On Symmetric Higher (U,R)-n-Derivation of Prime Rings

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Recommended Citation
DOI: 10.29350/2411-3514.1194
Available at: https://qjps.researchcommons.org/home/vol25/iss2/5

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On Symmetric Higher (U, \mathcal{R})-n-Derivation of Prime Rings

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Article History
Received on: 31/1/2020
Revised on: 26/2/2020
Accepted on: 29/3/2020

Keywords:
Prime ring, semiprime
\mathcal{R} n-derivation
\langle U, \mathcal{R} \rangle n-derivation
\langle U, \mathcal{R} \rangle n-derivation
higher derivation

DOI: https://doi.org/10.29350/jops.2020.25.2.1056

ABSTRACT
The main aim of this paper is to generalize Awtar’s Theorem concerning derivation on Lie ideal to symmetric higher \langle U, \mathcal{R} \rangle n-derivation by introducing many concepts have played an important role in obtaining this result such as \langle U, \mathcal{R} \rangle n-derivation, Jordan \langle U, \mathcal{R} \rangle n-derivation and highern-derivation.

MSC: 30C45, 30C50

1. Introduction
Throughout, \mathcal{R} will represent an associative ring with center Z(\mathcal{R}). A ring \mathcal{R} is said to be n-torsion free where n is nonzero integer if na = 0 with \forall a \in \mathcal{R} implies that a = 0. \mathcal{R} is said to be prime ring if \forall a \in \mathcal{R} b=0 implies that either a = 0 or b = 0 for all a, b \in \mathcal{R}, and \mathcal{R} is semiprime if \forall a \in \mathcal{R} a = 0 then a = 0 [8].

For all u, v \in \mathcal{R}, the commutator of u and v is [u, v] = ur - rv. An additive subgroup U of \mathcal{R} is said to be a Lie ideal of \mathcal{R} if [u, v] \in U for all u \in U, v \in \mathcal{R} [9]. A mapping d: \mathcal{R} \rightarrow \mathcal{R} is a derivation of \mathcal{R} if d is an additive mapping that satisfies the following d(uv) = d(u)v + ud(v) for all u, v \in \mathcal{R} and d is said to be Jordan derivation if u = r and this means d(u^2) = d(u^2) + u d(u) [12]. In general every derivation is a Jordan derivation, but the converse is not true. In [4] Herstein proved that every Jordan derivation of a prime ring \mathcal{R} of char ≠ 2 is a derivation. Awtar in [2] generalized Herstein’s result to Lie ideal, he proved that in a prime ring \mathcal{R} of char ≠ 2, every Jordan derivation d_u: U \rightarrow \mathcal{R} is a derivation, where U is a Lie ideal of \mathcal{R} such that u^2 \in U for all u \in U. A family of additive mappings D = (d_n)_{n \in \mathbb{N}} of \mathcal{R} into itself where d_0 = id_{\mathcal{R}} is called a higher derivation (resp. Jordan higher derivation), if d_n(uv) = \sum_{i+j=n} d_i(u) d_j(v) for all u, v \in \mathcal{R}, n \in \mathbb{N}, where N is the set of natural number [6].

According to [5], Ferrero and Haetinger were extended Herstein’s result to higher derivations, they proved that every Jordan higher derivation of 2-torsion free semiprime ring is a higher derivation. In [7] Haetinger extended Awtar’s theorem to higher derivation on Lie ideal. Further, Faraj, Haetinger and Majeed were expanded this result to higher \langle U, \mathcal{R} \rangle -derivation [6]. In this paper, new generalization...
of Herstein’s theorem to symmetric higher \((U, R)\)-\(n\)-derivation will introduce as different kind from previous generalizations, where \(U\) is a Lie ideal of \(R\) by introducing many concepts that play role to arrive main goal. Throughout this paper \(d_0\) is defined to be the identity mapping \(id_R: R \to R\).

2. Symmetric \((U, R)\)-\(n\)-Derivation

The concept of symmetric \((U, R)\)-\(n\)-Derivation is introduced and studied throughout this section.

We begin with following definition:

**Definition (2.1) [10]:**

A mapping \(d: R \times \ldots \times R \to R\) is called symmetric if the equation
\[d(u_1, u_2, \ldots, u_n) = d(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)})\]
holds, for all \(u_i \in R\) and for every permutation \(\{\pi(1), \pi(2), \ldots, \pi(n)\}\).

The concept of \((U, R) - n -derivation as a generalization of the concept \((U, R) - derivation can be introduced as follows

**Definition (2.2):**

Let \(U\) be a Lie ideal of \(R\). An \(n\)-additive mapping \(d: R \times R \times \ldots \times R \to R\) is said to be \((U, R)\)-\(n\)-derivation of \(R\), if the following equations are equivalent for all \(u, u_2, \ldots, u_n \in U, r, s \in R\)
\[d(u_1 r + su_1, u_2, \ldots, u_n) = d(u_1 u_2 \ldots u_n) r + u_1 d(r u_2 \ldots u_n) + d(s, u_2 \ldots u_n) u_1 + s d(u_1, u_2 \ldots u_n)\]
\[\ldots\]
\[d(u_1 u_2 \ldots u_n r + su_n) = d(u_1 u_2 \ldots u_n r + u_n d(u_1 u_2 \ldots r) + d(u_1 u_2 \ldots s) u_n + s d(u_1 u_2 \ldots u_n)\]

Moreover, \(d\) is said to be a symmetric \((U, R)\)-\(n\)-derivation if the above equations are equivalent to each other.

