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An Efficient Technique for solving Lane-Emden Equation

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Abstract:

The orthogonal Boubaker polynomials and their operational matrix of derivatives were deduced, introduced a new efficient approximate method for solving Lane-Emden equation with initial conditions via collocation method. Some numerical examples were given to demonstrate the applicability of this method. The results have been compared with exact solutions to show that they achieved a high accuracy.

Key words: Lane-Emden equation, collocation method, Boubaker polynomial, Gram-Schmidt rule.

1-Introduction

Recently, a lot of numerical methods have been utilized for solving the Lane-Emden equations. The formula of standard Lane-Emden equation is

$$u''(\tau)+\frac{s}{\tau}\,u'(\tau)+u(\tau)=g(\tau),\quad 0<\tau\leq 1,\quad s\geq 0,\quad \ldots.(1)$$

with initial conditions $u(0) = A,\ u'(0)=0,$ where A is constant

where $u(\tau)$, represents continuous real function and $g(\tau)$ is an analytic function.

These types of equations describe the variation of a gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics. Obviously, in past decades there is a remarkable interest in orthogonal polynomials for solving linear and nonlinear problems in physics and engineering. Lane-Emden equations were the focus of many research studies.

Parand and Taghavi (2008) solved nonlinear Lane-Emden on semi-infinite interval [1]. Adibi and Rismani (2010) utilized Legendre-spectral for solving singular initial value problem of Lane-Emden [2]. Some researchers utilized Laplace transform with Legendre wavelets [3-4]. Gelik (2013) used Chebyshev wavelet via collocation method [5]. Acar and Aysegul (2014) Bernestein polynomials were used of nonlinear differential equation of Lane-Emden type equation [6]. Youssri et al. (2015) introduced a new shifted ultra-spherical for solving linear and nonlinear differential equations of Lane-Emden [7]. Doha et al., proposed third and fourth kinds Chebyshev - Galerkin to solve such problem[8]. Yalcin (2018) used the collocation method with Chebyshev polynomials to solve systems of Lane-Emden type equations [9]. Many researchers used new other methods for solving this type of equations [10-14].In this paper, a new function based on Boubaker polynomial was deduced and applied as a new technique for solving Lane-Emden type equation.

This paper is arranged as follows, in section2, the construction of orthogonal Boubaker polynomials and the proposed method have been described, in section3. Some numerical examples have been applied for different kinds of Lane-Emden equation, in section4 then the results were compared with exact solutions. Finally, in section5, conclusions.

2 –Orthogonal Boubaker polynomials

Boubaker polynomials *Bo* have been first appeared by Boubaker et al, for solving different equations in physical applications and applied sciences …etc. see [15-18].

and is presented as in the following equation

$$Bo_m(\tau) = \sum_{s=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(m-4s)}{(m-s)} \binom{m-s}{s} (-1)^s \tau^{m-2s},$$
m = 0,1,2,… ... (3)

The first three terms of Boubaker polynomials are

$Bo_0(\tau) = 1, \quad Bo_1(\tau) = \tau, \quad Bo_2(\tau) = \tau^2 + 2, \ldots$

with recurrence relation $Bo_m(\tau) = \tau Bo_{m-1}(\tau) - Bo_{m-2}(\tau) \quad m > 2$

Since Boubaker polynomials are not orthogonal, the Gram-Schmit method has been applied to find the orthogonal Boubaker polynomials**.**

The first six orthogonal Boubaker polynomials denoted by $B_m(\tau)$ were found to be:

$B_0(\tau) = 1$,

$B_1(\tau) = \frac{1}{2}(2\tau - 1)$,

$B_2(\tau) = \frac{1}{6}(6\tau^2 - 6\tau + 1)$,

$B_3(\tau) = \frac{1}{20}(20\tau^3 - 30\tau^2 + 12\tau - 1)$,

$B_4(\tau) = \frac{1}{70}(70\,\tau^4 - 140\tau^3 + 90\tau^2 - 20\tau + 1)$,

$B_5(\tau) = \frac{1}{252}(252\tau^5 - 630\,\tau^4 + 560\tau^3 - 210\tau^2 + 30\tau - 1), \ldots$

