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A NEW SUBCLASS OF STARLIKE HARMONIC FUNCTIONS DEFINED BY SUBORDINATION

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ABSTRACT:
In this current work, by using a relation of subordination, we define a new subclass of starlike harmonic functions. We get coefficient bounds, distortion theorems, extreme points, convolution and convex combinations for this class of functions. Moreover, some relevant connections of the results presented here with diverse known results are briefly denoted.

KEYWORDS: Harmonic functions, starlike functions, subordination

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1. INTRODUCTION

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if both u and v are real harmonic in D. Consider the functions U and V analytic in D so that $u = \operatorname{Re} U$ and $v = \operatorname{Im} V$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)} \quad (z \in D),$$

where $h = (U+V)/2$ and $g = (U-V)/2$. We call h the analytic part and g co-analytic part of f. If g is identically zero then f reduces to the analytic case. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|g'(z)| < |h'(z)| \quad (z \in D)$ (see Clunie and Sheil-Small [2]).

Let S_H denote the class of functions $f = h + \overline{g}$ which are harmonic sense-preserving, and univalent in the open unit disk $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ with $f(0) = f_z(0) - 1 = 0.$ Thus, any function $f \in S_H$ can be written in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \quad (|b_1| < 1). \tag{1}$$

Also, let TS_H denote the subclass of S_H consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = \sum_{n=1}^{\infty} |b_n| \overline{z}^n \quad (|b_1| < 1). \tag{2}$$

Recently, Öztürk et al. [7], studied a family of complex valued harmonic starlike univalent functions related to uniformly convex analytic functions, denoted by $S_H^*(\lambda, \alpha)$ $(0 \le \lambda < 1, 0 \le \alpha < 1)$ so that $f = h + \overline{g} \in S_H^*(\lambda, \alpha)$ if

$$\operatorname{Re}\left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \right\} > \alpha.$$

When $\lambda = \alpha = 0$ and $\lambda = 0,$ this class is denoted by S_H^* and $S_H^*(\alpha),$ respectively. These classes have been studied by Silverman [9], Avcı and Zlotkiewicz [1], Öztürk and Yalçın [8], Jahangiri [6], Yalçın [10].

We say that an analytic function f is subordinate to an analytic function g and write $f \prec g,$ if there exists a complex valued function w which maps E into oneself with $w(0) = 0, |w(z)| < 1$ such that $f(z) = g(w(z)) \quad (z \in E).$

Now, by using a relation of subordination, we define a new subclass of starlike harmonic functions.

1.1 Definition. A function f given by (1) is said to be in the class $S_H^*(\lambda, A, B)$ if the following condition is satisfied

$$\frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \prec \frac{1+Az}{1+Bz}, \tag{3}$$

where $-1 \le B \le -B < A \le 1$ and $0 \le \lambda < 1.$

Also, we let $TS_H^*(\lambda, A, B) \equiv S_H^*(\lambda, A, B) \cap TS_H.$

By suitably specializing the parameters, the class $S_H^*(\lambda, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $S_H^*(0, A, B) = S_H^*(A, B)$ (see [5]),

(ii) $S_H^*(\lambda, 1-2\alpha, -1) = S_H^*(\lambda, \alpha)$ (see [7]),

(iii) $S_H^*(0, 1-2\alpha, -1) = S_H^*(\alpha)$ (see [1],[8],[6]),

(iv) $S_H^*(0, 1, -1) = S_H^*$ (see [9]).

Making use of the techniques and methodology used by Dziok (see [3], [4]), Dziok et al. [5], in this paper, we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $TS_H^*(\lambda, A, B).$

2. MAIN RESULTS

For functions $f_1, f_2 \in S_H$ of the form

$$f_m(z) = z + \sum_{n=2}^{\infty} a_{m,n} z^n + \sum_{n=1}^{\infty} \overline{b_{m,n} z^n} \quad (z \in E, m = 1, 2),$$

we define the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=1}^{\infty} \overline{b_{1,n} b_{2,n} z^n} \quad (z \in E).$$

First we state and prove the necessary and sufficient conditions for harmonic functions in $S_H^*(\lambda, A, B).$