Further, if \(R\) is 2-torsion free and \(r = s = u\), the above equations can be reduced to
\[d(u^2, u_2, \ldots u_n) = d(u, u_2, \ldots u_n u) = d(u, u_2, \ldots u_n) + u d(u, u_2, \ldots u_n)\]

Recall that a Lie ideal \(U\) of a ring is called a square closed Lie ideal if \(u^2 \in U\) for all \(u \in U\) and \(U\) is called an admissible Lie ideal of \(R\) if \(U\) is no central closed Lie ideal [6]

**Lemma (2.3):[6]**

Let \(R\) be a prime ring of char \(\neq 2\) and \(U\) be an admissible Lie ideal of \(R\) such that \(tw^2 + w^2 t = 0\) for all \(w \in U, t \in R\). Then \(t = 0\).

**Remark (2.4):**

Let \(D = \{d_n\} \in N\) be a family of additive symmetric mappings of \(R \times R \times \ldots \times R\). We set \(\phi_n(u, u_2, \ldots, u_n)\) for
\[d_n(u, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n)d_j(r, u_2, \ldots, u_n)\] for all \(u, u_2, \ldots, u_n \in U, r \in R, n \in N\).

**Lemma (2.5):** Let \(U\) be a Lie ideal of 2-torsion free ring and \(d: R \times R \times \ldots \times R \to R\) be a symmetric \((U, R)\)-\(n\)-derivation. Then
\[d(u_r u, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n) r u + u d(r, u_2, \ldots, u_n) u + u d(u_2, \ldots, u_n)\] for all \(u, u_2, \ldots, u_n \in U, r \in R\).
**Proof:** Substituting \( r = s = (2u)r + r(2u) \) in the Definition (2.2), then \( H = d(2u)(2u)r + r(2u)(2u) + ru, u_2, \ldots, u_n) \)

\[
2(du, u_2, \ldots, u_n)ru + ud(u, u_2, \ldots, u_n)r + u^2d(r, u_2, \ldots, u_n)u + 
urdu, u_2, \ldots, u_n + d(u, u_2, \ldots, u_n)ru + u d(r, u_2, \ldots, u_n)u + d(r, u_2, \ldots, u_n)u^2 + rd(u_2, \ldots, u_n)u + 
urdu, u_2, \ldots, u_n)ru + ur(u, u_2, \ldots, u_n)
\]

On the other hand, \( H = 2d(du, u_2, \ldots, u_n)ru + 2ud(u, u_2, \ldots, u_n)r + 2d(r, u_2, \ldots, u_n)u^2 + 2rd(u_2, \ldots, u_n)u + 2du, u_2, \ldots, u_n + 4d(r, u_2, \ldots, u_n). \) Compare the both sides of \( H \) and since \( R \) is 2-torsion free, then we get the required result.

The following corollary is immediate from Lemma (2.9).

**Corollary (2.6):**

Let \( U, R \) be a Lie ideal of 2-torsion free ring and \( d : R \times R \times \ldots \times R \to R \) be a symmetric \((U, R)\)-\( n \)-derivation of \( R \). Then for all \( w, u, u_2, \ldots, u_n \in U, r, s \in R \) Then \( d(uw + wr, u_2, \ldots, u_n) \)

\[
d(u, u_2, \ldots, u_n)ru + ud(r, u_2, \ldots, u_n)w + urd(w, u_2, \ldots, u_n) \\
+ d(w, u_2, \ldots, u_n)su + wrd(s, u_2, \ldots, u_n)u + wsd(u_2, \ldots, u_n)
\]

**Lemma (2.7):**

Let \( R \) be a prime ring of \( \text{char}(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d : R \times R \times \ldots \times R \to R \) be a symmetric mapping such that \( d(u_2, u_3, \ldots, u_n) = d(u, u_2, \ldots, u_n)u + ud(u, u_2, \ldots, u_n) \) for all \( u, u_2, \ldots, u_n \in U \) Then \( \phi(uw, u_2, \ldots, u_n)[u, v] = 0 \) for all \( u, v, u_2, \ldots, u_n \in U \).

**Proof:** Since \( U \) be an admissible Lie ideal of \( R \), then \( 2uv \in U \) for all \( u, v \in U \). So by Definition (2.2), \( d((uv)^2, u_2, \ldots, u_n) = d(uv, u_2, \ldots, u_n)uv + ud(0u_2, \ldots, u_n) 

Replace \( w \) by \( 2uv \) in Corollary (2.6) to get

\[
H = 2d(uw(0u) + (0u)w, u_2, \ldots, u_n) = 2d(u, u_2, \ldots, u_n)w + ud(v, u_2, \ldots, u_n)v + 0ud(u, u_2, \ldots, u_n) + uvd(v, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n)v
\]

But \( H = 2d(uw(0u) + (0u)w, u_2, \ldots, u_n) = 2d((uv)^2 + uv^2, u_2, \ldots, u_n) 

\[
2d((uv)^2 + uv^2, u_2, \ldots, u_n) + ud(0u_2, \ldots, u_n) + ud(u_2, \ldots, u_n)v + ud(0u_2, \ldots, u_n) + ud(u_2, \ldots, u_n)v + ud(u_2, \ldots, u_n)v + ud(u_2, \ldots, u_n)v + ud(u_2, \ldots, u_n)v
\]

Comparing both sides of \( H \) and since \( \text{char} (R) \neq 2 \), then \( \phi(uw, u_2, \ldots, u_n)[u, v] = 0 \) for all \( u, v, u_2, \ldots, u_n \in U \).

