. They also found conditions so that the function \( q \) is the largest function with this property, called the best subordinant of the superordination (1.3). See [1,2,3,4,5,6], the authors studied differential subordination results for multivalent functions for other classes.

We define the integral operator \( \mathcal{A}_p^g(\Psi, \Phi, T)f(z) \), \( f(z) \in A(p) \) as follows:

\[ \mathcal{A}_p^0(\Psi, \Phi, T)f(z) = f(z) \]

\[ \mathcal{A}_p^1(\Psi, \Phi, T)f(z) = \mathcal{A}_p(\Psi, \Phi, T)f(z) = \left( \frac{p + \Phi}{\Psi + T} \right) z^{p-(\frac{p+\Phi}{\Psi+T})} \int_{0}^{z} t^{(\frac{p+\Phi}{\Psi+T})-(p+1)} f(t) \, dt \]
\[
\mathcal{A}_p^q(\Psi, \Phi, T)f(z) = \left( \frac{p + \Phi}{\Psi + T} \right) z^{p-\frac{(p+\Phi)}{\Psi+T}} \int_0^z t^{\frac{(p+\Phi)}{\Psi+T} - (p+1)} a_p^1(\Psi, \Phi, T)f(t) dt 
\]

and, in general
\[
\mathcal{A}_p^q(\Psi, \Phi, T)f(z) = \left( \frac{p + \Phi}{\Psi + T} \right) z^{p-\frac{(p+\Phi)}{\Psi+T}} \int_0^z t^{\frac{(p+\Phi)}{\Psi+T} - (p+1)} a_p^{q-1}(\Psi, \Phi, T)f(t) dt 
\]

\[
(f(z) \in A(p); q \in N_0; z \in U). 
\]

We see that for \( f(z) \in A(p) \), we have that
\[
\mathcal{A}_p^q(\Psi, \Phi, T)f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{p + \Phi}{p + \Phi + (\Psi + T)(k - p)} \right)^q a_{k}z^k, 
\]

\[
0 < \Psi < 1, \Phi, T \geq 0; p \in N, q \in N_0. 
\]

From (1.5), it easy to verify that
\[
(\Psi + T)z \left( \mathcal{A}_p^{q+2}f(z) \right)' = (\Phi + p) \left( \mathcal{A}_p^{q+1}f(z) \right) - \left( \Phi + p(1 - (\Psi + T)) \right) \left( \mathcal{A}_p^{q+2}f(z) \right). 
\]

We note that:

1- \( \mathcal{A}_p^q(\Psi, 0, 0)f(z) = l_{\alpha}^{-q}f(z) \) (see [17])

2- \( \mathcal{A}_1^q(1, 1, 0)f(z) = l^qf(z) \) (see [11]).

3- \( \mathcal{A}_p^q(1, 1, 0)f(z) = l_1^qf(z) \) (see [19]).

4- \( \mathcal{A}_1^q(1, 1, 0)f(z) = D^qf(z) \) (see [16]).

5- \( \mathcal{A}_1^q(1, 1, 0)f(z) = l^qf(z) \) (see [10]).

6- \( \mathcal{A}_1^q(1, 0, 0)f(z) = l^qf(z) \) (see [18]).

Also we note that:

1- \( \mathcal{A}_p^q(1, 0, 0)f(z) = j^q_p f(z) \)

\[
= \left\{ f(z): j^q_p f(z) = z^p + \sum_{k=n+p}^{\infty} \left( \frac{p}{k} \right)^q a_{k}z^k, q \in N_0, z \in U \right\}.
\]

2- \( \mathcal{A}_p^q(1, l, 0)f(z) = j^q_p (l)f(z) = \)

\[
= \left\{ f(z): j^q_p (l)f(z) = z^p + \sum_{k=n+p}^{\infty} \left( \frac{p + l}{k + l} \right)^q a_{k}z^k, q \in N_0, l > 0, z \in U \right\}
\]
3. \( A_p^q(\lambda, 0, 0)f(z) = J_{p,\lambda}^q f(z) \)

\[
= \left\{ f(z) : J_{p,\lambda}^q f(z) = z^p + \sum_{k=n+p}^{\infty} \left( \frac{p}{k + \lambda(k-p)} \right)^q a_k z^k, q \in N_0, \lambda \geq 0, z \in U \right\}.
\]

4. \( A_p^q(\lambda, \alpha \delta, 0)f(z) = I_p^q(\lambda, \alpha, \delta) f(z) \) [1]

\[
= \left\{ f(z) : I_{p,\lambda}^q f(z) = z^p + \sum_{k=n+p}^{\infty} \left( \frac{p + \alpha \delta}{p + \alpha \delta + \lambda(k-p)} \right)^q a_k z^k, q \in N_0, \lambda \geq 0, \alpha, \delta > 0, z \in U \right\}.
\]

In this paper, we shall determine some properties on the admissible functions defined with operator \( A_p^q(\Psi, \Phi, T) \).