2.1 Some important properties of orthogonal Boubaker polynomial $B_m(\tau)$

2.1.1 Power of τ in term of orthogonal Boubaker polynomial

$\tau^0 = B_0(\tau)$,

$\tau^1 = \left(\frac{1}{2}\right) B_0(\tau) + B_1(\tau)$,

$\tau^2 = \left(\frac{1}{3}\right) B_0(\tau) + B_1(\tau) + B_2(\tau)$,

$\tau^3 = \left(\frac{1}{4}\right) B_0(\tau) + \left(\frac{9}{10}\right) B_1(\tau) + \left(\frac{2}{3}\right) B_2(\tau) + B_3(\tau)$,

$\tau^4 = \left(\frac{1}{5}\right) B_0(\tau) + \left(\frac{4}{5}\right) B_1(\tau) + \left(\frac{12}{7}\right) B_2(\tau) + 2B_3(\tau) + B_4(\tau)$,

$\tau^5 = \left(\frac{1}{6}\right) B_0(\tau) + \left(\frac{5}{7}\right) B_1(\tau) + \left(\frac{25}{14}\right) B_2(\tau) + \left(\frac{25}{9}\right) B_3(\tau) + \left(\frac{5}{2}\right) B_4(\tau) + B_5(\tau)$.

2.1.2 Matrix of Derivative for orthogonal Boubaker polynomial *D* ($B(\tau)$)

In this step we introduce the terms of derivatives of $B(\tau)$ denoted by $\dot{B}(\tau)$ *for m* = 0, 1, 2,…,5, we can write them down as follows:

$\dot{B}_0(\tau) = 0,$
$\dot{B}_1(\tau) = B_0(\tau),$
$\dot{B}_2(\tau) = 2B_1(\tau),$
$\dot{B}_3(\tau) = 3B_2(\tau) + \left(\frac{1}{10}\right)B_0(\tau),$
$\dot{B}_4(\tau) = 4B_3(\tau) + \left(\frac{6}{35}\right)B_1(\tau),$
$\dot{B}_5(\tau) = 5B_4(\tau) + \left(\frac{5}{21}\right)B_2(\tau) +$
$\left(\frac{1}{126}\right)B_0(\tau), \ldots$

$$\frac{dB(\tau)}{d\tau} = \dot{B}(\tau) = D_B B(\tau) \quad \ldots. (4)$$

where the matrix of D_B (6x6) represents the matrix of the first derivatives of orthogonal Boubaker polynomials

$$\begin{bmatrix} \dot{B}_0(\tau) \\ \dot{B}_1(\tau) \\ \dot{B}_2(\tau) \\ \dot{B}_3(\tau) \\ \dot{B}_4(\tau) \\ \dot{B}_5(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 3 & 0 & 0 & 0 \\ 0 & \frac{6}{35} & 0 & 4 & 0 & 0 \\ \frac{1}{126} & 0 & \frac{5}{21} & 0 & 5 & 0 \end{bmatrix} * \begin{bmatrix} B_0(\tau) \\ B_1(\tau) \\ B_2(\tau) \\ B_3(\tau) \\ B_4(\tau) \\ B_5(\tau) \end{bmatrix}$$