The proof of following lemma is similar to the proof of Lemma (2.7):

**Lemma (2.8):**

Let \( R \) be a prime ring of \( \text{char}(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d : R \times R \times \ldots \times R \to R \) be a symmetric mapping such that \( d(u_2, u_3, \ldots, u_n) = d(u, u_2, \ldots, u_n)u + ud(u, u_2, \ldots, u_n) \) for all \( u, u_2, \ldots, u_n \in U \). Then \( [u, v] \phi(uw, u_2, \ldots, u_n) = 0 \) for all \( u, v, u_2, \ldots, u_n \in U \).

**Lemma (2.9):**

Let \( R \) be a prime ring of \( \text{char}(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d : R \times R \times \ldots \times R \to R \) be a symmetric mapping such that \( d(u_2, u_3, \ldots, u_n) \)
\[ d(u, u_2, \ldots, u_n)u + ud(u, u_2, \ldots, u_n) \text{ for all } u, u_2, \ldots, u_n \in U \text{ and for all } u \in C(U) = \{ x \in R : [x, U] = 0 \}. \text{ Then } d(u, u_2, \ldots, u_n) \in Z(R) \text{ for all } u, u_2, \ldots, u_n \in U. \]

**Proof:** Since \( U \in Z(R) \), then [4, Lemma 2] implies that \( C(U) = Z(R) \) which gives \( u \in Z(R) \). In view of Definition (2.2) we have

\[ d(2uv, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n) v + vd(u, u_2, \ldots, u_n) + 2udv, u_2, \ldots, u_n. \] ...

(1)

Put \( v = wv + wv \) in equation (1), where \( w \in U \), then

\[ d(2u(vw + wv), u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n) (vw + wv) + (vw + wv) d(u, u_2, \ldots, u_n) + 2ud(vw + wv, u_2, \ldots, u_n) \] ...

(2)

Since \( u \in Z(R) \) and by Corollary (2.6) we obtain

\[ d(2u(wv + wv), u_2, \ldots, u_n) = 2d(u, u_2, \ldots, u_n)vw + d(u, u_2, \ldots, u_n) vw + 2u(d(v, u_2, \ldots, u_n)w + v d(w, u_2, \ldots, u_n)) \] ...

(3)

Comparing equations (2) and (3) in this case we have

\[ d(u, u_2, \ldots, u_n)(vw - wv) = (vw - wv)d(u, u_2, \ldots, u_n) \] and this means that \( d(u, u_2, \ldots, u_n)[U, U] = 0 \) for all \( u, u_2, \ldots, u_n \in U \). Hence, \( d(u, u_2, \ldots, u_n) \in C[U, U] = C(U) \) by [4, Lemma 3]. But, as above, \( C(U) = Z(R) \). Therefore, \( d(u, u_2, \ldots, u_n) \in Z(R) \) for all \( u, u_2, \ldots, u_n \in U \).

**Lemma (2.10):**

Let \( R \) be a prime ring of \( \text{char}(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d : R \times R \times \ldots \times R \rightarrow R \) be a symmetric mapping such that \( d(u^2, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n)u + ud(u, u_2, \ldots, u_n) \) for all \( u, u_2, \ldots, u_n \in U \) and \( uv = vu \) for some \( u \in U \). Then \( \phi_1(uw, u_2, \ldots, u_n) = 0 \), for all \( u, v, u_2, \ldots, u_n \in U \).

**Proof:** As an application of Corollary (2.6), we have

\[ d(uv, wu_2, \ldots, u_n) = d(u, u_2, \ldots, u_n)vw + ud(v, u_2, \ldots, u_n)w + uvwd(w, u_2, \ldots, u_n) + d(w, u_2, \ldots, u_n)v + wd(v, u_2, \ldots, u_n)u + wvud(u, u_2, \ldots, u_n). \] ...

(1)

Since \( uv = vu \) and by applying Definition (2.2), we get

\[ d(uv, wu_2, \ldots, u_n) = d(uv, u_2, \ldots, u_n)w + uvwd(w, u_2, \ldots, u_n) + d(w, u_2, \ldots, u_n)v + wd(uv, u_2, \ldots, u_n). \] ...

(2)

Subtracting equations (1) and (2) to get

\[ (d(uv, u_2, \ldots, u_n) - d(u^2, u_2, \ldots, u_n) - uud(v, u_2, \ldots, u_n)w + ud(v, u_2, \ldots, u_n)w - vud(u, u_2, \ldots, u_n)w - vud(u, u_2, \ldots, u_n)w) = 0 \]

and this means

\[ \phi_1(uw, u_2, \ldots, u_n)w + \phi_1(vu, u_2, \ldots, u_n)w = 0. \] By Remark (2.4), \( \phi_1(vu, u_2, \ldots, u_n)w = - \phi_1(vu, u_2, \ldots, u_n)w \) for all \( u, v, u_2, \ldots, u_n \in U \). Hence \( \phi_1(uw, u_2, \ldots, u_n)w = 0 \) for all \( u, v, u_2, \ldots, u_n \in U \). By Lemma 2 and Lemma 3 of [4], we have \( \phi_1(uw, u_2, \ldots, u_n) \in Z(R) \) for all \( u, v, u_2, \ldots, u_n \in U \). Since \( u^2 \in U \) for all \( u \in U \) and \( u^2v = vu^2 \).

So, \( d(u^2v, u_2, \ldots, u_n) - d(u^2, u_2, \ldots, u_n)v - uud(v, u_2, \ldots, u_n)w + ud(v, u_2, \ldots, u_n)w - vud(u, u_2, \ldots, u_n)w - vud(u, u_2, \ldots, u_n)w = 0 \)

\[ -uud(u, u_2, \ldots, u_n)v - uud(v, u_2, \ldots, u_n)w + ud(v, u_2, \ldots, u_n)w - \phi_1(uw, u_2, \ldots, u_n)w = 0. \]

Since \( uv = vu \) and \( 2uv \in U \), then \( (2uv)w = u(2uv) \), so we get

\[ d(u(2uv), u_2, \ldots, u_n) - 2d(u, u_2, \ldots, u_n)w - 2ud(u, u_2, \ldots, u_n)w - ud(u, u_2, \ldots, u_n)w = 0. \]

Since \( \phi_1(uw, u_2, \ldots, u_n)w = 0 \) for all \( u, v, u_2, \ldots, u_n \in U \), then \( (2uv)w = u(2uv) \), so we get

\[ d(u(2uv), u_2, \ldots, u_n) - 2d(u, u_2, \ldots, u_n)w - ud(u, u_2, \ldots, u_n)w = 0. \]

Comparing equations (3) and (4) and \( \text{char}(R) \neq 2 \) yields \( u\phi_1(uw, u_2, \ldots, u_n) \in Z(R) \) for all \( u, v, u_2, \ldots, u_n \in U \).
If \( u \phi_1(u, v, u_2, \ldots, u_n) \neq 0 \), since \( R \) is prime and \( \phi_1(u, v, u_2, \ldots, u_n) \in Z(R) \), then we can conclude that \( u \in Z(R) \) and in view of Lemma (2.9), \( d(u, u_2, \ldots, u_n) \in Z(R) \). Hence, \( 2d(u, v, u_2, \ldots, u_n) = d(u + vu, u_2, \ldots, u_n) \)
\[ \begin{align*}
&= 2(d(u, u_2, \ldots, u_n)v + vd(u, u_2, \ldots, u_n)). \\
&\text{Since char} \neq 2, \text{the last equation becomes} \\
d(u, u_2, \ldots, u_n)v + vd(u, u_2, \ldots, u_n) \quad \text{for all} \quad u, u_2, \ldots, u_n \in U. \\
&\text{That is} \quad \phi_1(u, v, u_2, \ldots, u_n) = 0 \quad \text{for all} \quad u, v, u_2, \ldots, u_n \in U \\
&\text{and this is a contradiction. Therefore,} \quad \phi_1(u, v, u_2, \ldots, u_n) = 0 \quad \text{for all} \quad u, v, u_2, \ldots, u_n \in U. \\

\textbf{Theorem (2.15):}

Let \( R \) be a prime ring of \( \text{char} \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d: R \times R \times \ldots \times R \rightarrow R \) be a symmetric mapping such that \( d(u^2, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n)u + ud(u, u_2, \ldots, u_n) \) for all \( u, u_2, \ldots, u_n \in U \). Then \( d(u, v, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n)v + ud(v, u_2, \ldots, u_n) \) for all \( u, v, u_2, \ldots, u_n \in U \).

\textbf{Proof:} Linearizing Lemmas (2.7) and (2.8) on \( v \), then
\[ \phi_1(u, v, u_2, \ldots, u_n)[u, v] = -\phi_1(u, u_2, \ldots, u_n)[u, v] \quad \text{for all} \quad u, w, u_2, \ldots, u_n \in U. \] ... (1)

Also, \( [u, w]\phi_1(u, v, u_2, \ldots, u_n) = -[u, v]\phi_1(u, u_2, \ldots, u_n) \quad \text{for all} \quad u, w, u_2, \ldots, u_n \in U. \] ... (2)

Multiply the left hand side of equation (1) by \( [u, w_1] \) and using equations (2) and (1), then
\[ [u, v]\phi_1(u, w, u_2, \ldots, u_n)[u, w_1] = -[u, w_1]\phi_1(u, u_2, \ldots, u_n)[u, v][u, v]. \] ... (3)

Replace \( w_1 \) by \( 2w_1v_1 \) in Equation (3) to get
\[ [u, v]\phi_1(u, w, u_2, \ldots, u_n)[u, w_1]v_1 + [u, w_1]v_1 \phi_1(u, w, u_2, \ldots, u_n)[u, v] = -[u, v]\phi_1(u, u_2, \ldots, u_n)w_1 [v, v_1] = -[u, v]\phi_1(u, u_2, \ldots, u_n)[u, r]. \] ... (4)