2. Preliminaries:

In order to prove our results, we shall make use of the following known results.

**Lemma (2.1)[9]**: Let \( q \) be univalent in \( U, \zeta \in \mathbb{C}^* \) and suppose that

\[
Re \left\{ 1 + \frac{z \hat{q}(z)}{q(z)} \right\} > \max\left\{ 0, -Re \left( \frac{1}{\zeta} \right) \right\}. \tag{2.1}
\]

If \( p(z) \) is analytic in \( U \), with \( p(0) = q(0) \) and

\[
p(z) + \zeta \hat{p}(z) < q(z) + \zeta \hat{q}(z), \tag{2.2}
\]

then \( p(z) < q(z) \), and \( q(z) \) is the best dominant.

**Lemma (2.2)[17]**: Let the function \( q(z) \) be univalent in the unit disk, and let \( \theta, \varphi \) be analytic in domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set

\[
Q(z) = z \hat{q}(z) \varphi(q(z)) \text{ and } h(z) = \theta(q(z)) + Q(z).
\]

1. \( Q \) is starlike univalent in \( U \).

2. \( Re \left\{ \frac{zh(z)}{Q(z)} \right\} > 0 \) for \( z \in U \).

If \( p \) is analytic with \( p(0) = q(0) \), \( p(U) \subseteq D \) and

\[
\theta(p(z)) + z \hat{p}(z) \varphi(p(z)) < \theta(q(z)) + z \hat{q}(z) \varphi(q(z)), \tag{2.3}
\]

then \( p < q \), and \( q(z) \) is the best dominant.

**Lemma (2.3)[7]**: Let \( q(z) \) be convex in \( U, q(0) = a \) and \( \zeta \in \mathbb{C}, Re(\zeta) > 0 \).

If \( p \in H[a, 1] \) and \( p(z) + \gamma z \hat{q}(z) \) is univalent in \( U \), then

\[
q(z) + \zeta \hat{q}(z) < p(z) + \zeta \hat{p}(z), \tag{2.4}
\]
implies $q(z) < p(z)$, and $q(z)$ is the best subordinant.

**Lemma (2.4)**[8]: Let $q(z)$ be convex univalent in the unit disk $U$ and let $\theta, \varphi$ be analytic in a domain $D$ containing $q(U)$. Suppose that

1- $\text{Re} \left( \frac{\dot{q}(q(z))}{q'(q(z))} \right) > 0$, for $z \in U$.

2- $z\dot{q}(z)\varphi(q(z))$ is starlike univalent in $U$.

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + z\dot{p}(z)\varphi(p(z))$ is univalent in $U$, and

$$\theta(q(z)) + z\dot{q}(z)\varphi(q(z)) < \theta(p(z)) + z\dot{p}(z)\varphi(p(z)),$$

then $q(z) < p(z)$, and $q(z)$ is the best subordinant.

**3- Main results:**

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\Psi > 0, \Phi, T \geq 0; p \in N, q \in N_0 = N \cup \{0\}; z \in U$ and the powers are understood as principle values.

**Theorem (3.1):** Let $q(z)$ be univalent in $U$ with $q(0) = 0, \gamma > 0$ and suppose that

$$\text{Re} \left\{ 1 + \frac{z\dot{q}(z)}{q(z)} \right\} > \max \left\{ 0, -\text{Re} \left( \frac{\alpha(\Phi + p)}{T + \Psi} \right) \right\}. \quad (3.1)$$

If $f \in A(p)$ satisfies the subordination

$$\left( \frac{\mathcal{A}_p^q f(z)}{z^p} \right)^\sigma + \left( \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma \left( \frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right) < q(z) + \frac{T + \Psi}{\alpha(\Phi + p)} z\dot{q}(z), \quad (3.2)$$

then

$$\left( \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < q(z)$$

and $q(z)$ is the best dominant.