2.1.3 Matrix of Derivative for orthogonal Boubaker polynomial $D^2(B(\tau))$

The operational matrix of second derivative can be written as:

$$\frac{d^2B(\tau)}{d\tau^2} = D_B\big(D_B B(\tau)\big) = D_B^2 B(\tau) = \ddot{B}(\tau) \quad \ldots (5)$$

where D^2 represents the matrix (6x6) of the second derivative of $B(\tau)$

$$\begin{bmatrix} \ddot{B}_0(\tau) \\ \ddot{B}_1(\tau) \\ \ddot{B}_2(\tau) \\ \ddot{B}_3(\tau) \\ \ddot{B}_4(\tau) \\ \ddot{B}_5(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ \frac{4}{7} & 0 & 12 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 20 & 0 & 0 \end{bmatrix} * \begin{bmatrix} B_0(\tau) \\ B_1(\tau) \\ B_2(\tau) \\ B_3(\tau) \\ B_4(\tau) \\ B_5(\tau) \end{bmatrix}$$

3-The Method for Solving Lane-Emden equations with orthogonal Boubaker polynomial:

The aim of this solution is to transform Eq.1 into a system of algebraic equations, then finding the coefficients of the approximate solution.

The procedure is as follows:

- Solving Eq.1 by orthogonal Boubaker polynomial with unknown function $u(\tau)$ defined on (0, 1] considering the function as a linear combination as follows:

$$u(\tau) \approx \sum_{i=0}^{m} a_i B_i(\tau) \approx a^T B(\tau) \quad \ldots.. (6)$$

where

$$B(\tau) = [B_0(\tau) \quad B_1(\tau) \quad B_2(\tau) \quad B_3(\tau) \quad B_4(\tau) \quad \ldots B_m(\tau)]^T$$

which represent orthogonal Boubaker polynomials, and a_i's, $i = 0,1,2,\ldots,m$ are the coefficients .

- Multiplying the Eq.1 by τ, we obtain the following equation

$$\tau u''(\tau) + s\, u'(\tau) + \tau u(\tau) = \tau g(\tau), \quad 0 < \tau \le 1, \quad s \ge 0, \qquad \ldots (7)$$

- Substituting Eqs. (4-6) into Eq.7 we get

$$\tau\, D^2 a^t B(\tau) + D a^t B(\tau) + \tau\, a^t B(\tau) = d^t B(\tau) \qquad \ldots (8)$$

where d is the power of orthogonal Boubaker polynomials of $g(\tau)$,

Now finding $\tau\, B(\tau)$ for six term of $B(\tau)$ as follows

$$\tau B_0(\tau) = \left(\frac{1}{2}\right) B_1(\tau) + \left(\frac{1}{2}\right) B_0(\tau),$$

$$\tau B_1(\tau) = \left(\frac{1}{6}\right) B_0(\tau) + \left(\frac{1}{2}\right) B_1(\tau) + \left(\frac{1}{3}\right) B_2(\tau),$$

$$\tau B_2(\tau) = \left(\frac{1}{5}\right) B_1(\tau) + \left(\frac{1}{2}\right) B_2(\tau) + \left(\frac{3}{10}\right) B_3(\tau),$$

$$\tau B_3(\tau) = \left(\frac{3}{14}\right) B_2(\tau) + \left(\frac{1}{2}\right) B_3(\tau) + \left(\frac{2}{7}\right) B_4(\tau),$$

$$\tau B_4(\tau) = \left(\frac{2}{9}\right) B_3(\tau) + \left(\frac{1}{2}\right) B_4(\tau) + \left(\frac{5}{18}\right) B_5(\tau),$$

$$\tau B_5(\tau) = \left(\frac{5}{22}\right) B_4(\tau) + \left(\frac{1}{2}\right) B_5(\tau) + \left(\frac{3}{11}\right) B_6(\tau).$$

Also compute $\tau\, D^2 B(\tau)$ as follows

$$\tau D^2 B_0(\tau) = 0, \qquad \tau D^2 B_1(\tau) = 0$$

$$\tau D^2 B_2(\tau) = 6B_0(\tau) + 6B_1(\tau),$$

$$\tau D^2 B_3(\tau) = 10B_0(\tau) + 30B_1(\tau) + 20B_2(\tau),$$

$$\tau D^2 B_4(\tau) = 20\, B_0(\tau) + 48\, B_1(\tau) + 70B_2(\tau) + 42\, B_3(\tau),$$

$$\tau D^2 B_5(\tau) = 28\, B_0(\tau) + 84\, 6B_1(\tau) + 110B_2(\tau) + 126\, B_3(\tau) + 72\, B_4(\tau).$$

use initial conditions

$$\sum_{i=0}^{m} a_i\, B_i(0) = A\,, \text{ and } \sum_{i=0}^{m} a_i\, \dot{B}_i(0) = 0 \qquad \ldots (9)$$

- Using collocation method by replacing τ in m points to get a system of linear equations which can be easily solved to find the required coefficients.