Now applying equations (1) and (2) to equation (3), it is easily observed that
\[ [u, v]\phi_1(u, w, u_2, \ldots, u_n)[u, w_1] = [u, w_1] \phi_1(u, w, u_2, \ldots, u_n)[u, w], \quad [u, w]\phi_1(u, w, u_2, \ldots, u_n)[u, w_1] = [u, w_1] \phi_1(u, w, u_2, \ldots, u_n)[u, v]. \]
\[ \phi_1(u, w, u_2, \ldots, u_n)[u, v] = \phi_1(u, w, u_2, \ldots, u_n)[u, v]. \]

And using these in equation (4), next by using equations (1) and (2) we have \( [u, w_1] \phi_1(u, v, u_2, \ldots, u_n)[u, w]v_1 - v_1 \phi_1(u, u_2, \ldots, u_n)[u, v] \)
\[ = -([u, r]\phi_1(u, u_2, \ldots, u_n)w_1 - w_1[u, w]\phi_1(u, u_2, \ldots, u_n)[u, r_1]). \]

Or, \( [u, w_1]\phi_1(u, u_2, \ldots, u_n)[u, w], v_1 = -[[u, v]\phi_1(u, u_2, \ldots, u_n), w_1] [v, v_1]. \) ... (5)

Putting \( v_1 = 2v_1u_1 \) in equation (5) and use it, then \( [u, w_1]\phi_1(u, u_2, \ldots, u_n)[u, w], v_1 = -[[u, v]\phi_1(u, u_2, \ldots, u_n), w_1] [v, v_1]. \) ... (6)

As proof of [2, Theorem] one can complete the proof to get the required result. Hence, \( \phi_1(u, v, u_2, \ldots, u_n) = 0 \) for all \( u, v, u_2, \ldots, u_n \in U. \)

\textbf{Lemma (2.16):}

Let \( R \) be a prime ring of \( \text{char}(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( d: R \times R \times \ldots \times R \rightarrow R \) be a symmetric \((U, R)\)-n-derivation. Then \( \phi_1(u^2r, u_2, \ldots, u_n) = 0 \) for all \( u, u_2, \ldots, u_n \in U, r \in R. \)

\textbf{Proof:} By Theorem (2.15), \( \phi_1(u, v, u_2, \ldots, u_n) = 0 \) for all \( u, v, u_2, \ldots, u_n \in U. \) Then
\[ 0 = \phi_1(u(u^2r - ruu), u_2, \ldots, u_n) = d(u^2r - ruu, u_2, \ldots, u_n) - d(u, u_2, \ldots, u_n)(u^2r - ruu) \]
\[ = d(u^2r, u_2, \ldots, u_n) - d(u, u_2, \ldots, u_n)(ur - ru) - d((ur - ru), u_2, \ldots, u_n)u \]
\[ = d(u^2r, u_2, \ldots, u_n) - (d(u, u_2, \ldots, u_n)ur + ud(u_2, \ldots, u_n)r + u^2d(r, u_2, \ldots, u_n)) = \phi_1(u^2r, u_2, \ldots, u_n). \]

The following result is a generalization of [2, Theorem] to \((U, R)\)-n-derivation.

\textbf{Theorem (2.17):}
Let $\mathcal{R}$ be a prime ring of $\text{char}(\mathcal{R}) \neq 2$, $U$ be a Lie ideal of $\mathcal{R}$ and $d: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ be a symmetric $(U, \mathcal{R})$-n-derivation. Then $\phi_1(u_1, u_2, \ldots, u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in U$, $r \in \mathcal{R}$.

**Proof:** Since $d$ is a symmetric $(U, \mathcal{R})$-n-derivation, then replacing $r$ by $ur$ in Definition (2.2) yields

$$H = d(ur + uru, u_2, \ldots, u_n) = d(u, u_2, \ldots, u_n)ur + ud(ur, u_2, \ldots, u_n) + d(ur, u_2, \ldots, u_n)u + urd(u, u_2, \ldots, u_n)$$

On the other hand, $H = d(u^2r, u_2, \ldots, u_n) + d(uru, u_2, \ldots, u_n)$

By Lemma (2, 16) and Lemma (2, 5), the last equation can be reduced to

$$H = d(u, u_2, \ldots, u_n)ur + ud(ur, u_2, \ldots, u_n)r + u^2d(r, u_2, \ldots, u_n)$$

$$+ d(u, u_2, \ldots, u_n)r + ud(r, u_2, \ldots, u_n)u + urd(u, u_2, \ldots, u_n)$$

Comparing both sides of $H$ implies that

$$u \phi_1(u_1, u_2, \ldots, u_n) + \phi_1(u_1, u_2, \ldots, u_n)u = 0 \text{ for all } u_1, u_2, \ldots, u_n \in U, r \in \mathcal{R}. \quad (1)$$

Substituting $u = u + w$ where $w \in U$ in the equation (1) satisfies

$$u \phi_1(u_1, u_2, \ldots, u_n) + w \phi_1(u_1, u_2, \ldots, u_n) + \phi_1(u_1, u_2, \ldots, u_n)w + \phi_1(\omega, u_2, \ldots, u_n)u = 0. \quad (2)$$

Put $w = w^2$ in equation (2), we get

$$u \phi_1(w^2r, u_2, \ldots, u_n) + w^2 \phi_1(u_1, u_2, \ldots, u_n) + \phi_1(u_1, u_2, \ldots, u_n)w^2 + \phi_1(w^2r, u_2, \ldots, u_n)u = 0.$$

As application of Lemma (2.16) to the last equation implies that

$$w^2 \phi_1(u_1, u_2, \ldots, u_n) + \phi_1(u_1, u_2, \ldots, u_n)w^2 = 0 \text{ for all } u, w, u_2, \ldots, u_n \in U, r \in \mathcal{R}.$$ Now we have two cases:

Case 1: If $U \not\subset Z(\mathcal{R})$, by Lemma (2.3), the last equation gives

$$\phi_1(u_1, u_2, \ldots, u_n) = 0 \text{ for all } u_1, u_2, \ldots, u_n \in U, r \in \mathcal{R}.$$

Case 2: If $U \subset Z(\mathcal{R})$ and since $\text{char}(\mathcal{R}) \neq 2$, then $w^2 \phi_1(u_1, u_2, \ldots, u_n) = 0$ for all $u, w, u_2, \ldots, u_n \in U, r \in \mathcal{R}$ and this implies that $0 = c w^2 \phi_1(u_1, u_2, \ldots, u_n) = w^2 c \phi_1(u_1, u_2, \ldots, u_n) = 0$ the primness of $\mathcal{R}$ implies that either $w^2 = 0$ or $\phi_1(u_1, u_2, \ldots, u_n) = 0$. Since $U \neq 0$, hence $\phi_1(u_1, u_2, \ldots, u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in U, r \in \mathcal{R}$.

### 3. Symmetric Higher $(U, \mathcal{R})$-n-Derivation

The concept of symmetric higher $(U, \mathcal{R})$-n-Derivation is introduced and studied to extend the results of section 1.

Motivated by Definition (2.2) and the definition of higher derivation, the concept of symmetric higher $(U, \mathcal{R})$-n-derivation can be defined as follows

**Definition (3.1):**

Let $U$ be a Lie ideal of a ring $\mathcal{R}$. A family of additive mappings $D = (d_n)_{n \in N}$ of $\mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$ into $\mathcal{R}$ is said to be a symmetric higher $(U, \mathcal{R})$-n-derivation if for all $u, u_2, \ldots, u_n \in U, r, s \in \mathcal{R}, n \in N$, the following equations are equivalent

$$d_n(u_1, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u_1, u_2, \ldots, u_n)d_j(r, u_2, \ldots, u_n) + d_i(s, u_2, \ldots, u_n)d_j(u_1, u_2, \ldots, u_n)$$

.$$.$$.$$.
\[ d_n(u_1, u_2, \ldots, u_n r + s u_n) = \sum_{i+j=n} d_i(u_1, u_2, \ldots, u_n) d_j(u_1, u_2, \ldots, r) + d_i(u_1, u_2, \ldots, s) d_j(u_1, u_2, \ldots, u_n) \]

Further, if \( \mathcal{R} \) is a 2-torsion free and \( r = s = u_n \) for all \( n \in \mathbb{N} \) then
\[ d_n(u^2, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \text{ for all } u, u_2, \ldots, u_n \in U. \] \hfill \ldots (\ast)

The following example explains the concept of symmetric higher \((U, \mathcal{R})\)-\textit{n}-derivation.

**Example (3.2):**

Let \( S \) be a commutative ring of characteristic 2. Numbers. Consider \( \mathcal{R} = M_2(S) \) is the ring of all \( 2 \times 2 \) matrices over \( S \) with addition and multiplication of matrices. It is simple matter to check that \( U = \left\{ \begin{pmatrix} u & v \\ v & u \end{pmatrix} \mid u, v \in S \right\} \) is a Lie ideal of \( \mathcal{R} \). Define \( d_n : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R} \) as follows

\[
d_n \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) = \begin{pmatrix} 0 & -b_1 b_2 \ldots b_n \\ n c_1 c_2 \ldots c_n & 0 \\ 0 & n \end{pmatrix}, \text{ for } n = 1, 2 \]

\[
\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}, \text{ for } n \geq 3
\]

For all \( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{R}, n \in \mathbb{N} \), then \( d_n \) is a symmetric higher \((U, \mathcal{R})\)-\textit{n}-derivation of \( \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \).

**Lemma (3.3):**

Let \( \mathcal{R} \) be a prime ring of \( char(\mathcal{R}) \neq 2 \), \( U \) be an admissible Lie ideal of \( \mathcal{R} \) and \( D = (d_n)_{n \in \mathbb{N}} \) be a family of mappings satisfying equation \((\ast)\). Then
\[
d_n(u w + w u, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n) d_j(w, u_2, \ldots, u_n) + d_i(w, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \text{ for all } \]
\[ u, w, u_2, \ldots, u_n \in U, n \in \mathbb{N}. \]

**Proof:** By hypothesis
\[
d_n(u^2, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n), \text{ for all } u, u_2, \ldots, u_n \in U, n \in \mathbb{N}. \] \hfill \ldots (1)

Putting \( u = u + w \) in, equation \((1)\) where \( w \in U \) we obtain
\[
d_n((u + w)^2, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n) d_j(w, u_2, \ldots, u_n) + d_i(w, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \]
\[ + d_i(w, u_2, \ldots, u_n) d_j(w, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \] \hfill \ldots (2)

On the other hand, we have
\[
d_n((u + w)^2, u_2, \ldots, u_n) = d_n(u^2, u_2, \ldots, u_n) + d_n(u w + w u, u_2, \ldots, u_n) + d_n(w^2, u_2, \ldots, u_n). \] \hfill \ldots (3)

Compare the Equations \((2)\) and \((3)\) yields
\[
d_n(u w + w u, u_2, \ldots, u_n) = \sum_{i+j=n} d_i(u, u_2, \ldots, u_n) d_j(w, u_2, \ldots, u_n) + d_i(w, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \text{ for all } u, w, u_2, \ldots, u_n \in U. \]

**Lemma (3.4):**

Let \( \mathcal{R} \) be a 2-torsion-free ring and \( D = (d_n)_{n \in \mathbb{N}} \) be an higher \((U, \mathcal{R})\)-\textit{n}-derivation of \( \mathcal{R} \). Then
\[
d_n(u r u, u_2, \ldots, u_n) = \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \text{ for all } u, u_2, \ldots, u_n \in U, w r \in \mathcal{R}, n \in \mathbb{N}. \]

**Proof:** Let \( r = s = (2u) r + r(2u) \) in Definition \((3.1)\) to get
\[
d_n(u((2u) r + r(2u)) + ((2u) r + r(2u)) u u_2, \ldots, u_n) \]
\[ = \sum_{i+j+r} d_i(u, u_2, \ldots, u_n) d_j(w + r u, u_2, \ldots, u_n) + d_i(w + r u, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) \]
\[ = \sum_{i+j+r} d_i(u, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) d_y(r, u_2, \ldots, u_n) + \]
\[ d_i(u, u_2, \ldots, u_n) d_n(r, u_2, \ldots, u_n) d_q(u, u_2, \ldots, u_n) + 2 \sum_{p+q+j=n} d_p(u, u_2, \ldots, u_n) d_q(r, u_2, \ldots, u_n) d_j(u, u_2, \ldots, u_n) + d_p(r, u_2, \ldots, u_n) d_q(u, u_2, \ldots, u_n) d_f(u, u_2, \ldots, u_n)) \]

... (1)

On the other hand,

\[ d_m(u((2u)r + r(2u)) + ((2u)r + r(2u))u_2, \ldots, u_n) = d_n((2u^2)r + r(2u^2))u_2, \ldots, u_n) + 4d_n(uruu_2, \ldots, u_n) \]

\[ = 2 \sum_{a+b+j=n} d_u(u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) + 2 \sum_{i+j+k=n} d_i(u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) + 4d_n(uruu_2, \ldots, u_n). \]

... (2)

By comparing equations (1) and (2) and since \( R \) is 2-torsion-free, we get

\[ d_n(uruu_2, \ldots, u_n) = \sum_{i+j+k=n} d_i(u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n). \]

... (3)

The following corollary is a linearization of equation (3) on \( u \).

**Corollary (3.5):**

Let \( R \) be a 2-torsion-free ring and \( D = (d_n/n \in N) \) be a higher \((U, R)_n\)-derivation of \( R \). Then

\[ d_n(uruu_2, \ldots, u_n) = \sum_{i+j+k=n} d_i(u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \]

**Lemma (3.6)** [5, Lemma 2.2]: Let \( R \) be a 2-torsion free prime ring and \( U \) be an admissible Lie ideal of \( R \). If \( a \in U, \)

\( b \in R \) are such that \( axb = bxa = 0 \) for all \( x \in R \), then \( axb = bxa = 0 \) for all \( x \in R \).

**Lemma (3.7)** [5, Lemma 2.2.1]:

Let \( R \) be a 2-torsion free prime ring and \( U \subset R \) be a Lie ideal of \( R \). If \( a, b \in R \) such that \( aUb = 0 \), then either \( a = 0 \) or \( b = 0 \).

**Lemma (3.8):**

Let \( R \) be a prime ring of \( char(R) \neq 2, U \) be an admissible Lie ideal of \( R \) and \( D = (d_n/n \in N) \) be a symmetric higher \((U, R)_n\)-derivation. Then \( \phi_n(uw, u_2, \ldots, u_n) = 0 \) for all \( u, w, u_2, \ldots, u_n \in U, n \in N \).

**Proof:** Since \( \phi_0(uw, u_2, \ldots, u_n) = 0 \) and by Theorem (2.15), \( \phi_n(uw, u_2, \ldots, u_n) = 0 \) for all \( u, w, u_2, \ldots, u_n \in U, n \in N \). Assume by induction on \( n \in N \), that \( \phi_n(uw, u_2, \ldots, u_n) = 0 \) for all \( u, w, u_2, \ldots, u_n \in U, n \in N \).

Let \( H = d_n(4(uwwv + uvvw), u_2, \ldots, u_n) \), by Lemma (3.6) we conclude that

\[ H = 4\{d_n(uwvw + wuvu, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \}

\[ + d_i(uw, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \}

By Lemma (3.5) one can compute \( H \) in other way.

\[ H = 4\{uwv \sum_{3 \leq k \leq n} d_k(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \}

\[ + wuv \sum_{3 \leq k \leq n} d_k(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \}

\[ + wuv \sum_{3 \leq k \leq n} d_k(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \}

\[ + wuv \sum_{3 \leq k \leq n} d_k(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(v, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \].
Since $\phi_n(uw,u_2,\ldots,u_n) = 0$ for all $u,w,u_2,\ldots,u_n \in U$, $m \leq N, m < n$ and $\text{char}(R) \neq 2$, comparing both sides of $H$ implies that

$$\phi_n(uw,u_2,\ldots,u_n) v[w] + [u,w] v\phi_n(uw,u_2,\ldots,u_n) = 0$$

For all $u,w,u_2,\ldots,u_n \in U, n \in N$. By Lemma (3.6), $\phi_n(uw,u_2,\ldots,u_n) U[v,w] = 0$ for all $u,w,u_2,\ldots,u_n \in U, n \in N$.

In view of Lemma (3.7) and since $U \varsubsetneq Z(R)$, then $\phi_n(uw,u_2,\ldots,u_n) = 0$ for all $u,w,u_2,\ldots,u_n \in U, n \in N$.

**Lemma (3.9):**

Let $R$ be a prime ring of $\text{char}(R) \neq 2$, $U$ be an admissible Lie ideal of $R$ and $D = (d_n)_{n \in N}$ be an higher $(U,R)$-n-derivation of $R$, then $\phi_n(u^2w,u_2,\ldots,u_n) = 0$ for all $u,w,u_2,\ldots,u_n \in U, r \in R, n \in N$.

**Proof:** we have $\phi_n(uw,u_2,\ldots,u_n) = 0$ for all $u,w,u_2,\ldots,u_n \in U, n \in N$.

Putting $w = ur - ru$ in the last relation, we have $0 = \phi_n(u(ur - ru),u_2,\ldots,u_n) = d_n(u^2r,u_2,\ldots,u_n) - d_n(uw,u_2,\ldots,u_n) + \sum_{i+j=n} d_i(u,u_2,\ldots,u_n) d_j(r,u_2,\ldots,u_n)$

$$d_j(ru - ru,u_2,\ldots,u_n) = d_n(u^2r,u_2,\ldots,u_n)$$

Now, we can prove our main theorem which a generalization of [4, Theorem 2.1] and [2, Theorem].

**Theorem (3.10):**

Let $R$ be a prime ring of $\text{char}(R) \neq 2$, $U$ be an admissible Lie ideal of $R$ and $D = (d_n)_{n \in N}$ be a symmetric higher $(U,R)$-n-derivation. Then $\phi_n(uw,u_2,\ldots,u_n) = 0$ for all $u,w,u_2,\ldots,u_n \in U, r \in R, n \in N$.

**Proof:** Since $\phi_n(uw,u_2,\ldots,u_n) = 0$, by Theorem (2.17) $\phi_n(uw,u_2,\ldots,u_n) = 0$, then we can assume that $\phi_n(uw,u_2,\ldots,u_n) = 0$. Since $D$ is higher $(U,R)$-n-derivation in this case we have

$$d_n(uuw + uwu,u_2,\ldots,u_n) = \sum_{i+j=n} d_i(u,u_2,\ldots,u_n) d_j(r,u_2,\ldots,u_n)$$

$$= ud_n(uw,u_2,\ldots,u_n) + \sum_{i+j=n} d_i(u,u_2,\ldots,u_n) d_j(r,u_2,\ldots,u_n)$$

Now, also on the other hand and by Lemma (3.4) and Lemma (3.9) we get

$$d_n(uuw - uwu,u_2,\ldots,u_n) = \sum_{i+j=n} d_i(u^2,u_2,\ldots,u_n) d_j(r,u_2,\ldots,u_n)$$

$$= \sum_{i+j=n} d_i(u,u_2,\ldots,u_n) d_j(r,u_2,\ldots,u_n) d_k(u,u_2,\ldots,u_n)$$

where $i,j,k \leq n$.
\[ \Sigma_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) = \sum_{g+h+j=n} d_g(u, u_2, \ldots, u_n) d_h(u, u_2, \ldots, u_n) \]

\[ d_j(r, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \]

\[ = d_n(u, u_2, \ldots, u_n) ur + u \sum_{h+j=n} d_h(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) + \sum_{g+h+j=n} d_g(u, u_2, \ldots, u_n) d_h(u, u_2, \ldots, u_n) \]

\[ d_k(r, u_2, \ldots, u_n) + u r d_n(u, u_2, \ldots, u_n) + \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \]

\[ \sum_{i+j+k=n} d_i(u, u_2, \ldots, u_n) d_j(r, u_2, \ldots, u_n) d_k(u, u_2, \ldots, u_n) \] ... (2)

By comparing equations (1) and (2) to get

\[ \phi_n(ur, u_2, \ldots, u_n) u + u \phi_n(ur, u_2, \ldots, u_n) = 0 \] ... (3)

A linearization Equation (3) yield \[ \phi_n(ur, u_2, \ldots, u_n) w + \phi_n(wr, u_2, \ldots, u_n) ur + u \phi_n(wr, u_2, \ldots, u_n) + \phi_n(u, u_2, \ldots, u_n) = 0 \] for all \( u, u_2, \ldots, u_n \in U, r \in R, n \in N \). ... (4)

Let \( w = w^2 \) in Equation (4), then using Lemma (3.9) we get \( \phi_n(ur, u_2, \ldots, u_n) = 0 \) for all \( u, u_2, \ldots, u_n \in U, r \in R, n \in N \).

References