**Proof:** If we consider the analytic function

$$\left( \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma, \sigma > 0, z \in U. \quad (3.3)$$

Differentiating (3.3) logarithmically with respect to $z$ and using the identity (1.6) in the resulting equation, we have
\[
\frac{z \dot{p}(z)}{p(z)} = \frac{\sigma(\Phi + p)}{\Psi + T} \left( \frac{\mathcal{A}_{p}^{q+1} f(z)}{\mathcal{A}_{p}^{q+2} f(z)} - 1 \right),
\]

(3.4)

that is

\[
\frac{\Psi + T}{\sigma(\Phi + p)} z \dot{p}(z) = \left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} \left( \frac{\mathcal{A}_{p}^{q+1} f(z)}{\mathcal{A}_{p}^{q} f(z)} - 1 \right).
\]

Thus, the subordination (3.2) is equivalent to

\[
p(z) + \frac{\Psi + T}{\sigma(\Phi + p)} z \dot{p}(z) \prec q(z) + \frac{\Psi + T}{\sigma(\Phi + p)} z \dot{q}(z).
\]

(3.5)

Applying Lemma (2.1), with \( \zeta = \frac{\Psi + T}{\sigma(\Phi + p)} \), the proof of Theorem (3.1) is complete.

Taking the convex function \( q(z) = \frac{1 + Az}{1 + Bz} \), in the Theorem (3.1), we have the following corollary.

**Corollary (3.1):** Let \( A, B \in \mathbb{C}, A \neq B, |B| < 1, \ \Re(\zeta) > 0 \) and \( \sigma > 0 \). If \( f(z) \in A(p) \) satisfies the subordination

\[
\left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} + \left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} \left( \frac{\mathcal{A}_{p}^{q+1} f(z)}{\mathcal{A}_{p}^{q+2} f(z)} - 1 \right) < \frac{1 + Az}{1 + Bz} + \frac{\Psi + T}{\sigma(\Phi + p)} \frac{(A - B)z}{(1 + Bz)^2},
\]

then

\[
\left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} < \frac{1 + Az}{1 + Bz}
\]

and \( \frac{1 + Az}{1 + Bz} \) is the best dominant.

Taking \( q = 0 \) in Theorem (3.1), we obtain the following result:

**Corollary (3.2):** Let \( q(z) \) be univalent in \( U \), with \( q(0) = 1, \sigma > 0 \), and suppose that (3.1) holds. If \( f(z) \in A(p) \) satisfies the subordination

\[
\left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} + \left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} \left( \frac{\mathcal{A}_{p}^{q+1} f(z)}{\mathcal{A}_{p}^{q+2} f(z)} - 1 \right) < q(z) + \frac{\Psi + T}{\sigma(\Psi + p)} z \dot{q}(z),
\]

then

\[
\left( \frac{\mathcal{A}_{p}^{q+2} f(z)}{z^p} \right)^{\sigma} < q(z).
\]

and \( q(z) \) is the best dominant.
Taking $\Phi = \Psi = 1$ in the Theorem (3.1), we have the following result.

**Corollary (3.3):** Let $q(z)$ be univalent in $U$, with $q(0) = 1$, $\beta \in \mathbb{C}^*$, $\sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination
\[
\left(\frac{A_p^q f(z)}{z^p}\right)^\sigma + \left(\frac{A_p^{q+2} f(z)}{z^p}\right)^\sigma \left(\frac{A_p^{q+1} f(z)}{\sigma A_p^{q+2} f(z)} - 1\right) < q(z) + \frac{1 + T}{\sigma(1 + p)} z\dot{q}(z),
\]
then
\[
\left(\frac{A_p^{q+2} f(z)}{z^p}\right)^\sigma < q(z).
\]
and $q(z)$ is the best dominant.

**Theorem (3.2):** Let $q(z)$ be univalent in $U$, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, let $\lambda, \sigma \in \mathbb{C}^*$, $f \in A(p)$ and suppose that $f$ and $q$ satisfy the next conditions:
\[
\frac{A_p^q f(z)}{z^p} \neq 0, \tag{3.6}
\]
and
\[
\Re \left\{1 + \frac{z\dot{q}(z)}{\dot{q}(z)} - \frac{z\dot{q}(z)}{q(z)}\right\} > 0, \quad (z \in U). \tag{3.7}
\]
If
\[
\frac{A_p^{q+1} f(z)}{A_p^{q+2} f(z)} < 1 + \frac{(\Psi + T)zq(z)}{\sigma(\Phi + p)q(z)}, \tag{3.8}
\]
then
\[
\left(\frac{A_p^{q+2} f(z)}{z^p}\right)^\sigma < q(z)
\]
and $q(z)$ is the best dominant of (3.6).