4-Numerical Examples

Numerical examples were used to illustrate the applicability of this method. The approximate solution and the exact solution were compared to show that the approximate solutions are closer to the exact solutions graphical illustrations were used for this purpose.

Example1: Testing the following Lane-Emden type equation: [8]

$u''(\tau) + \frac{8}{\tau}\acute{u}(\tau) + \tau u(\tau) = \tau^5 - \tau^4 + 44\tau^2 - 30\tau$, $0 < \tau \leq 1$

with initial conditions $u(0) = 0$, $\acute{u}(0) = 0$, and $u_{exact}(\tau) = \tau^4 - \tau^3$

we can easily deduce the solution as

$u(\tau) = 0.05 * B_0(\tau) + 0.1 * B_1(\tau) + 0.214285714285715 * B_2(\tau) + 1 * B_3(\tau) + 1 * B_4(\tau)$

This problem was introduced by (Abd - Elhameed (2016)),our results coincide with the exact result obtained by him [8].

Table (1) shows that the numerical results of Ex.1 are conforming with exact, also the results are illustrated graphically in Fig.1.

Table1: some numerical results of Ex.1

τ	$u_{appro.}(\tau)$	$u_{exact}(\tau)$	error
0	0.000000000000000	0.000000000000000	0.000000000000000
0.1	-0.000900000000000	-0.000900000000000	0.000000000000000
0.2	-0.006400000000000	-0.006400000000000	0.000000000000000
0.3	-0.018900000000000	-0.018900000000000	0.000000000000000
0.4	-0.038400000000000	-0.038400000000000	0.000000000000000
0.5	-0.062500000000000	-0.062500000000000	0.000000000000000
0.6	-0.086400000000000	-0.086400000000000	0.000000000000000
0.7	-0.102900000000000	-0.102900000000000	0.000000000000000
0.8	-0.102400000000000	-0.102400000000000	0.000000000000000
0.9	-0.072900000000000	-0.072900000000000	0.000000000000000
1.0	0.000000000000000	0.000000000000000	0.000000000000000

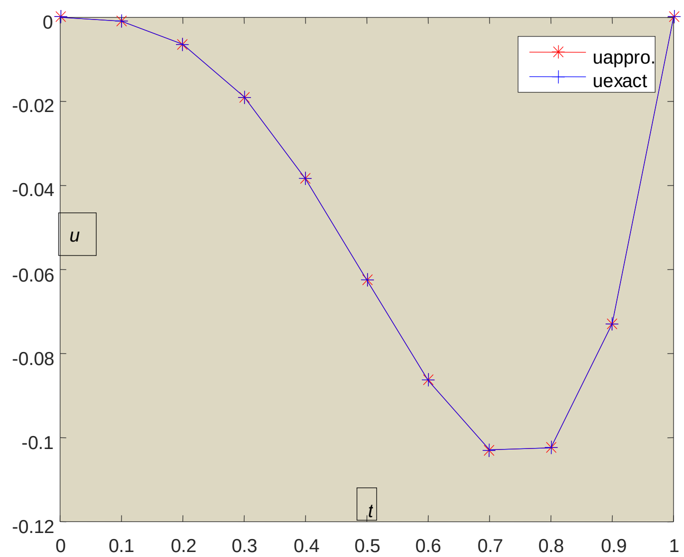

Fig.1 Approximate and exact solutions of Ex.1

Example 2: consider the following homogenous Lane-Emden equation

$$u''(\tau)+\frac{2}{\tau}\acute{u}(\tau)+u(\tau)=0 \quad , 0<\tau\leq 1$$

with $u(0)=1$, $\acute{u}(0)=0$,and $u_{exact}(\tau)=\frac{\sin(\tau)}{\tau}$

By applying the same procedure we deduced that

$u(\tau)=0.946072922225827*B_0(\tau)-0.160149848790850*B_1(\tau)-0.152772815539786*B_2(\tau)+0.015842951770037*B_3(\tau)+0.007450582338353*B_4(\tau)$

Table (2) shows the numerical results for Ex. (2) and compared with exact solution, graphically illustrated in Fig. 2.

Table 2: some numerical results of Ex.2

τ	$u_{app}(\tau)$	$u_{exact}(\tau)$	Error
0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	0.998332107614872	0.998334166468282	0.000002058853409
0.2	0.993341138306667	0.993346653975306	0.000005515668639
0.3	0.985059564894363	0.985067355537799	0.000007790643435
0.4	0.973537741594550	0.973545855771626	0.000008114177076
0.5	0.958843904021431	0.958851077208406	0.000007173186975
0.6	0.941064169186819	0.941070788991726	0.000006619804906
0.7	0.920302535500141	0.920310981768130	0.000008446267989
0.8	0.896680882768436	0.896695113624403	0.000014230855968
0.9	0.870338972196352	0.870363232919426	0.000024260723074
1.0	0.841434446386154	0.841470984807897	0.000036538421742

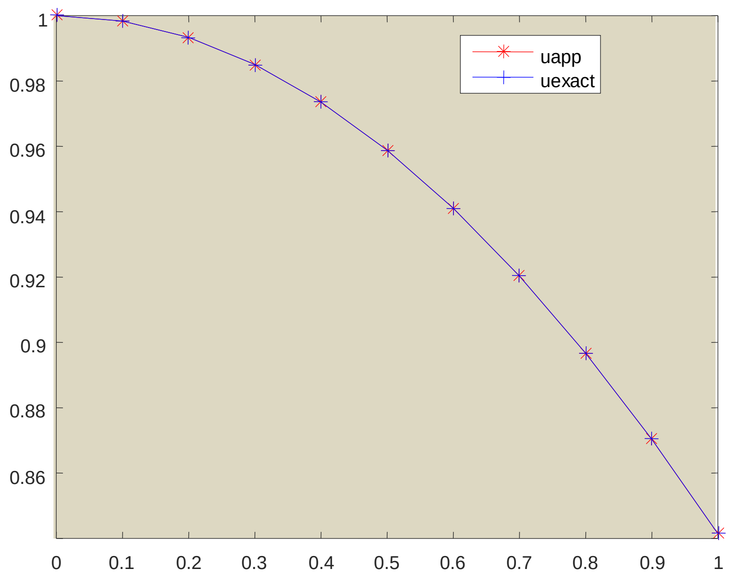

Fig.2 Approximate and exact solution of Ex2

Example 3: Consider the second order non homogenous Lane-Emden differential equation:

$$u''(\tau) + \frac{2}{\tau}\acute{u}(\tau) + u(\tau) = 6 + 12\tau + \tau^2 + \tau^3 \quad , 0 < \tau \leq 1$$

with $u(0) = 0$, $\acute{u}(0) = 0$, and $u_{exact}(\tau) = \tau^2 + \tau^3$

After applying the same procedure in the above examples, we deduced that

$u(\tau) = 0.5833333333333333 * B_0(\tau) + 1.9 * B_1(\tau) + 2.500000000000001 * B_2(\tau) + 1 * B_3(\tau) - 0.000000000000002 * B_4(\tau)$

Table (3) show the numerical results of Ex.3 and compared with exact solution, graphically show in Fig.3.

Table 3: some numerical results of Ex.3

τ	$u_{appro}(\tau)$	$u_{exact}(\tau)$	Error
0	0.000000000000	0.0000000000000	0.0000000000000
0.1	0.011000000000	0.0110000000000	0.0000000000000
0.2	0.048000000000	0.0480000000000	0.0000000000000
0.3	0.117000000000	0.1170000000000	0.0000000000000
0.4	0.224000000000	0.2240000000000	0.0000000000000
0.5	0.375000000000	0.3750000000000	0.0000000000000
0.6	0.576000000000	0.5760000000000	0.0000000000000
0.7	0.833000000000	0.8330000000000	0.0000000000000
0.8	1.152000000000	1.1520000000000	0.0000000000000
0.9	1.539000000000	1.5390000000000	0.0000000000000
1.0	2.000000000000	2.0000000000000	0.0000000000000

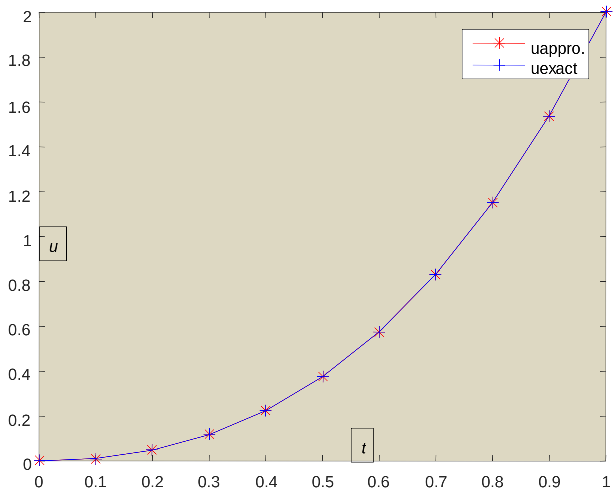

Fig.3 Approximate and exact solution of Ex3

Example4: Consider the following Lane-Emden equation:

$$u''(\tau) + \frac{2}{\tau}\acute{u}(\tau) + u^0(\tau) = 0 \quad , 0 < \tau \leq 1$$

with $u(0) = 1$, $\acute{u}(0) = 0$, and $u_{exact}(\tau) = 1 - \frac{1}{3!}\tau^2$

the solution of this example is :

$u(\tau) = 0.944444444444444 * B_0(\tau) - 0.166666666666667 * B_1(\tau) - 0.166666666666667 * B_2(\tau) + 0 * B_3(\tau) + 0 * B_4(\tau)$

Table (4) shows the numerical results for Ex.4 and compared with exact solution, graphically illustrated in Fig.4.

Table 4: Some Numerical Results of Ex.4

τ	$u_{appro}(\tau)$	$u_{exact}(\tau)$	Error
0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	0.998333333333333	0.998333333333333	0.000000000000000
0.2	0.993333333333333	0.993333333333333	0.000000000000000
0.3	0.985000000000000	0.985000000000000	0.000000000000000
0.4	0.973333333333333	0.973333333333333	0.000000000000000
0.5	0.958333333333333	0.958333333333333	0.000000000000000
0.6	0.940000000000000	0.940000000000000	0.000000000000000
0.7	**0.918333333333333**	**0.918333333333333**	0.000000000000000
0.8	0.893333333333333	0.893333333333333	0.000000000000000
0.9	0.864999999999999	0.865000000000000	0.000000000000001
1.0	0.833333333333333	0.833333333333333	0.000000000000001

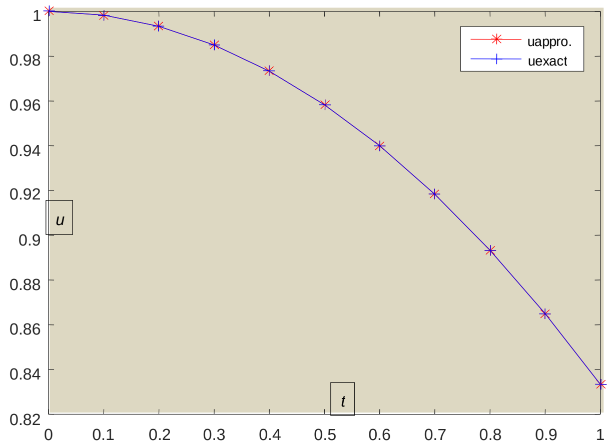

Fig.4 Approximate and exact solution of Ex4

Example5: Consider the following nonlinear Lane-Emden differential equation:

$$u''(\tau) + \frac{6}{\tau}\acute{u}(\tau) + 14u(\tau) = -4u(\tau)\ln(u(\tau)) \quad , 0 < \tau \leq 1$$

with initial conditions $u(0) = 1$, $\acute{u}(0) = 0$, and exact solution $u(t) = e^{-\tau^2}$

$u(\tau) = 0.744557844103311 * B_0(\tau) - 0.694531210494534 * B_1(\tau) - 0.398758539276440 * B_2(\tau) + 0.528752962179964 * B_3(\tau) + 0.075176871314594 * B_4(\tau)$

Table (5) shows the results for Ex.5 and compared with exact solution, graphically illustrated in Fig.5.

Table 5: Some Numerical Results of Ex.5

τ	$u_{appro}(\tau)$	$u_{\text{exact}}(\tau)$	error
0	1.000000000000001	1.000000000000000	0.000000000000001
0.1	0.989433596855264	0.990049833749168	0.000616236893905
0.2	0.959338196544834	0.960789439152323	0.001451242607489
0.3	0.912254831122748	0.913931185271228	0.001676354148480
0.4	0.850904957134199	0.852143788966211	0.001238831832012
0.5	0.778190455615533	0.778800783071405	0.000610327455872
0.6	0.697193632094253	0.697676326071031	0.000482693976778
0.7	0.611177216589016	0.612626394184416	0.001449177595400
0.8	0.523584363609635	0.527292424043049	0.003708060433414
0.9	0.438038652157076	0.444858066222941	0.006819414065865
1.0	0.358344085723463	0.367879441171442	0.009535355447979

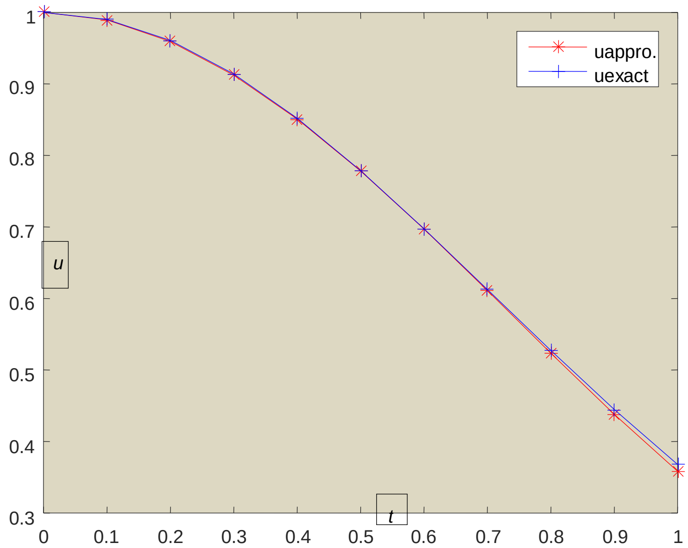

Fig.5 Approximate and exact solution of Ex.5

5- Conclusions:

In this paper, we found that the proposed method using the approximation of orthogonal Boubaker series and the collocation method for solving linear and nonlinear second order Lane-Emden type equations is powerful and computationally efficient. This explicit solution of the orthogonal Boubaker polynomials has been proved simple and accurate with small number of grid points which can be clearly seen by the numerical results illustrated and compared with the exact. This leads to a wide future work in solving many problems in science and engineering.

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