2.1. Theorem. Let $f \in S_H.$ Then $f \in S_H^*(\lambda, A, B)$ if and only if

$$f(z) * \phi(z;\xi) \ne 0 \qquad (\xi \in \mathbb{C}, |\xi| < 1, z \in E \setminus \{0\}),$$

where

$$\phi(z;\xi) = \frac{(A-B)\xi z - (1+A\xi)(1-\lambda)z^2}{(1-z)^2} + \frac{\left[2 + (A+B)\xi - 2(1+A\xi)\lambda\right]\overline{z} - (1+A\xi)(1-\lambda)\overline{z}^2}{(1-\overline{z})^2}.$$

Proof. Let $f \in S_H$ be of the form (1). Then $f \in S_H^*(\lambda, A, B)$ if and only if it satisfies (3) or equivalently

$$\frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \ne \frac{1+A\xi}{1+B\xi}, \tag{4}$$

where $\xi \in \mathbb{C}, |\xi| < 1$ and $z \in E \setminus \{0\}.$ Since

$$h(z) = h(z) * \frac{z}{1-z}, \quad g(z) = g(z) * \frac{z}{1-z}$$

and

$$zh'(z) = h(z) * \frac{z}{(1-z)^2}, \quad zg'(z) = g(z) * \frac{z}{(1-z)^2},$$

the inequality (4) yields

$$\begin{aligned}
&(1+A\xi)\left[\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})\right] \\
&-(1+B\xi)\left[zh'(z) - \overline{zg'(z)}\right] \\
&= h(z) * \left\{\left[(1+A\xi)\lambda - (1+B\xi)\right]\frac{z}{(1-z)^2} + (1+A\xi)(1-\lambda)\frac{z}{1-z}\right\} \\
&+ \overline{g(z)} * \left\{\left[(1+B\xi) - \lambda(1+A\xi)\right]\frac{\overline{z}}{(1-\overline{z})^2} + (1+A\xi)(1-\lambda)\frac{\overline{z}}{1-\overline{z}}\right\} \\
&= h(z) * \frac{(A-B)\xi z - (1+A\xi)(1-\lambda)z^2}{(1-z)^2} \\
&+ \overline{g(z)} * \frac{\left[2 + (A+B)\xi - 2(1+A\xi)\lambda\right]\overline{z} - (1+A\xi)(1-\lambda)\overline{z}^2}{(1-\overline{z})^2} \\
&= f(z) * \phi(z;\xi) \ne 0.
\end{aligned}$$

Here we state a result due to Silverman [9], which we will use throughout this paper.

2.2. Theorem. Let f be of the form (1). If

$$\sum_{n=1}^{\infty} n(|a_n|+|b_n|) \le 2, \tag{5}$$

then f is harmonic, sense preserving, univalent in E and $f \in S_H^*$. The condition (5) is also necessary if $f \in S_H^* \cap TS_H$.

Now we state and prove a sufficient coefficient bound for the class $S_H^*(\lambda, A, B)$.

2.3. Theorem. Let f be of the form (1). If $-1 \le B \le -B < A \le 1$, $0 \le \lambda < 1$ and

$$\sum_{n=1}^{\infty} (\Phi_n |a_n| + \Psi_n |b_n|) \le 2(A-B), \tag{6}$$

where

$$\Phi_n = (A\lambda - B)n + (1-\lambda)(n-1+A) \tag{7}$$

and

$$\Psi_n = (A\lambda - B)n + (1-\lambda)(n+1-A) \tag{8}$$

then f is harmonic, sense preserving, univalent in E and $f \in S_H^*(\lambda, A, B)$.

Proof. Since $n(A-B) \le (A\lambda - B)n + (1-\lambda)(n-1+A)$ and $n(A-B) \le (A\lambda - B)n + (1-\lambda)(n+1-A)$ for $0 \le \lambda < 1$ and $-1 \le B \le 0 < A \le 1$, it follows from Theorem 2.2 that $f \in S_H^*$ and hence f is sense preserving and starlike univalent in E. Now, we only need to show that if (3) holds then $f \in S_H^*(\lambda, A, B)$..

By definition of subordination, $f \in S_H^*(\lambda, A, B)$. if and only if there exists a complex valued function $w;\ w(0)=0, |w(z)|<1\ (z \in E)$ such that

$$\frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

or equivalently

$$\left| \frac{(1-\lambda)(zh'(z) - \overline{zg'(z)} - h(z) - \overline{g(z)})}{(A\lambda - B)(zh'(z) - \overline{zg'(z)}) + A(1-\lambda)(h(z) + \overline{g(z)})} \right| < 1\,. \tag{9}$$

Substituting for $zh'(z)$ and $zg'(z)$ in (9), we obtain

$$\begin{aligned}
&\left|(1-\lambda)(zh'(z) - \overline{zg'(z)} - h(z) - \overline{g(z)})\right| \\
&- \left|(A\lambda - B)(zh'(z) - \overline{zg'(z)}) + A(1-\lambda)(h(z) + \overline{g(z)})\right| \\
&= \left| \sum_{n=2}^{\infty} (1-\lambda)(n-1)a_n z^n - \sum_{n=1}^{\infty} (1-\lambda)(n+1)\overline{b_n z^n} \right| \\
&- \left| (A-B)z + \sum_{n=2}^{\infty} [(A\lambda - B)n + A(1-\lambda)]a_n z^n \right. \\
&\left. - \sum_{n=1}^{\infty} [(A\lambda - B)n - A(1-\lambda)]\overline{b_n z^n} \right| \\
&\le \sum_{n=2}^{\infty} (1-\lambda)(n-1)|a_n||z|^n + \sum_{n=1}^{\infty} (1-\lambda)(n+1)|b_n||z|^n \\
&-(A-B)|z| \\
&+ \sum_{n=2}^{\infty} [(A\lambda - B)n + A(1-\lambda)]|a_n||z|^n \\
&+ \sum_{n=1}^{\infty} [(A\lambda - B)n - A(1-\lambda)]|b_n||z|^n \\
&\le |z| \sum_{n=2}^{\infty} [(A\lambda - B)n + (1-\lambda)(n-1+A)]|a_n| \\
&+ |z| \sum_{n=1}^{\infty} [(A\lambda - B)n + (1-\lambda)(n+1-A)]|b_n| - |z|(A-B) \\
&< 0,
\end{aligned}$$

by (6).

The harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{A-B}{\Phi_n} x_n z^n + \sum_{n=1}^{\infty} \frac{A-B}{\Psi_n} y_n \bar{z}^n, \tag{10}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by in Theorem 2.3 is sharp.

Next we show that the bound (6) is also necessary for $TS_H^*(\lambda, A, B)$.

2.4. Theorem. Let $f = h + \bar{g}$ with h and g of the form (2). Then $f \in S_H^*(\lambda, A, B)$. if and only if the condition (6) holds.

Proof. In view of Theorem 2.3, we only need to show that $f \notin TS_H^*(\lambda, A, B)$ if condition (6) does not hold. We note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (2) to be in $TS_H^*(\lambda, A, B)$ is that the coefficient condition (6) to be satisfied. Equivalently, we must have

$$\left| \frac{-\sum_{n=2}^{\infty} (n-1)(1-\lambda)|a_n| z^n - \sum_{n=1}^{\infty} (n+1)(1-\lambda)|b_n| \bar{z}^n}{(A-B)z - \sum_{n=2}^{\infty} [(A\lambda - B)n + A(1-\lambda)]|a_n| z^n - \sum_{n=1}^{\infty} [(A\lambda - B)n - A(1-\lambda)]|b_n| \bar{z}^n} \right| < 1.$$

For $z = r < 1$, we obtain

$$\frac{\sum_{n=2}^{\infty} (n-1)(1-\lambda)|a_n| r^{n-1} + \sum_{n=1}^{\infty} (n+1)(1-\lambda)|b_n| r^{n-1}}{(A-B) - \sum_{n=2}^{\infty} [(A\lambda - B)n + A(1-\lambda)]|a_n| r^{n-1} - \sum_{n=1}^{\infty} [(A\lambda - B)n - A(1-\lambda)]|b_n| r^{n-1}} < 1. \tag{11}$$

If condition (6) does not hold then condition (11) does not hold for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient (11) is greater than 1. This contradicts the required condition for $f \in S_H^*(\lambda, A, B)$. and so the proof is completed.

2.5. Theorem. Let $f \in S_H^*(\lambda, A, B)$. Then for $|z| = r < 1$, we have

$$|f(z)| \le (1 + |b_1|)r + \left(\frac{A-B}{2(A\lambda - B) + (1-\lambda)(1+A)} - \frac{A\lambda - B + (1-\lambda)(2-A)}{2(A\lambda - B) + (1-\lambda)(1+A)} |b_1| \right) r^2$$

and

$$|f(z)| \ge (1 - |b_1|)r - \left(\frac{A-B}{2(A\lambda - B) + (1-\lambda)(1+A)} - \frac{A\lambda - B + (1-\lambda)(2-A)}{2(A\lambda - B) + (1-\lambda)(1+A)} |b_1| \right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in TS_H^*(\lambda, A, B)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\le (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\
&\le (1 + |b_1|)r + \frac{1}{2(A\lambda - B) + (1-\lambda)(1+A)} \sum_{n=2}^{\infty} (\Phi_n |a_n| + \Psi_n |b_n|) r^2 \\
&\le (1 + |b_1|)r + \frac{A - B - [(A\lambda - B) + (1-\lambda)(2-A)]|b_1|}{2(A\lambda - B) + (1-\lambda)(1+A)} r^2.
\end{aligned}$$

The following covering results follows from the left hand inequality in Theorem 2.5.

2.6. Corollary. Let $f = h + \overline{g}$ with h and g of the form (2). If $f \in TS_H^*(\lambda, A, B)$, then

$$\left\{ w : |w| < \frac{\left[(A-1)\lambda - B + 1\right] - \left[(2-\lambda)(A-1) - B + 1\right]|b_1|}{2(A\lambda - B) + (1-\lambda)(1+A)} \right\} \subset f(E).$$

2.7. Theorem. Set

$$h_1(z) = z, \quad h_n(z) = z - \frac{A-B}{\Phi_n} z^n \quad (n = 2, 3, \ldots)$$

and

$$g_n(z) = z + \frac{A-B}{\Psi_n} \overline{z}^n \quad (n = 1, 2, 3, \ldots).$$

Then $f \in TS_H^*(\lambda, A, B)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z)),$$

where $x_n \geq 0,\ y_n \geq 0,\ \sum_{n=1}^{\infty}(x_n + y_n) = 1.$ In particular, the extreme points of $TS_H^*(\lambda, A, B)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose that

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$

$$= \sum_{n=1}^{\infty} (x_n + y_n) z - \sum_{n=2}^{\infty} \frac{A-B}{\Phi_n} x_n z^n + \sum_{n=2}^{\infty} \frac{A-B}{\Psi_n} y_n \overline{z}^n.$$

Then

$$\sum_{n=2}^{\infty} \Phi_n |a_n| + \sum_{n=1}^{\infty} \Psi_n |b_n| = (A-B) \sum_{n=2}^{\infty} x_n + (A-B) \sum_{n=1}^{\infty} y_n$$

$$= (A-B)(1-x_1) \leq A-B$$

and so $f \in TS_H^*(\lambda, A, B)$. Conversely, if $f \in TS_H^*(\lambda, A, B)$, then

$$|a_n| \leq \frac{A-B}{\Phi_n} \quad \text{and} \quad |b_n| \leq \frac{A-B}{\Psi_n}.$$

Set

$$x_n = \frac{\Phi_n}{A-B} |a_n| \quad (n = 2, 3, \ldots) \quad \text{and}$$

$$y_n = \frac{\Psi_n}{A-B} |b_n| \quad (n = 1, 2, \ldots).$$

Then note by Theorem 2.4, $0 \leq x_n \leq 1$ $(n = 2, 3, \ldots)$ and $0 \leq y_n \leq 1$ $(n = 1, 2, \ldots)$.
We define

$$x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n$$

and note that by Theorem 2.4, $x_1 \geq 0$. Consequently, we obtain $f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$ as required.

2.8. Theorem. The class $TS_H^*(\lambda, A, B)$. is closed under convex combination.

Proof. For $i = 1, 2, \ldots$ let $f_i \in TS_H^*(\lambda, A, B)$. where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z}^n.$$

Then by (6), we get

$$\sum_{n=1}^{\infty} (\Phi_n |a_{n_i}| + \Psi_n |b_{n_i}|) \leq 2(A-B). \tag{12}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1,$ the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}| \right) \overline{z}^n.$$

Then by (12), we can write

$$\sum_{n=1}^{\infty} \left(\Phi_n \sum_{i=1}^{\infty} t_i |a_{n_i}| + \Psi_n \sum_{i=1}^{\infty} t_i |b_{n_i}| \right) = \sum_{n=1}^{\infty} t_i \left(\sum_{i=1}^{\infty} (\Phi_n |a_{n_i}| + \Psi_n |b_{n_i}|) \right)$$

$$\leq 2(A-B) \sum_{i=1}^{\infty} t_i = 2(A-B).$$

This is the condition required by (6) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in TS_H^*(\lambda, A, B).$

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