**Proof:** Let
\[
p(z) = \left(\frac{\sigma A_p^{q+2} f(z)}{z^p}\right)^\sigma, \quad z \in U. \tag{3.9}
\]
According to (3.4) the function $p(z)$ is analytic in $U$, and differentiating (3.9) logarithmically with respect to $z$, we obtain
\[
\frac{z\dot{p}(z)}{p(z)} = \frac{\sigma(\Phi + p)}{\Psi + T} \left( \frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right).
\] (3.10)

In order to prove our result we will use Lemma (2.2). In this lemma consider

\[ \theta(w) = 1 \text{ and } \varphi(w) = \frac{\Psi + T}{\sigma(\Phi + p)}w, \]

then \( \theta \) is analytic in \( \mathbb{C} \) and \( \varphi(w) \neq 0 \) is analytic in \( \mathbb{C}^* \). Also if we let

\[ Q(z) = z\dot{q}(z)\varphi(q(z)) = \frac{(\Psi + T)z\dot{q}(z)}{\sigma(\Phi + p)q(z)}, \]

and

\[ h(z) = \theta(q(z)) + Q(z) = 1 + \frac{(\Psi + T)z\dot{q}(z)}{\sigma(\Phi + p)q(z)}, \]

from (3.7), we see that \( Q(z) \) is a starlike function in \( U \). We also have

\[ \text{Re} \left( \frac{z\dot{h}(z)}{Q(z)} \right) = \text{Re} \left( 1 + \frac{z\dot{q}(z)}{q(z)} \right) > 0, \quad (z \in U) \]

and then, by using Lemma (2.2), we deduce that the subordination (3.6) implies

\[ p(z) < q(z) \]

and the function \( q(z) \) is the best dominant of (3.8).

Taking \( q(z) = \frac{1 + Az}{1 + Bz} \) \((-1 \leq B < A \leq 1)\) in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

**Corollary (3.4):** Let \( \sigma \in \mathbb{C}^* \). Let \( f(z) \in A(p) \) and suppose that

\[ \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \quad (z \in U). \]

If

\[ \frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} < 1 + \frac{(\Psi + T)z(A - B)}{\sigma(\Phi + p)(1 + Az)(1 + Bz)}, \]

then

\[ \left( \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^{\sigma} < \frac{1 + Az}{1 + Bz} \]

and \( q(z) = \frac{1 + Az}{1 + Bz} \) is the best dominant.
Taking \( q(z) = \frac{1+z}{1-z} \) in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

**Corollary (3.5):** Let \( \sigma \in \mathbb{C}^* \), \( f(z) \in A(p) \) and suppose that

\[
\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \quad (z \in U).
\]

If

\[
\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} < 1 + \frac{2(\Psi + T)z}{\sigma(\Phi + p)(1 - z)(1 + z)},
\]

then

\[
\left( \frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < \frac{1 + z}{1 - z}
\]

and \( q(z) = \frac{1+z}{1-z} \) is the best dominant.

**Theorem (3.3):** Let \( q(z) \) be univalent in \( U \), with \( q(0) = 1 \), let \( \sigma \in \mathbb{C}^* \), and let \( \psi, \nu, \eta \in \mathbb{C} \) with \( \nu + \eta \neq 0 \). Let \( f \in A(p) \) and suppose that \( f \) and \( q \) satisfy the next conditions:

\[
\frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \neq 0, \quad (z \in U) \tag{3.11}
\]

and

\[
Re \left\{ 1 + \frac{z \dot{q}(z)}{q(z)} \right\} > \max\{0, -Re(\psi)\}, \quad (z \in U). \tag{3.12}
\]

If

\[
K(z) = Y \left[ \frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma + \sigma \left[ \frac{\nu z (\mathcal{A}_p^{q+1} f(z)) + \eta z (\mathcal{A}_p^{q+2} f(z))}{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)} - p \right] \tag{3.13}
\]

and

\[
K(z) < Y q(z) + \frac{z \dot{q}(z)}{q(z)}, \tag{3.14}
\]

then

\[
\left[ \frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma < q(z)
\]
and \( q(z) \) is the best dominant of (3.11).

**Proof**: Let

\[
p(z) = \left[ \frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma, \quad z \in U.
\] (3.15)

According to (3.8) the function \( p(z) \) is analytic in \( U \), and differentiating (3.15) logarithmically with respect to \( z \), we obtain

\[
\frac{zp'(z)}{p(z)} = \sigma \left[ \frac{\nu z(A_p^{q+1} f(z)) + \eta z(A_p^{q+2} f(z))}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right],
\] (3.16)

and hence

\[
zp'(z) = \sigma \left[ \frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma \left[ \frac{\nu z(A_p^{q+1} f(z)) + \eta z(A_p^{q+2} f(z))}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right].
\]

In order to prove our result, we will use Lemma (2.2). In this lemma consider

\[
\theta(w) = Yw \text{ and } \varphi(w) = \frac{1}{w},
\]

then \( \theta \) is analytic in \( C \) and \( \varphi(w) \neq 0 \) is analytic in \( C^* \). Also if we let

\[
Q(z) = zq(z)\varphi(q(z)) = \sigma \left[ \frac{\nu z(A_p^{q+1} f(z)) + \eta z(A_p^{q+2} f(z))}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right]
\]

and

\[
h(z) = \theta(q(z)) + Q(z)
\]

\[
\begin{align*}
&= Y \left[ \frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma + \sigma \left[ \frac{\nu z(A_p^{q+1} f(z)) + \eta z(A_p^{q+2} f(z))}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right],
\end{align*}
\]

from (3.11), we see that \( Q(z) \) is a starlike function in \( U \). We also have

\[
Re \left( \frac{zh(z)}{Q(z)} \right) = Re \left( Y + 1 + \frac{zq(z)}{\varphi(z)} \right) > 0, \quad (z \in U)
\]

and then, by using Lemma (2.2), we deduce that the subordination (3.14) implies

\[
p(z) < q(z).
\]

Taking \( q(z) = \frac{1+A_z}{1+Bz} \) \((-1 \leq B < A \leq 1) \) in Theorem (3.3) and according to(3.4), the condition (3.12) becomes
\[
\max\{0, -\Re(Y)\} \leq \frac{1 - |B|}{1 + |B|}.
\]

Hence, for the special case \( \nu = 1 \) and \( \eta = 0 \), we obtain the following result.

**Corollary (3.6)**: Let \( Y \in \mathcal{C} \) with
\[
\max\{0, -\Re(Y)\} \leq \frac{1 - |B|}{1 + |B|}.
\]

Let \( f(z) \in A(p) \) and suppose that
\[
\frac{A^{q+1}_p f(z)}{z^p} \neq 0, \quad (z \in U).
\]

If
\[
Y \left[ \frac{\nu A^{q+1}_p f(z)}{z^p} \right] + \sigma \left[ \left( \frac{z(A^{q+1}_p f(z))}{A^{q+1}_p f(z)} - p \right) \right] < Y \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Az)(1 + Bz)},
\]
then
\[
\left( \frac{A^{q+1}_p f(z)}{z^p} \right)^Y < \frac{1 + Az}{1 + Bz}
\]
and \( q(z) = \frac{1 + Az}{1 + Bz} \) is the best dominant.

Taking \( p = \nu = q = 1, \eta = 0 \) and \( q(z) = \frac{1 + z}{1 - z} \) in Theorem (3.3), we obtain the next result.

**Corollary (3.7)**: Let \( f(z) \in A(p) \) and suppose that
\[
\frac{A^2_pf(z)}{z^p} \neq 0, \quad (z \in U).
\]

and \( \sigma \in \mathcal{C}^* \). If
\[
Y \left[ \frac{A^2f(z)}{z} \right] + \sigma \left[ \left( \frac{z(A^2f(z))}{A^2f(z)} - 1 \right) \right] < Y \frac{1 + z}{1 - z} + \frac{2z}{(1 + z)(1 - z)},
\]
then
\[
\left( \frac{A^2f(z)}{z} \right)^Y < \frac{1 + z}{1 - z}
\]
and \( q(z) = \frac{1 + z}{1 - z} \) is the best dominant.
References:


